Language of the billiard inside the cube

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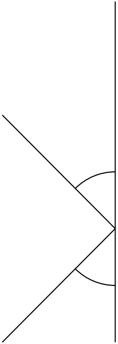
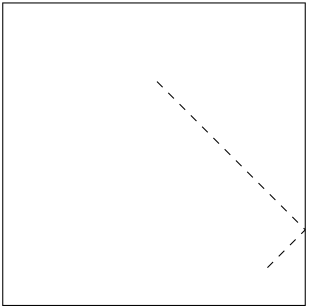
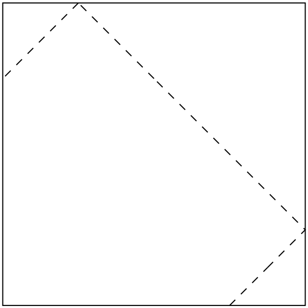
Law of Descartes. Reflection of a line.



Billiard inside a polygon



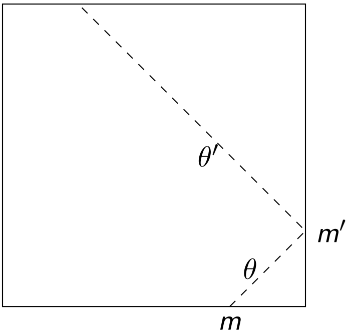
Billiard inside a polygon



Definition

Point m on the boundary and a direction θ.
The image is (m', θ') such that

- m' on the boundary
- (mm') is parallel to θ
- θ' is obtained from θ by reflection along the part of the boundary containing m'



The image of (m, θ) is (m', θ').

We obtain a dynamical system

$$T : \begin{array}{ccc} X & \to & X \\ (m,\theta) & \mapsto & (m',\theta') \end{array}$$

where X is a subset of $\partial P \times \mathbb{R}^2$.

General method to code a dynamical system (X, T) into a subshift.

It allows us to study infinite words.

Let (X, T) be a dynamical system. Consider $\mathcal{P}$ a finite partition of X and a coding

$$\begin{array}{ccc} X & \to & \mathcal{A}^{\mathbb{N}} \\ x & \mapsto & \phi(x) = (u_n)_{\mathbb{N}} \end{array}$$

where $u_n = i$ if $T^n x \in P_i$ and $|\mathcal{A}| = card\mathcal{P}$. We denote (for the usual topology)

$$\Sigma = \overline{\phi(X)}.$$

We have a **subshift** (Σ, S), where S is the shift map, and a language $\mathcal{L}$.

Proposition

We have the following relation

$$\phi \circ T(x) = S \circ \phi(x).$$

This relation shows the relation between the two dynamical systems: they are semi conjugated.

We apply the coding to the billiard:

Square: one letter for parallel faces. We have a 2-letter alphabet.

Same thing inside a cube and a 3-letter alphabet.

Different subshifts and languages (set of finite words which appears in one coding sequence).

There is a subshift $\overline{\{S^n\phi(m,\theta) \mid n \in \mathbb{N}\}}$

And another one with the union on all points m with fixed direction.

$$\overline{\{S^n\phi(m,\theta) \mid n \in \mathbb{N}, m \in \partial P\}}$$

And the one with all the possibilities.

We have

$$\mathcal{L}_{m,\theta} \subset \mathcal{L}_\theta \subset \mathcal{L}$$

Direction $\theta = \begin{pmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{pmatrix}$.

Definition

θ is a minimal direction: for every point m, the orbit of (m, θ) is dense inside the table.

Proposition

The direction θ is a minimal direction if and only if $\theta_1, \ldots, \theta_d$ are linearly independant over $\mathbb{Q}$.

Three remarks:

Projective result: θ and $\lambda\theta$ are the same.

Square:

Minimal direction if and only if $\frac{\theta_2}{\theta_1}$ is irrational.

Cube: more complicated.

Square

Proposition

If θ is minimal, $\mathcal{L}_{m,\theta} = \mathcal{L}_{\theta}$.
If not then we have, for each point m, a periodic word.

$$\begin{cases} p(n,\theta) = n+1 & \textit{minimal direction} \\ p(n,\theta) \leq C \end{cases}$$

Square

Recall that a sturmian word has complexity $n + 1$.

Theorem (Coven-Hedlund)

The word u is a Sturmian word if and only if there exists (m, θ) such that $u = \phi(m, \theta)$

Cube

A trajectory can be:

- dense in the cube $\begin{pmatrix} 1 \\ \sqrt{2} \\ \pi \end{pmatrix}$
- dense on a finite number of planes $\begin{pmatrix} 1 \\ \sqrt{2} \\ 2 \end{pmatrix}$
- Periodic trajectory $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

Theorem

The value of $p(n,m,\theta)$ can be

$$\begin{cases} n^2+n+1 \quad (H) \\ \sim C_\theta n^2 \quad \text{minimal and not (H)} \\ = \Theta(n) \quad \text{non minimal and non periodic} \\ = \Theta(1) \quad \text{periodic trajectory} \end{cases}$$

Hypothesis (H): $\theta_1, \theta_2, \theta_3$ are linearly independant over $\mathbb{Q}$, and $\theta_1^{-1}, \theta_2^{-1}, \theta_3^{-1}$ are linearly independant over $\mathbb{Q}$.

History:

Arnoux-Mauduit-Shiokawa-Tamura 1994: Proof false for minimal direction.

Baryshnikov 1995 (dimension d) maximal value.

Bédaride 2003, 2007, 2009.

Goal

Better understanding of the language of one trajectory inside the cube.

Link with another notion of combinatorics on words.

Let u be an infinite word on the alphabet$\mathcal{A}$.

Let us write a factorization with words of length k

$u = w_0 w_1 w_2 w_3 \cdots w_i \cdots$ avec $w_i \in L_k(u)$, $i \in \mathbb{N}$.

We add a letter c between two consecutive factors
$v = c w_0 c w_1 c w_2 c w_3 c \cdots c w_i c \cdots$

We denote this word $v = I_k^c(u)$.

Example

Consider the Fibonacci word

$$\mathbf{f} = abaababaabaababaababaababaababaabaababaa\cdots.$$

$$I_2^c(\mathbf{f}) = cabcaacbacbacabcaacbacbacabcabcabcaacbacbacabcaacbacbaca\cdots$$

$$I_3^c(\mathbf{f}) = cabacabacbaacbaacbabcaabcabcabacabacbaacbaacbabcaa\cdots.$$

Theorem (Barro 00)

Let u be a sturmian word and $v = I_k^c(u)$. Then we obtain

$$\mathrm{P}_v(n) = \begin{cases} n^2 + n + 1 & si \quad n \leq k \\ kn + k + 1 & si \quad n > k \end{cases}$$

Theorem

Let u be a sturmian word and $v = I_k^c(u)$. Then v is 2-balanced.

Theorem

Let u be a sturmian word and $v = I_k^c(u)$. Then the abelian complexity is given by

$$\mathrm{P}_v^{ab}(n) = \begin{cases} 3 & n = 1 \\ 2 \quad \textit{if} \quad n = 0 & \mod k+1 \\ 4 & \textit{otherwise} \end{cases}$$

Let u be the Fibonacci word, let c be a letter (either a, b or another one) and $v = I_k^c(u)$.

Then remark that $\pi_c(v) = u$ and the frequency of c is equal to $\frac{1}{k+1}$ for some fixed $k \geq 1$.

Consider the orthogonal projection on one face of the cube.

The projection of a billiard trajectory in the cube is a billiard trajectory inside the square.

We have two infinite words u and v on 2 and 3 letters alphabets and $\pi(v) = u$.

$$v = abcabcabcc\dots$$

$$\pi(v) = ababab\dots$$

All the following on one example with u the Fibonacci word.

Same thing for all sturmian words:
We can do the same for all other non minimal direction.

If u is the Fibonacci word, then it is a billiard word inside the square.

We can imagine that v is a billiard word inside a cube.

Find $v = I_k^c(u)$?

One projection is the Fibonacci word.

We look for a trajectory θ which projects on $\begin{pmatrix} 1 \\ \varphi \end{pmatrix}$

And the frequency of the last letter must be $\frac{1}{k+1}$

$$\begin{pmatrix} 1 \\ \varphi \\ x \end{pmatrix}$$

with $\frac{x}{x+\varphi^2} = \frac{1}{k+1}$.

We consider the direction ($k = 2$)

$$\begin{pmatrix} 1 \\ \varphi \\ \frac{\varphi+1}{2} \end{pmatrix}$$

We have a billiard trajectory inside the cube. This trajectory is not dense.

Description of the language ?

Example

Fibonacci word on the alphabet $\{b, c\}$

$$bc.b.bc.bc.b.bc.b.bc.bc.b\dots$$

Insertion of a

$$abc.abb.acb.acb.abc.abb.acv.acb\dots$$

Recoding

$$a_2.a_4.a_1.a_1.a_2\dots$$

List of results

For each non minimal direction:

Value of $p(n,\theta)$.

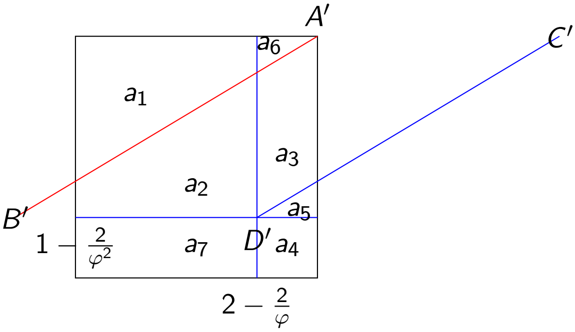
Description of $\mathcal{L}_{m,\theta}$ and the subshift:

For each (m,θ) its coding is the image by a morphism of a (bad) coding of a translation on the torus $\mathbb{T}^1$.

Method

First return map on the face of the cube coded by a. It gives the first return words of a.

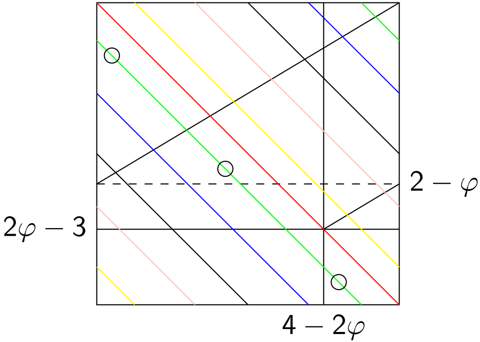
Linear flow on a fixed direction. It explains how to concatenate the return words.



Partition of the face coded by a. Each region corresponds to a return word.

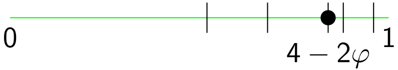
The morphism

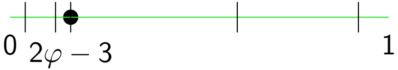
$m \in P_{a_i}$	a_1	a_2	a_3	a_4	a_5	a_6	a_7
$\Phi(a_i)$	acb	abc	$abcb$	abb	$abbc$	$acbb$	ab



Linear flow $z \mapsto z + 2\varphi \mod 1$ on each orbit.

Rotation by $2\varphi - 3$ coded by several ($6 \geq 2$ for this example) intervals.





The green segment intersects 6 polygons. One orbit of the rotation is coded by a word on the alphabet $\{a_1, a_2, a_7, a_4, a_6, a_3\}$.
Then we apply Φ and obtain the cubic billiard word.

If we consider the red segment, we have a rotation coded by 3 letters, and we apply Φ to obtain the billiard word. In this case we obtain Barro results.

Future

Description of the language for a minimal direction.

Billiard inside the hypercube.

Work of M. Andrieu . . .