The class of groups which have a subgroup of index 2 is not elementary

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Abstract

F. Oger proved that if A is a finite group, then the class of groups which are abelian-by-A can be axiomatized by a single first order sentence. It is established here that, in Oger's result, the word abelian cannot be replaced by group.

In [2] it was proved that, if A is a finite group, then the class of groups G which have an abelian subgroup H such that G/H is isomorphic to A can be axiomatized by a single first order sentence. Professor G. Sabbagh suggested that in this result the word abelian cannot be deleted. This, and more, is established in the present note.

We consider exclusively the case where A is the group with two elements, hence the title of this note.

For any group G and any integer n we denote by G^n the subgroup generated by the n^{th}-powers of elements of G. We denote by $\mathcal{C}$ the class of groups which have a subgroup of index 2.

Theorem 1 *The classe $\mathcal{C}$ is not elementary. More precisely $\mathcal{C}$ is not closed under elementary substructures.*

It is clear that $\mathcal{C}$ is closed under ultraproducts.

The first step of the proof of theorem 1 is the following very simple characterisation of $\mathcal{C}$.

Lemma 2 *For any group G, G is in $\mathcal{C}$ if and only if $G^2 \neq G$.*
Proof : Let G be in $\mathcal{C}$ and N be a subgroup of index 2 in G, then N is normal, and $G^2 \subset N \subsetneq G$. Conversely, if $G^2 \neq G$, then G/G^2 is a $\mathbb{Z}/2\mathbb{Z}$ vector space of dimension ≥ 1 and contains an hyperplane of the form H/G^2. H is a subgroup of G of index 2. □

For every integer i let G_i be the free group on countably many generators: $x_{i,1}, \ldots x_{i,k}, \ldots$. Consider the embeddings : $G_i \longrightarrow G_{i+1}$, $x_{i,k} \longmapsto {x_{i+1,k}}^2$. Define $G_\infty = \varinjlim G_i$.

Obviously, G_∞ is not in $\mathcal{C}$ since ${G_\infty}^2 = G_\infty$.

We will need the next proposition which may be found in [1] p. 53.

Proposition 3 *Let $N > 1$ and let $u_1, \ldots, u_m$ be elements of a free group F which satisfy ${u_1}^N \cdots {u_m}^N = 1$. Then the subgroup generated by $u_1, \ldots, u_m$ has a rank $\leq m/2$.* □

We are going to use a corollary of this result :

Corollary 4 *In the free group, F, on generators $x_1, \ldots, x_n$, for each $k > 0$, the element ${x_1}^{2^k} {x_2}^{2^k} \cdots {x_n}^{2^k}$ is not a product of less than n squares.*

<u>Proof</u> : Suppose ${x_1}^{2^k} {x_2}^{2^k} \cdots {x_n}^{2^k} {u_1}^2 \cdots {u_m}^2 = 1$, where $u_1, \ldots, u_m$ are elements of F. Proposition 3 implies that the rank r of the subgroup H generated by ${x_1}^{2^{k-1}}, \ldots, {x_n}^{2^{k-1}}, u_1, \ldots, u_m$ is at most equal to $(n+m)/2$.

Consider the image of H by the quotient homomorphism : $F \longrightarrow F/F'$ where F' denotes the derived subgroup of F. This image is clearly a free abelian group of rank n. It follows that H cannot be generated by less than n elements and hence $r \geq n$. This implies $m \geq n$ and the corollary. □

It follows from corollary 4 that $x_{0,1} \cdots x_{0,n}$ is not the product of less than n squares in G_∞.

Let $\mathcal{U}$ be a non-principal ultrafilter on ω the set of natural numbers. Let $G = {G_\infty}^{\mathcal{U}}$. We have $G_\infty \prec G$ and $G \in \mathcal{C}$. Indeed, the element of G which is the class of $(x_{0,1} x_{0,2} \cdots x_{0,n})_{n \in \omega}$ is not a product of squares.

This proves the theorem. □

References

[1] Lyndon, R., Schupp, P.: Combinatorial group theory. New-York: Springer 1977.

[2] Oger, F.: Axiomatization of abelian-by-G groups for a finite group G. To appear.

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