PARTIAL ACTION OF GROUPS ON RELATIONAL STRUCTURES : A CONNECTION BETWEEN MODEL THEORY AND PROFINITE TOPOLOGY

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1 Introduction

This note is the text of a lecture delivered by the author at the workshop "Model Theory, Profinite Topology and Semigroups" held in Coimbra (Portugal) in June 2001. The aim is to give here an overview of the links between the study of the profinite topology of free groups and the technics of extension of partial isomorphisms and partial action of groups. There are no proofs in this note as most of the results are published elsewhere as indicated in the text. Moreover if we give all the definitions used here, it may be useful for the reader to go back to the original articles to get more details and examples.

The second goal of this note is to explain the perspectives of this area. Therefore we include at the end a few open problems.

The origin of this research comes from the interaction between model theory, profinite topology of groups and formal languages. Indeed it came out from the conjecture of J.-É Pin on the profinite topology of free groups (which is now known as Ribes-Zalesskiĭ's theorem) that he needed in order to solve the type II conjecture of J. Rhodes. Then B. Herwig and D. Lascar gave another proof of this conjecture using some model theory associated with automorphisms of first order structures as it has been done by E. Hrushovski in his proof that the class of graphs has the extension property (see below).

In fact the question of extending partial ismoprhisms of structures and of studying automorphism groups was already popular among model theorists. It is quite natural while trying to classify first-order structures to look at their automorphism groups. This had been done in various manners. One can refer to the article by D. Lascar[1] to obtain information in this direction and to learn about the small index property.

Section 2 is devoted to define relational structures and their partial isomorphisms. Then we introduce the extension property as it was defined by

B. Herwig and D. Lascar[2]. The reader should refer to their work for all that concern extension property and profinite topology of free groups. In section 3 we deal with the profinite topology of groups. We describe several properties of free groups and we give examples of groups having the same properties. Partial actions are the core of section 4. We present there the results that link extension properties and the profinite topology of groups. A precise and detailed presentation of these topics can be found in works by the author[3,4]. The last two sections present perspectives of this research field. We consider in section 5 a wide class of extension properties by defining $\mathcal{T}$-free structures. They relate to the profinite topology of groups through left systems. For $\mathcal{T}$-free structures and left systems our main sources are the work of B. Herwig and D. Lascar[2] and that of J. Almeida and M. Delgado[13]. The last section contains some open problems.

2 Relational structures

2.1 Some definitions

A **language** is a (finite) set of symbols together with their arity.

A **relational structure** M is a set endowed with interpretations of relational symbols from a given language. For a given symbol R we denote by R_M or simply R its interpretation in M.

Example 1 *A **graph** for us is a relational structure in the language $\mathcal{L} = \{R\}$ where R is a binary relation. With this definition a graph is oriented and it is said unoriented if the relation R is symmetric.*

A **substructure** A of M is a subset of M where each symbol of relation is interpreted as the restriction of its interpretation in M.

A **partial isomorphism** of a relational structure is an isomorphism between two substructures.

We denote by $\mathrm{PI}(M)$ the inverse monoid of all partial isomorphisms of a relational structure M. This inverse monoid is equipped with the usual partial order, which can be defined alternatively by the inclusion of graphs : we say that q is an **extension** of p if the underlying graph of the function p is a subset of the graph of the function q.

If A is a substructure of a relational structure M and p is in $\mathrm{PI}(M)$ we define the restriction of p to A and we denote it by $p\lceil A$ as follows :

$$\forall x, y \in A,\ p\lceil A(x) = y \iff p(x) = y.$$

$\mathrm{Aut}(M)$ is the subgroup of $\mathrm{PI}(M)$ of all automorphisms of the relational structure M.

In this note we will use a particular kind of relational structures that were invented by B. Herwig and D. Lascar : **the n-partitionned structures**. The language for these structures is $\mathcal{L} = \{\longrightarrow, U_1, \ldots, U_n\}$ where $\longrightarrow$ is a binary relation and $U_1, \ldots, U_n$ unary predicates. An $\mathcal{L}$-structure M is said to be n-partitionned if the predicates U_i defines a partition of M and if the realtion $\longrightarrow$ holds between elements satisfying U_i and elements satisfying U_{i+1} $(i = 1, \ldots, n, n+1 = 1)$.

A n-partitionned structure is **n-cycle free** if it does not contains n elements $s_1, \ldots, s_n$ such that

$$s_1 \longrightarrow s_2 \longrightarrow \cdots \longrightarrow s_n \longrightarrow s_1.$$

2.2 Extension properties

To link these relational structures and their partial isomorphisms to the profinite topology of free groups, we use the extension property for a class of structures.

A class $\mathcal{C}$ of structures has the **extension property** if given any finite structure A of $\mathcal{C}$ and a finite collection $p_1, \ldots, p_r$ of partial isomorphisms of A, the following properties are equivalent :

1. there exists an extension M of A (possibly infinite) in $\mathcal{C}$ and automorphisms $\bar{p_1}, \ldots, \bar{p_r}$ of M that extend $p_1, \ldots, p_r$;
2. there exists a finite extension B of A in $\mathcal{C}$ and automorphisms $\tilde{p_1}, \ldots, \tilde{p_r}$ of B that extend $p_1, \ldots, p_r$.

It is obvious that the class of sets have the extension property. Indeed one can always take B equals to A and extends partial isomorphisms of a finite set to automorphisms.

The second result concerning extension property was proved by E. Hrushovski[5] : graphs have the extension property.

In these two first examples the first condition of the definition is always satisfied, which means that the second is as well. The usefulness of the first condition will only become clear in the following examples.

It is a result of B. Herwig and D. Lascar[2] that n-cycle free, n-partitionned structures have the extension property. This is a complicated result which was proved in order to get a new proof of the result of L. Ribes and P. Zalesskiĭ that will be mentionned in the sequel. Here condition 1 of the definition is not always satisfied.

B. Herwig and D. Lascar[2] obtained some even stronger result for the extension property. They proved that the class of $\mathcal{T}$-free structures (of which

n-cycle free n-partitionned structures is a particular case) has the extension property. We will wait until section 5.2 before giving the definition of $\mathcal{T}$-free structures.

3 Profinite topology

The goal of this note is to recall the deep link between the profinite topology of groups and extension of partial isomorphisms. We begin with free groups which may be more familiar and easier. Then we will deal with the general case of groups and for that we will introduce the notion of partial action of a group on a structure.

The profinite topology on a group G is the coarsest topology for which any mapping from G into a finite discrete group is continuous. A basis of clopen neighborhoods of 1 for this topology is given by the finite index subgroups.

3.1 Free groups

Many results are known on the profinite topology of free groups. It is known to be Hausdorff. This means that a free group is residually finite, in other words that for any non-trivial element x of a free group F there exists a finite group G and a morphism π from F to G that maps x to a non trivial element.

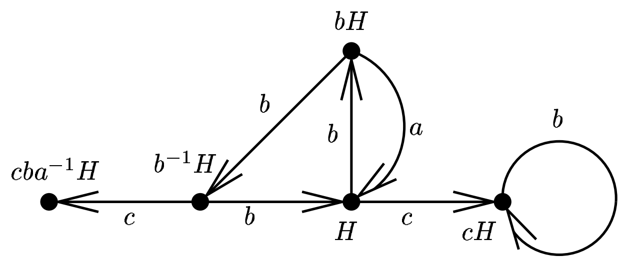
It is also known that any finitely generated subgroup of a free group is closed with respect to its profinite topology. This means that a free group is LERF (locally extended residually finite), in other words that for any finitely generated subgroup H of a free group F and any element x of F which is not in H, there exists a finite group G and a morphism π from F to G which separates H and x :

$$\pi(x) \notin \pi(H).$$

The second of these results was obtained in the pioneering article of M. Hall[6] where the profinite topology is introduced.

This second result is tightly linked with the extension property for the class of sets. Indeed let F be a free group on an alphabet Σ. To any subgroup H of F we can associate the set M of left cosets of H and for any letter of the alphabet define an automorphism of M by left multiplication. If H is finitely generated and x is an element of F, we can define the finite subset A of M whose elements are the cosets wH where w is a subword of one of the generators of H or of x. To any letter we can now define a partial isomorphism of A which is the restriction of the automorphism of M.

Figure 1. An example of partial isomorphisms associated with a finitely generated subgroup of a free group.



The extension property for sets gives us a finite set B containing A and for each letter of the alphabet an automorphism of B that extends the corresponding partial isomorphism of A. This is enough to define a morphism π from F into the symmetric group on B which is a finite group. And it is easy to check that

$$\pi(x) \notin \pi(H).$$

Example 2 *Let* $F = < a, b, c >$, $H = < ab, bbb, c^{-1}bc >$ *and* $x = cba^{-1}$. *Figure 1 shows the set* A *and the partial isomorphisms.*

The sketch of this proof that free groups are LERF and example 2 were given here because they illustrates the correspondance between finitely generated subgroups of free groups and partial isomorphisms. They show how extension of partial isomorphisms can be used to prove that some subsets of free groups are closed for the profinite topology.

After these old results on the profinite topology of free groups, R. Gitik and E. Rips proved :

Theorem 1 ([7]) *Let* H *and* K *be finitely generated subgroups of a free group* F. *The double coset* HK *is closed for the profinite topology of* F.

We say that a free group is double coset separable or RZ_2.

This result is tightly linked with the extension property for the class of graphs. Indeed it is a corollary of Hrushovski's result.

L. Ribes et P. Zalesskiĭ proved an even stronger result :

Theorem 2 ([8]) *Let* $H_1, \ldots, H_n$ *be finitely generated subgroups of a free group* F. *The product set* $H_1 \cdots H_n$ *is closed for the profinite topology of* F.

We say that a free group is RZ_n.

In turn this result is tightly linked with the extension property for the class of n-cycle free n-partionned graphs. B. Herwig and D. Lascar used the extension property for n-cycle free n-partitionned structures to give a new proof of theorem 2.

3.2 Profinite topology of groups

We are here interested in groups whose profinite topology has similar property to that of free groups.

A group is said **residually finite (RF)** if its profinite topology is Hausdorff. It is said **LERF** if all its finitely generated subgroups are closed for its profinite topology. And, it is $\mathbf{RZ}_n$ if for all finitely generated subgroups $H_1, \ldots, H_n$, the product set $H_1 \cdots H_n$ is closed for the profinite topology. We say that a group is **RZ** if it is RZ_n for all integer n.

It is clear that we have a hierarchy of properties. Every group which is RZ_{n+1} is also RZ_n. Properties RZ_1 and LERF are equivalent and every LERF group is RF. We give here various examples of groups with the properties of their profinite topology.

Example 3

- *It is obvious that finite groups and finitely generated abelian groups are RZ.*
- *It is a result of L. Ribes and P. Zalesskii*[8] *that free groups are RZ.*
- *An easy corollary of the previous example is that* $GL_2(\mathbb{Z})$ *and* $SL_2(\mathbb{Z})$ *are RZ.*
- *As a corollary of results by the author*[3] *one gets that surface groups are RZ;*
- *J. Lennox and J. Wilson*[9] *proved that polycyclic-by-finite groups are* RZ_2;
- *The author proved in his thesis*[3] *that free metabelian groups are LERF;*
- *It is a result of K. Gruenberg*[10] *that free solvable groups are RF.*

4 Partial actions

For a group G with a set of generators S to define a morphism into an automorphism group it is not enough to give an automorphism for each element of S. Therefore we need a stronger definition than the one of partial isomorphism. This is the role of our notion of partial action.

Let $\bar{\varphi}$ be an action of a group G on a relational structure M. Let A be a finite substructure of M and S a finite subset of G. The **partial action** φ of G on A induced by $\bar{\varphi}$ and S is the application φ from G into $\mathrm{PI}(A)$ such that for all g in G and for all a, a' in A, we have

$$\varphi(g)(a) = a'$$

if and only if there exists a finite collection $s_1, \ldots, s_r$ in S, $\epsilon_1, \ldots, \epsilon_r$ in $\{\pm 1\}$ and $a_0, \ldots, a_r$ in A such that

$$g = s_1^{\epsilon_1} \cdots s_r^{\epsilon_r},\ a_0 = a,\ a_r = a' \text{ and, } \bar{\varphi}({s_i}^{\epsilon_i})(a_{i-1}) = a_i.$$

Although the definition is technical, it is rather a natural notion as illustrated by the following properties of partial actions.

Property 1 *In the condition of the previous definition, we have :*

1. $\forall s \in S, \varphi(s) = \bar{\varphi}(s)\lceil A;$
2. $\forall g \in G, \varphi(g) \leq \bar{\varphi}(g)$ *and* $\varphi(g)^{-1} = \varphi(g^{-1});$
3. $\forall g, h \in G, \varphi(g) \circ \varphi(h) \leq \varphi(gh).$

Thanks to this three properties it is clear that the **stabilizer** of an element a of A which is defined as

$$\mathrm{Stab}_{\varphi}(a) = \{g \in G | \varphi(g)(a) = a\}$$

is a subgroup of G.

Moreover the requirement that S and A being finite enforces that a stabilizer is a finitely generated subgoup of G.

We can extend the order on the inverse monoid of partial isomorphisms into an order on partial actions. We say that $\tilde{\varphi}$ is an extension of φ if for all g in G, $\tilde{\varphi}(g)$ is an extension of $\varphi(g)$ as a partial isomorphism of A.

The preaction of G on A can now be also defined as the smallest application from G into $\mathrm{PI}(A)$ satisfying the three above properties.

Of course an action (of a finitely generated group G) is a special case of a preaction.

We use this definition of partial action to be able to define an extension property of a given group similar to that of partial isomorphisms.

We say that a group G has the **extension property** for a class of relational structures $\mathcal{C}$ if given any partial action φ of G on a finite element A of $\mathcal{C}$ which is induced by an action $\bar{\varphi}$ of G on a structure M of $\mathcal{C}$ there exists a finite extension B of A which is in $\mathcal{C}$ and an action $\tilde{\varphi}$ of G on B which extends φ.

The existence of $\bar{\varphi}$ and M plays here the role of the first condition in the definition of the extension property in section 2.2.

The extension property for a class of structures, as defined in the previous section, is the extension property of all free groups for this class of structures.

In the sequel of this note, we will try to understand the meaning of the extension properties for different class of structures. We will indeed try to translate these properties into statements about the profinite topology of groups.

Theorem 3 ([11]) *A group has the extension property for sets if and only if it is LERF.*

In fact R. Gitik does not use our terminology but that of labeled graphs and covers which are equivalent to our notions of partial action and extension.

Going a little further one can translate RZ_n into an extension property :

Theorem 4 ([4]) *1. A group has the extension property for graphs if and only if it is RZ_2.*

2. A group has the extension property for n-cycle free, n-partition structures if and only if it is RZ_n.

We will not go into the proofs (which have been published) of these three results. Although the first two one are quite elementary the last one is technical.

R. Gitik first used her result on LERF groups to give new proofs of results on LERF groups. Using the previous characterisation of groups having RZ_n property the author was also able to prove some new results :

Theorem 5 ([4,3]) *1. The free product of two RZ_n groups is RZ_n;*

2. Surface groups are RZ.

5 Various properties ?

The translation between these two settings, extension properties and properties of the profinite topology of groups, is not a simple matter. We were able to understand what is the extension property for n-cycle free, n-partitionned structures (recall that they have been created to prove that free groups are RZ_n), but it appears to be more difficult in other cases.

5.1 RZ_n hierarchy

We first want to stress out that it is not clear that the various RZ_n properties are a strict hierarchy. There are example of groups which are RF and not LERF (free solvable groups of class greater than 3[10,12]), of groups which are

LERF and note RZ_2 (free metabelian groups[3]) and of groups which are RZ_2 but not RZ_3 (free nilpotent group of class 3[9]).

Thereafter we conjecture that this hierarchy is strict, but we lack a proof. Going back to extension properties, the fact that this hierarchy is not strict, for example that RZ_3 and RZ_4 are equivalent would indicate that in a sense that is unclear one could encode 4-cycle free structures in 3-cycle free structures.

5.2 $\mathcal{T}$-free structures

B. Herwig and D. Lascar[2] obtained other extension properties (for free groups). They defined the classes of $\mathcal{T}$-free structures which contain the class of n-cycle free, n-partitionned structures.

We deal with a relational language $\mathcal{L}$ and $\mathcal{L}$-structures.

Let T and M be $\mathcal{L}$-structure. A **weak morphism** from T to M is a mapping f from T to M such that for every symbol R in $\mathcal{L}$ of arity r and every r-tuple $t_1, \ldots, t_r$ we have

$$R_T(t_1, \ldots, t_r) \Rightarrow R_M(f(t_1), \ldots, f(t_r)).$$

Let $\mathcal{T}$ be a finite set of finite structures, a structure M is **$\mathcal{T}$-free** if there is no weak morphism from an element of $\mathcal{T}$ into M.

Theorem 6 ([2]) *The class of $\mathcal{T}$-free structures has the extension property*

This result of B. Herwig and D. Lascar gives us informations on the profinite topology of free groups. This will be detailed in the next section. Indeed we are able to characterize groups which have the extension properties for all classes of $\mathcal{T}$-free structures.

But as before we can also focus on the class of groups which have the extension property for a given class of $\mathcal{T}$-free structures. And then the translation between the two settings is unclear. Moreover it is also unclear that these various extension properties are not equivalent. It is possible that an RZ_3 group has the extension property for all class of $\mathcal{T}$-free structures.

5.3 Left systems

To translate extension properties for $\mathcal{T}$-free structures within pure group theoretic settings, J. Almeida, M. Delgado[13], B. Herwig and D. Lascar introduced left systems of equations in a group.

A **left system** over a group G is a finite set of equations of the following forms :

$$x \equiv_i yc \text{ or } x \equiv_i c$$

where c is an element of G, x, y are variables from a set X and $i = 1, \ldots, n$.

For an n-tuple $\mathcal{H} = (H_1, \ldots, H_n)$ of subgroups of G, a **solution** of the left system $\mathcal{S}$ modulo $\mathcal{H}$ is a family $(v_x)_{x\in X}$ of element of G such that

$$\begin{cases} v_x H_i = v_y c H_i & \text{for all equation } \ x \equiv_i yc \text{ in } \mathcal{S} \\ \text{and} & \\ v_x H_i = c H_i & \text{for all equation } \ x \equiv_i c \text{ in } \mathcal{S} \end{cases}$$

A left system is **finitely approximable** in a group G if for all n-tuple $\mathcal{H} = (H_1, \ldots, H_n)$ of finitely generated subgroups of G there exists an n-tuple $\mathcal{K} = (K_1, \ldots, K_n)$ of finite index subgroups of G such that H_i is contained in K_i and such that $\mathcal{S}$ has a solution modulo $\mathcal{H}$ if and only if it has a solution modulo $\mathcal{K}$.

Details about these notions and the link with $\mathcal{T}$-free structures can be found in the article of B. Herwig and D. Lascar[2]. There they are used mainly for free groups, but we can generalize their work to all groups. The main result that can be proved using the work done by J. Almeida, M. Delagado, B. Herwig and D. Lascar, and inspired by the results of the author about RZ_n groups is the following.

Theorem 7 *A group G has the extension properties for all classes of $\mathcal{T}$-free structures if and only if all left systems are finitely approximable in G.*

6 Open questions

We already mentioned some problems about extension properties. The first one is to prove that the RZ_n hierarchy is strict, or more generally to understand the relative strength of extension properties for different classes of $\mathcal{T}$-free structures. In particular it will be very interesting to find groups having some extension properties and not others.

Another direction is to study groups having the **maximal extension property** that is to say extension property for all classes of $\mathcal{T}$-free structures. The class of group having the maximal extension property contains free groups, finite groups and finitely generated abelian groups. It is very likely that this class is closed under free products as is the class of RZ_n groups and that it contains surface groups.

We conclude this note with a well-known open question that we state as a conjecture.

A **tournament** is an oriented graph such that to different vertices have exactly one oriented edge linking them. It can be seen as a relational structure in a language $\mathcal{L}$ having only one binary relation symbol with the following requirement :

$$\forall x, y (R(x,y) \vee R(y,x)) \wedge \neg (R(x,y) \wedge R(y,x)).$$

Conjecture 1 *The class of tournaments has the extension property.*

The class of tournaments is not one of the classes of $\mathcal{T}$-free structures . But in this case we know how to translate this conjecture within a statement about profinite topology of free groups. Precisely it is equivalent to a conjecture about the oddadic topology of free groups.

The oddadic topology of a group G is the coarsest topology that makes continuous all morphisms from G into finite groups of odd cardinal. The normal subgroups of finite odd index are clopen for this topology. It is not true that all finitely generated subgroups are closed for this topology. A necessary condition for a subgroup H of a free group F to be closed is that for all element x in F, if x^2 is in H then x is in H. When this holds we say that H is closed for square roots.

Conjecture 1 is equivalent to saying that this necessary condition is sufficient :

Conjecture 2 *A finitely generated subgroup of a free group is closed for the oddadic topology of a free group if and only if it is closed for square roots.*

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