Avoiding doubled patterns

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Patterns are over $\Delta = \{A, B, \ldots\}$, alphabet of **variables**.
Words are over $\Sigma_k = \{0, 1, \ldots, k-1\}$, alphabet of **letters**.

An **occurrence** is defined by a morphism: $\phi : \Delta^* \to \Sigma^+$, $\phi(P)$ is a factor of w.
Ex: P = ABA occurs in w = 12001001 .

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A pattern is **k-avoidable** if there is an infinite word on k letters with no occurrence of it. Otherwise, it is **k-unavoidable**.
Ex: AA is 2-unavoidable, but AAA isn't.
Thue-Morse morphism: $\varphi : 0 \mapsto 01,\ 1 \mapsto 10$. $\varphi^\omega(0)$ is cube-free.

Doubled pattern: every variable occurs at least twice.
Not doubled: ABBCA .
Doubled: ABACBCA, ABCDBCBACBD .

Theorem 1 (Ochem 2016)

Every doubled pattern is 3-avoidable.

Conjecture 2 (Ochem & Pinlou, 2014)

There exist only finitely many 2-unavoidable doubled patterns.

Growth Rate: $GR_k(P) = \lim_{n\to\infty} |\{w \in \Sigma_k^n \text{ avoiding P}\}|^{1/n}$.

So $|\{w \in \Sigma_k^n \text{ avoiding P}\}| = \tilde{\Theta}(GR_k(P)^n)$.

Proposition 3

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  - P is 2-avoidable $\implies$ $GR_3(P) \geq \sqrt{2}$

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  - P is 2-avoidable $\implies$ $GR_3(P) \geq \sqrt{2}$
  - $GR_3(P) \geq GR_3(AA) \simeq 1.30$

Theorem 5

Every square-free doubled pattern with at most 4 variables is 2-avoidable.

Proof:

1. List all "minimal" relevant patterns.
2. Show that they are 2-avoidable.

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$Q \succcurlyeq P$ if:

$\begin{cases} \text{Q is P up to renaming variables, and } Q \geq_{lex} P \\ \\ \end{cases}$

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$$\begin{aligned} (ABCACDBD)^R &= DBDCACBA \\ &= ABACDCBD \end{aligned}$$

So:

$$ABCACDBD \succcurlyeq ABACDCBD$$

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$$ABCADCDBDAB \succcurlyeq ABCADCDB$$

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Also forbid occurrences of already known 2-avoidable patterns.
e.g: $ABACBC$ occurs in $ABABACDBCD$

2. Show that they are 2-avoidable.

- Morphic words: Rosenfeld's implementation of Cassaigne's algorithm.
  Ex: let $f_5 : \Sigma_5^* \to \Sigma_5^*$ be 01/23/4/21/0, and $c : \Sigma_5^* \to \Sigma_2^*$ be 00/10/111/01/011. Then $c(f_5^\omega(0))$ avoids ABCBDCABCD.

- Otherwise: images of a Dejean word (Ochem lemma, 2004).
  Ex: let w_4 be any Dejean word over Σ_4, and $c : \Sigma_4^* \to \Sigma_2^*$ be 1111/1101/0010/0000. Then $c(w_4)$ avoids ABCBDABCD.

Conjecture 2 (Ochem & Pinlou, 2014)

There exist only finitely many 2-unavoidable doubled patterns.

Conjecture 4

Every square-free doubled pattern is 2-avoidable.

$\rightarrow$ are those two equivalent ?

How much are 2-unavoidable patterns 2-unavoidable ?
→ AA: only three distincts squares needed: $00, 11, 0101$ (Fraenkel, Simpson, 1995)

Different Occurrences Number: number of occurrences of a pattern in a binary word.
Recall : occurrences are morphisms, not factors !
Ex :

ABBA in 000000

ABBA
$A \mapsto 0$
$B \mapsto 00$
$A \mapsto 00$
$B \mapsto 0$
$A \mapsto 0$
$B \mapsto 0$

So $DON(ABBA, 000000) = 3$

Minimal Occurrences Number: $MON(P) = \min_{w \in \Sigma^\omega} \{DON(P, w)\}$

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Theorem 6

Let P be a 2-unavoidable pattern. Then:

$$MON(P) < \infty \iff P \textit{ is doubled}$$

$\Longrightarrow$: (Ex:) P = ABACBAB is 2-unavoidable. Assume by contradiction that there is a word u such that $DON(P, u) = k \in \mathbb{N}^*$.

- $L = \{v \in \Sigma_2^\omega \text{ s.t. } DON(P, v) \leq k\}$ is factorial.
- So there is $w \in L$ recurrent.
  - P occurs at least once : $\phi : A \mapsto x, B \mapsto y, C \mapsto z$
  - So xyx and yxy appear infinitely many times.
  - So $DON(P, w) = \infty \not\leq k$. Contradiction.

Theorem 6

Let P be a 2-unavoidable pattern. Then:

$$MON(P) < \infty \iff P \textit{ is doubled}$$

$\Longleftarrow$: The binary word w_P such that $MON(P) < \infty$ avoids large repetitions of some fixed exponent. w_P exists by a result of Rumyantsev.

Pattern	conjectured MON (and lower bound)
AA	3
AABAB	1
AABCCB	4
ABAB	1
ABACCB	1
ABBA	8
ABBCCA	6
ABCBCA	1
ABCCAB	1
ABCCBA	7

Conjectured MON of 2-unavoidable doubled patterns with at most 3 variables.

(Conjectured observing large finite binary words.)

"The" morphic word for ABBA:

Let $f_5 : \Sigma_5^* \to \Sigma_5^*$ be :

$$\begin{aligned} f_5(0) &= 01, \\ f_5(1) &= 23, \\ f_5(2) &= 4, \\ f_5(3) &= 21, \\ f_5(4) &= 0, \end{aligned}$$

Let $c : \Sigma_5^* \to \Sigma_2^*$ be :

$$\begin{aligned} c(0) &= 0010111100, \\ c(1) &= 1101000011, \\ c(2) &= \epsilon, \\ c(3) &= 1101001100, \\ c(4) &= 0010110011, \end{aligned}$$

$c(f_5^\omega(0))$ shows $\mathrm{MON}(ABBA) = 8$.

ABBA
$A \mapsto 0$ $B \mapsto 0$
$A \mapsto 1$ $B \mapsto 0$
$A \mapsto 0$ $B \mapsto 1$
$A \mapsto 10$ $B \mapsto 11$
$A \mapsto 1$ $B \mapsto 1$
$A \mapsto 10$ $B \mapsto 00$
$A \mapsto 01$ $B \mapsto 11$
$A \mapsto 01$ $B \mapsto 00$

A lot of basic occurrences

A(BC)(BC)A
$A \mapsto 01$ $B \mapsto 0$ $C \mapsto 0$

Any questions ?