Taking and merging games as rewrite games

Eric Duchêne and Aline Parreau

LIRIS, CNRS, Université Lyon 1

Joint work with Victor Marsault and Michel Rigo

One World Combinatorics on Words Seminar, February 27th 2023

First part

Combinatorial games

Combinatorial games: definition

Berlekamp, Conway and Guy, *Winning Ways*, 1981

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- 2 players

Chess

Tarot

Othello

Checkers

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- 2 players
- Total information, no chance
- Finite number of turns, no draw
- Winner given by the last move.
  Normal Convention: the player who cannot play loses.

~~Chess~~

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“Taking and breaking” games

Board: Heaps of tokens
Rules: A player takes tokens in a single heap, with some constraints on the number, and possibly splits the heap.
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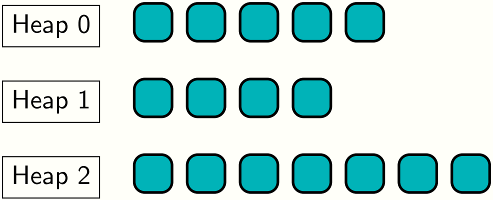

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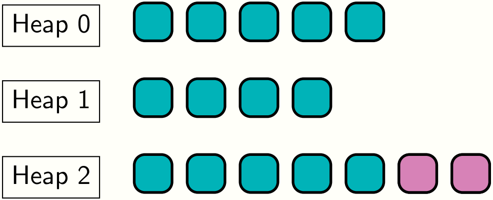

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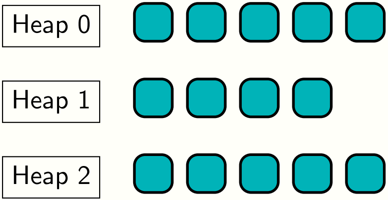

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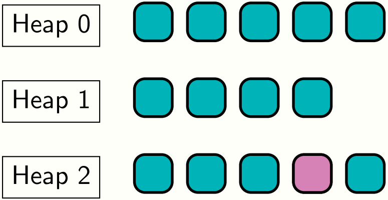

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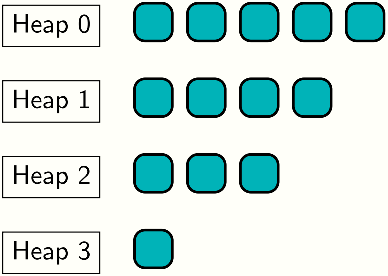

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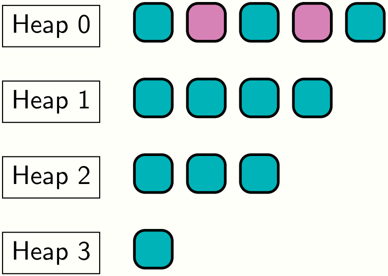

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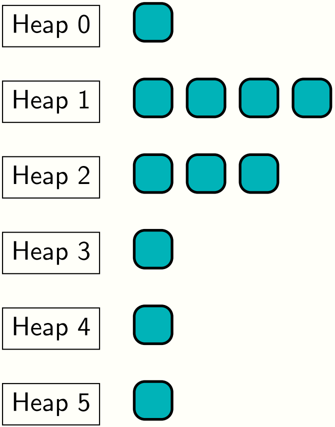

Subtraction games

One cannot split a heap $\rightarrow$ Subtraction game
Defined by a set $S \subseteq \mathbb{N}$:

- At his turn, a player removes $k \in S$ tokens from a heap, without breaking it.
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Example : $S = \{1, 2, 4\}$

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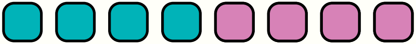

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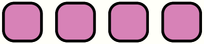

Existence of a winning strategy

Any combinatorial game can be represented by a finite DAG.

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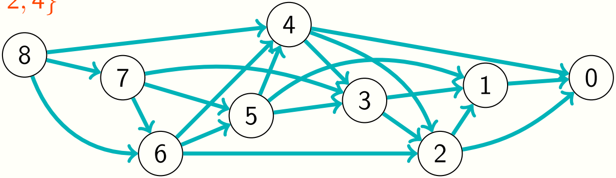

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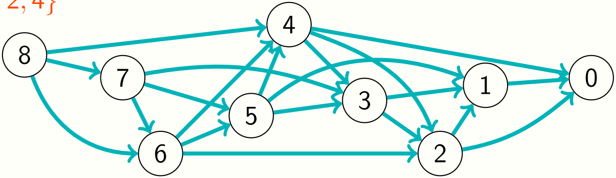

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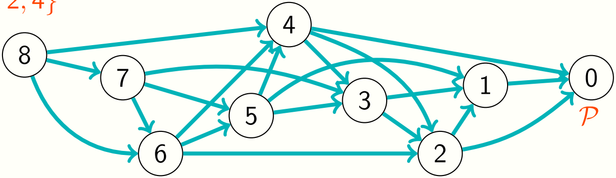

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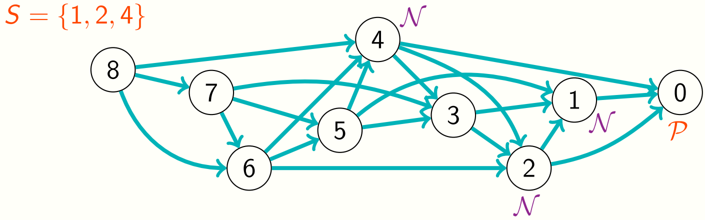

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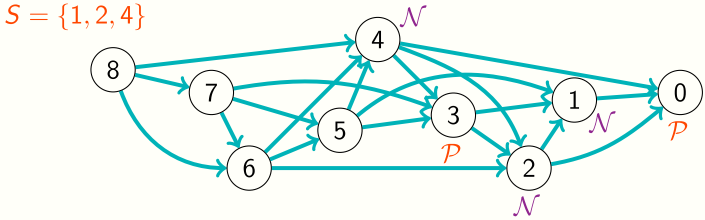

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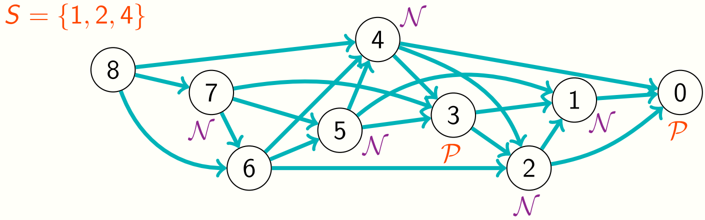

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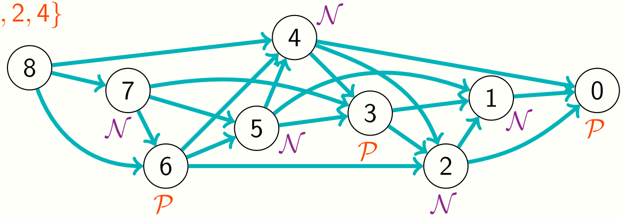

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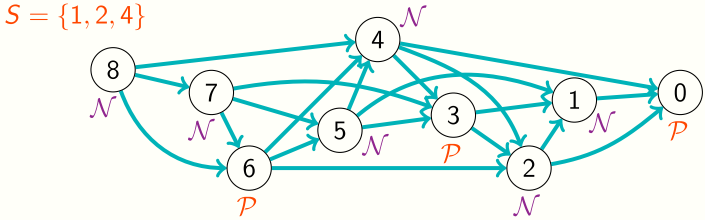

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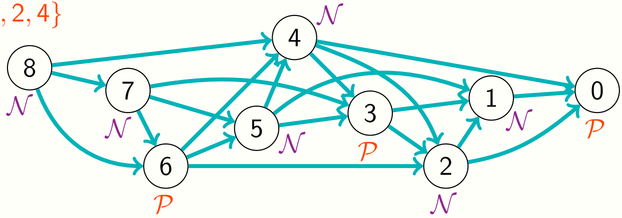

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Theorem

One of the players has a winning strategy.

Main issue

Outcome of the game

Input : Game position
Output : First ($\mathcal{N}$) or second ($\mathcal{P}$) player wins?

Winning strategy

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These two problems can be solved using the DAG...but its size is often exponential !

They are generally in PSPACE

A standard PSPACE problem

Quantified Boolean Formula (QBF)

Input : $Q_1 x_1 Q_2 x_2 \ldots Q_n x_n \Phi(x_1, \ldots, x_n)$: $Q_i \in \{\forall, \exists\}$
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QBF-game :

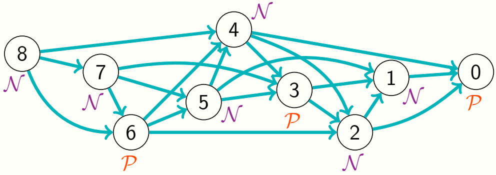
- Board: logic formula $\Phi(x_1, \dots, x_n)$
- Players assign boolean values to x_1,...,x_n, following this order.
- First player wins if at the end the formula is true.

Theorem Schaeffer, 1989 and Arora,Barak, 2009

Deciding if there is a winning strategy for the first player at QBF-game is PSPACE-complete.

Polynomiality of subtraction games

Consider the subtraction game $S = \{1, 2, 4\}$ on n token.



n	0	1	2	3	4	5	6	7	8	9	10	11	12
outcome	$\mathcal{P}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{P}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{P}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{P}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{P}$

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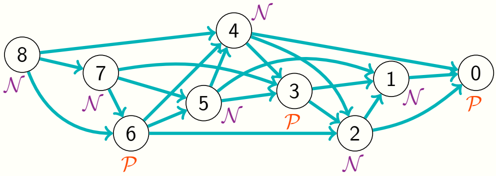

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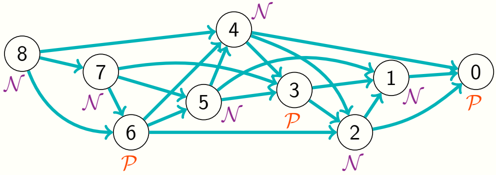

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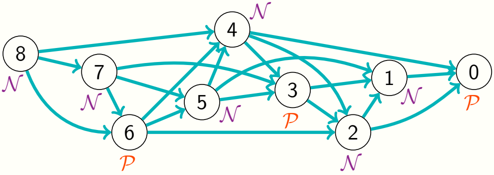

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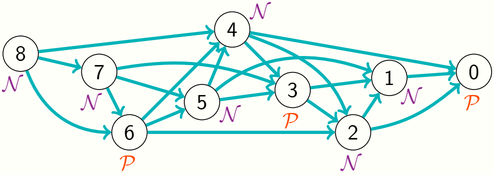

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Open

Size of the preperiod and the period in function of S?

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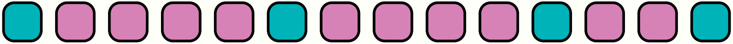

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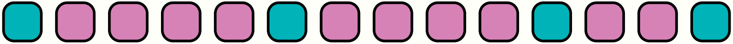

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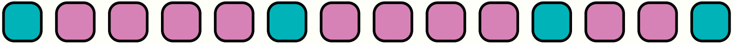
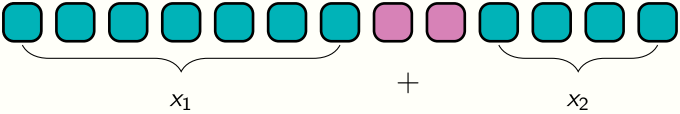


The number of positions is exponential, how to simplify?
$\rightarrow$ with sum of games



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With a $\mathcal{P}$-position:

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 $+$  $\equiv \mathcal{P}$

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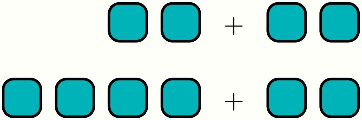

$\equiv \mathcal{P}$

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$\equiv \mathcal{P}$

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$+$	$\mathcal{P}$	$\mathcal{N}$
$\mathcal{P}$	$\mathcal{P}$	$\mathcal{N}$
$\mathcal{N}$	$\mathcal{N}$	$\mathcal{P}$ or $\mathcal{N}$

Grundy values

Let $I \subset \mathbb{N}$. MeX (minimum excluded value) of $I = \min \mathbb{N} \setminus I$.

$$\text{MeX}(\{0, 1, 3, 5\}) = 2, \quad \text{MeX}(\{2, 3, 6\}) = 0, \quad \text{MeX}(\emptyset) = 0.$$

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The Grundy value of a position x is given by

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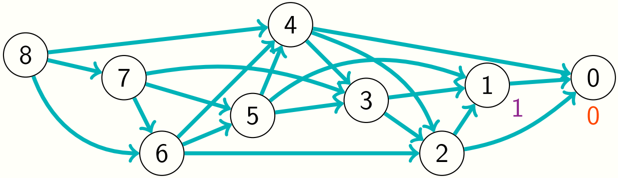

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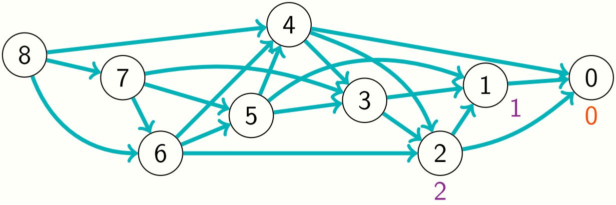

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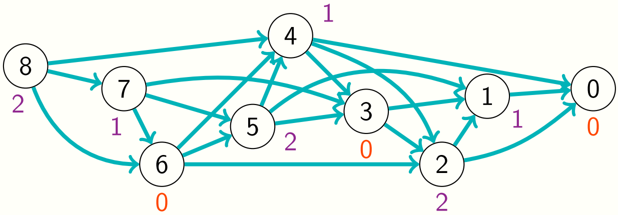

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Let x_1, x_2 be two game positions. Then:

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Corollary

The sum $x_1 + x_2$ is $\mathcal{P}$ if and only if $\mathcal{G}(x_1) = \mathcal{G}(x_2)$.

Grundy sequence

Grundy sequence: sequence of grundy values for 1,2,3,..., n tokens.
For the subtraction game $\{1, 2, 4\}$:

n	0	1	2	3	4	5	6	7	8	9	10	11	12
$\mathcal{G}(n)$	0	1	2	0	1	2	0	1	2	0	1	2	0

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Theorem Guy, Smith, 1956

The Grundy sequence of CRAM is periodic with period 34 and preperiod 53.

Octal games

Played on several heaps of tokens. A move consists in choosing a heap and, according to the rules:

- remove all the tokens from the heap, and delete this heap,
- remove some tokens from the heap, leaving 1 non-empty heap,
- remove some tokens from the heap, separating the remaining tokens into 2 non-empty heaps.

The number of tokens that can be removed is given by the game rules via an octal code

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Examples:

- The subtraction game $\{1, 2, 4\}$ is the octal game $0 \bullet 3303$.
- The game CRAM corresponds to the octal game $0 \bullet 07$.
- The game $0 \bullet 304$ allows you to remove 1 token without splitting the heap, or 3 tokens by necessarily dividing the heap.

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Conjecture Guy, 1956

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- The Grundy sequence of the game $0 \bullet 007$ (James Bond Game) is conjectured to be periodic (tested up to 2^{28})
- It is open for very simple games like $0 \bullet 6$!

Which octal games are in P ?

Nontrivial Octal-Games with at most 3 places

Game	sgv-sequence	type	bitstring	rare	last	max n	max G	index	lost	depth	period	preperiod	except
.6	0012012312340342...	0	0111011111111...	1584	20627	2^{33}	363	7775706554	14	1008823			
.04	0000111220331110...	0	0001110111111...	22476	5029984	2^{28}	1689	248902928	38	5218954			
.06	0001122031122334...	0	?			2^{24}	22097	16360327	37				
.14	0100102122104144...	0	0011111111111...	1896	178727	2^{32}	85	1839780623	172	576735			
.16	0100122140142140...	0	0111111111111...	53	13935	-	23	229790	7	21577	149459	105351	16
.36	0102102132132430...	0	0111111111111...	516	11798	2^{34}	208	1762187846	14	17168			
.37	0120123123403421...	0	0111011111111...	1583	20626	2^{33}	363	7775706553	13	1008822			
.45	0011223114432211...	0	1111111111111...	11	198	-	8	37	2	37	20	498	8
.56	0102241132446621...	0	1101101111111...	46	1795	-	64	22778	2	7405	144	326640	26
.64	0012341532154268...	0	0111110111111...	488	156751	2^{33}	262	1911635806	2	470403814			
.74	0101232414623215...	0	1101101111111...	1386	15929	2^{31}	512	76103606	2	137102			
.76	0102341623416732...	0	0000000110011...	219248	5208068	2^{24}	16814	4995486	2				
.004	0000011112220333...	0	0001111111111...	184854	15869181	2^{25}	6063	22057995	32				
.005	0001011222033411...	1	1110101001011...	95660	67070800	2^{26}	1059	3022366	-				
.006	0000111222033111...	0	0000000101111...	470413	16772624	2^{24}	6532	4798522	40				
.007	0001112203311104...	0	0001110111111...	22476	5029983	2^{28}	1689	248902927	37	5218954			
.014	0010010122123401...	0	0111111111111...	2037	64126	2^{31}	365	169860345	13	126438			
.015	0011010212230142...	0	0111111111111...	237	11973	2^{35}	101	2350397235	7	27036			
.016	0010122201014422...	0	0010111111111...	21439	102335997	2^{27}	1093	102705419	18	41416941			
.024	0001122304112532...	0	?			2^{25}	12371	30810166	26				
.026	0001122304112533...	0	?			2^{25}	37903	33220674	27				
.034	0011022314014312...	0	1111111111111...	1079	374473	2^{34}	256	26376	10	596840			
.054	0010122234411163...	0	1011111111111...	38	796	-	41	33671802	3	16284	10015179	193235616	18
.055	0011122231114443...	0	1111111111111...	6	43	-	8	51	2	20	148	259	2
.064	0001122334115533...	0	0111111111111...	6795	528569986	2^{29}	523	275511554	3	28677643			
.104	0100010221224104...	1	1110111111111...	20	284	-	29	186892397	-	4178	11770282	197769598	9
.106	0100012221440106...	1	1011011111111...	15	1103	-	31	1937780317	-	15343	328226140474	465384263797	25
.114	0110011202120411...	0	1111111101111...	100891	33547932	2^{25}	1610	20501458	11				
.125	0102110213011302...	0	?			2^{24}	44496	16775217	145				
.126	0100213321042503...	1	0111001110001...	20444	102973539	2^{28}	2222	265978	-	40637003			
.127	0102210441220144...	1	1000001111111...	693	27106	-	56	24734	1190	13551	4	46578	11
.135	0112011203110312...	0	?			2^{24}	27960	16768149	91				
.136	0110021302110223...	0	?			2^{24}	25272	15750407	40				
.142	0100222110332410...	1	1100011101111...	1357	117323	2^{34}	441	17142768844	-	411815			
.143	0101222010422150...	0	1011111111111...	9417	2561883	2^{27}	148	26789789	13	3047015			

http://wwwhomes.uni-bielefeld.de/achim/octal.html

Second Part

Rewriting games

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Rewriting games (Waldmann, 2002):

- Rewriting system (terminal)
- Starting from a word t, players alternate applying rules to the word.
- The player who can no longer apply a rule loses.

Example : R_1 : $\texttt{ab} \rightarrow \varepsilon$, R_2 : $\texttt{aaa} \rightarrow \texttt{b}$ and $t = \texttt{aabbbaabaaa}$

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aabbbaabaaa $\rightarrow$ aabbbaaaa $\rightarrow$ aabbbba $\rightarrow$ abbba $\rightarrow$ bba

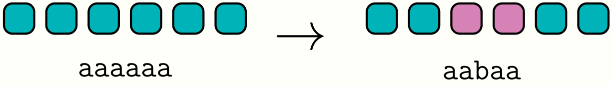
The first player can no longer play and loses.

Allows to model many games, including octal games.

Octal games as rewrite games

Alphabet on two letters: a (for tokens), b (to separate the heaps)

Example: Game CRAM can be modeled with rules $\mathtt{aa} \to \varepsilon$ and $\mathtt{aa} \to \mathtt{b}$.



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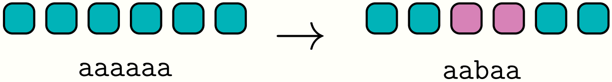

The three rules of octal games can be translated with rewrite rules:

- Emptying a heap of k tokens: $\mathtt{ba}^k\mathtt{b} \to \mathtt{b}$
- Removing k without emptying: $\mathtt{a}^{k+1} \to \mathtt{a}$
- Removing k and splitting in 2: $\mathtt{a}^{k+2} \to \mathtt{aba}$

Interpretation of Periodicity

For each word t, there is a corresponding Grundy value $\mathcal{G}(t)$.
Grundy class $\mathcal{L}_k$: words with value k.

Theorem Waldmann, 2002

The Grundy sequence of an octal game is ultimately periodic iff, in the associated rewriting game, there is a finite number of non-empty Grundy classes, and each class is rational.

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$\Rightarrow$ Find a DFA that determines if a given word $\mathtt{ba}^{x_1}\mathtt{ba}^{x_2}\cdots\mathtt{ba}^{x_n}\mathtt{b}$ satisfies $\mathcal{G}(x_1) \oplus \ldots \oplus \mathcal{G}(x_n) = k$.

- There exists a DFA that computes $\mathcal{G}(x_i)$ $\forall i$
- Before each new x_i, we keep in memory the previous sum $\mathcal{G}(x_1) \oplus \ldots \oplus \mathcal{G}(x_{i-1})$: possible because the number of Grundy classes is bounded (by M).
- The new sum can be computed by a DFA.

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$\Leftarrow$ The $\mathcal{L}_k$ are rational.

- $\mathcal{L}_k \cap \mathtt{ba}^*\mathtt{b}$ is rational.
- Rational language with one letter $\Leftrightarrow \bigcup \mathtt{ba}^{kp+\ell}\mathtt{b} : k \in \mathbb{N}$
- Partition of $\mathbb{N} \Rightarrow$ periods are multiple.

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With different rules? Rational classes?

What about other types of rules?

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Définition

A rewriting game is said to be "taking-and-merging" if all the rules are of the form $\mathtt{a}^k \rightarrow \varepsilon$ or $\mathtt{b}^\ell \rightarrow \varepsilon$

Notation: $\mathtt{a}^{k_1}, \mathtt{a}^{k_2}, \ldots, \mathtt{a}^{k_n}, \mathtt{b}^{\ell_1}, \mathtt{b}^{\ell_2}, \ldots \mathtt{b}^{\ell_m}$

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Question: Are the Grundy classes rational?

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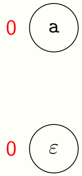
Rules: $\mathtt{aa} \to \varepsilon$ and $\mathtt{b} \to \varepsilon$
We construct the DAG starting with small words:



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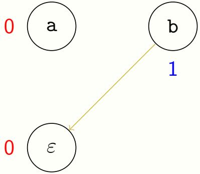

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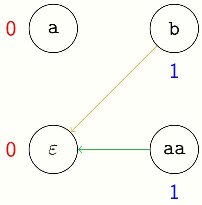

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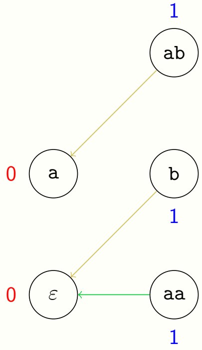

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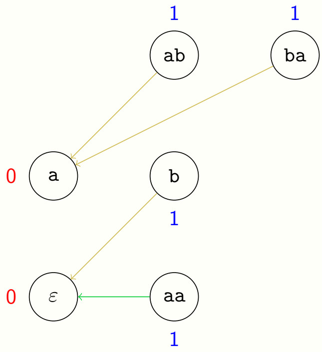

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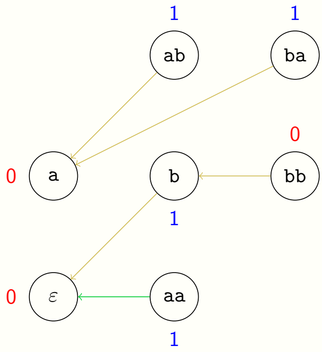

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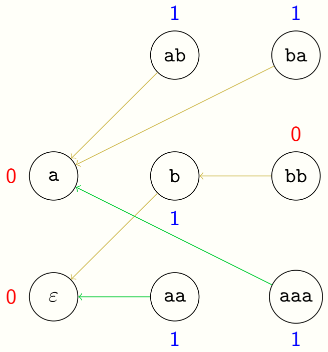

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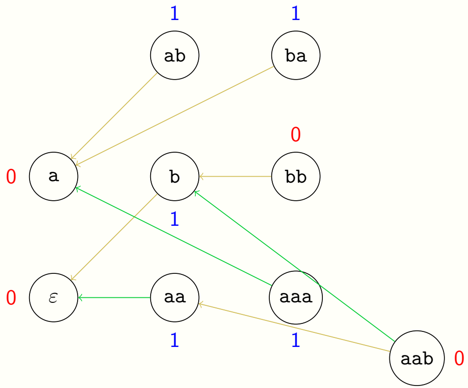

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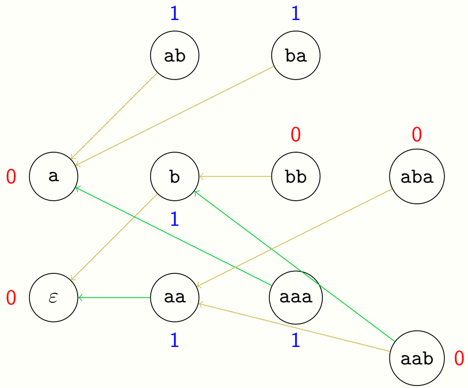

The quantity $|u|_a + 2|u|_b$ decreases by 2 after each move.

A first example: the game $\mathtt{a}^2, \mathtt{b}$

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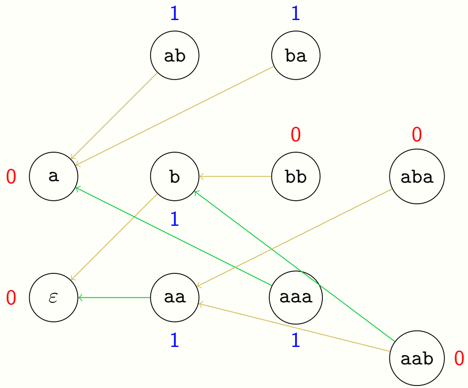


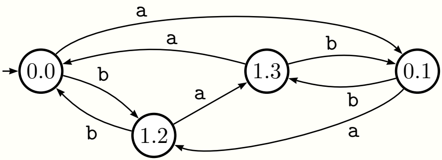
The quantity $|u|_a + 2|u|_b$ decreases by 2 after each move.
$\to \mathcal{G}(u) = 0$ iff $|u|_a + 2|u|_b \bmod 4 \in \{0, 1\}$

The game $\mathtt{a}^2, \mathtt{b}$ is rational

Theorem D., Marsault, P., Rigo, 2020

The game $\{a^2, b\}$ has two classes of Grundy values $\mathcal{L}_0$ and $\mathcal{L}_1$, each forming a rational language.

DFA computing $S(u) = (|u|_a + 2|u|_b) \bmod 4$:



State $g.s$ stands for Grundy value g and $S(u) = s$

Rationality of rewriting games?

Theorem D., Marsault, P., Rigo, 2020

Let G be the rewriting game $\{\mathtt{a}^{k_1}, \mathtt{a}^{k_2}, \cdots, \mathtt{b}^{\ell_1}, \mathtt{b}^{\ell_2}, \cdots\}$, where $1 < k_1 \leq k_2 \leq \ldots$ and $1 < \ell_1 \leq \ell_2 \leq \ldots$.
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Proof idea:

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- By induction: $\mathtt{b}^{\ell_1 - 1}(\mathtt{ab}^{\ell_1 - 1})^{\mathtt{i}}(\mathtt{ba}^{k_1 - 1})^{\mathtt{j}}$ from L is $\mathcal{P}$ iff $i \geq j$.
- The intersection of L and $\mathcal{L}_0$ is not rational, and thus $\mathcal{L}_0$ is not rational.

Games with two rules

Theorem D.,Marsault, P.,Rigo, 2020

The game $\{\mathtt{a}^k, \mathtt{b}^\ell\}$ has only two Grundy classes and the associated languages $\mathcal{L}_0$ and $\mathcal{L}_1$ are context-free.

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Remark: If $\ell = 1$ (or $k = 1$), the stack is not longer needed and $\mathcal{L}_0$, $\mathcal{L}_1$ are rational. The number of moves is: $|u|_{\mathtt{b}} + \lfloor \frac{|u|_{\mathtt{a}}}{k} \rfloor$.

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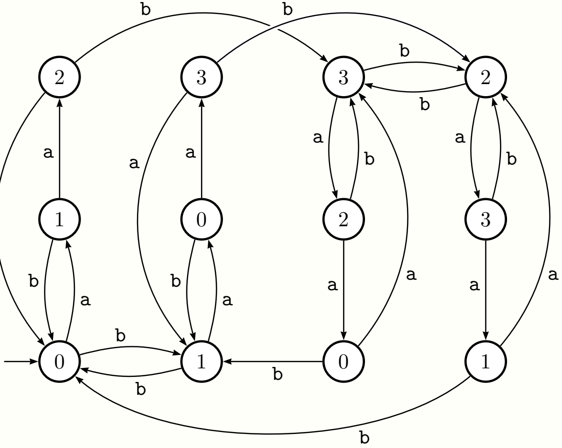

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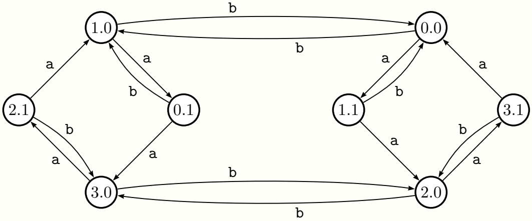

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$$(\max \mathcal{G}(u))_{|u|=0,1,2,\dots} =$$

$$0, 1, 2, 3, 4, 5, 5, 6, 7, 7, 7, 7, 7, 8, 9, 9, 10, 11, 11, 12, 13, 13, 13, 14$$

Question: Are the values for this game bounded?

Computing the rationality more easily

Need to examine all the $\mathcal{L}_i$'s ?

Theorem Waldmann, 2002

For an octal rewriting game:
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  - Question : For the game $\mathtt{a}, \mathtt{a}^2, \mathtt{b}, \mathtt{b}^2$, is $\mathcal{G}$ bounded? Is there an i such that $\mathcal{L}_i$ is not rational?

Deciding $\mathcal{P}$-position of a rewriting game is not decidable

Theorem DMPR,2020

Given a rewriting game and a rational language $\mathcal{L}_R$ of starting positions, it is undecidable to determine if there exists a $\mathcal{P}$-position in $\mathcal{L}_R$.

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Conjecture

Given a rewriting game, it is undecidable to determine if the $\mathcal{P}$-positions are rational.

Conclusion

- Rewriting games generalize a large set of combinatorial games with a nice equivalence between periodicity and regularity.
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Thank you !