A uniqueness result for quasilinear elliptic equations with measures as data

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Abstract We prove here a uniqueness result for Solutions Obtained as the Limit of Approximations of quasilinear elliptic equations with different kinds of boundary conditions and measures as data.

1 Introduction

1.1 Notations

In this paper, Ω is a bounded domain in $\mathbb{R}^N$ ($N \geq 2$), with a Lipschitz continuous boundary. The unit normal to $\partial\Omega$ outward to Ω is denoted by $\mathbf{n}$. We denote by $x \cdot y$ the usual Euclidean product of two vectors $(x, y) \in \mathbb{R}^N \times \mathbb{R}^N$; the associated Euclidean norm is written $|.|$. The Lebesgue measure of a measurable subset E in $\mathbb{R}^N$ is denoted by $|E|$; σ is the Lebesgue measure on $\partial\Omega$ (i.e. the $(N-1)$-dimensional Hausdorff measure). Γ_d and Γ_f are measurable subsets of $\partial\Omega$ such that $\partial\Omega = \Gamma_d \cup \Gamma_f$ and $\sigma(\Gamma_d \cap \Gamma_f) = 0$.
For $q \in [1, +\infty]$, we denote by q' the conjugate exponent of q (i.e. $q' = q/(q-1)$). $W^{1,q}(\Omega)$ is the usual Sobolev space, endowed with the norm $||u||_{W^{1,q}(\Omega)} = ||u||_{L^q(\Omega)} + ||\,|\nabla u|\,||_{L^q(\Omega)}$. $W^{1,q}_{\Gamma_d}(\Omega)$ is the space of functions of $W^{1,q}(\Omega)$ which have a null trace on Γ_d.
When $q = 2$, we write $H^1_{\Gamma_d}(\Omega)$ instead of $W^{1,q}_{\Gamma_d}(\Omega)$. The space of the traces of functions in $H^1_{\Gamma_d}(\Omega)$ is denoted by $H^{1/2}_{\Gamma_d}(\Omega)$ and it is endowed with the norm

$$||u||_{H^{1/2}_{\Gamma_d}(\Omega)} = \inf\{||f||_{H^1(\Omega)} \mid f \in H^1_{\Gamma_d}(\Omega)\,,\ f_{|\partial\Omega} = u\}.$$

The hypotheses on the function a that will define our quasilinear elliptic equation are the following:

$$a : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N \text{ is a Caratheodory function,} \tag{1.1}$$

$$\begin{gathered}\exists \gamma > 0\,,\ \Theta \in L^1(\Omega) \text{ such that } a(x,s,\xi)\cdot\xi \geq \gamma|\xi|^2 - \Theta(x)\\ \text{for a.e. } x \in \Omega, \text{ for all } (s,\xi) \in \mathbb{R}\times\mathbb{R}^N,\end{gathered} \tag{1.2}$$

$$\begin{gathered}\exists \beta > 0 \text{ and } h \in L^2(\Omega) \text{ such that } |a(x,s,\xi)| \leq h(x) + \beta|s| + \beta|\xi|\\ \text{for a.e. } x \in \Omega, \text{ for all } (s,\xi) \in \mathbb{R}\times\mathbb{R}^N,\end{gathered} \tag{1.3}$$

$$\begin{gathered}\exists \alpha > 0 \text{ such that } (a(x,s,\xi) - a(x,s,\eta))\cdot(\xi-\eta) \geq \alpha|\xi-\eta|^2\\ \text{for a.e. } x \in \Omega, \text{ for all } (s,\xi,\eta) \in \mathbb{R}\times\mathbb{R}^N\times\mathbb{R}^N,\end{gathered} \tag{1.4}$$

$$\begin{gathered}\exists \Lambda > 0 \text{ such that } |a(x,s,\xi) - a(x,s,\eta)| \leq \Lambda|\xi-\eta|\\ \text{for a.e. } x \in \Omega, \text{ for all } (s,\xi,\eta) \in \mathbb{R}\times\mathbb{R}^N\times\mathbb{R}^N,\end{gathered} \tag{1.5}$$

$$\begin{gathered}\exists \delta > 0 \text{ such that}\\ |a(x,s,\xi) - a(x,t,\xi)| \leq \delta|s-t| \text{ for a.e. } x \in \Omega,\\ \text{for all } (s,t,\xi) \in \mathbb{R}\times\mathbb{R}\times\mathbb{R}^N.\end{gathered} \tag{1.6}$$

Remark 1.1 *Hypotheses (1.1)—(1.3) are classical for the Leray-Lions operators in divergence form acting on $H^1(\Omega)$; Hypothesis (1.4) is a stronger form of the classical monotonicity hypothesis*

$$(a(x,s,\xi) - a(x,s,\eta))\cdot(\xi-\eta) > 0 \text{ for a.e. } x \in \Omega, \text{ for all } (s,\xi,\eta) \in \mathbb{R}\times\mathbb{R}^N\times\mathbb{R}^N \text{ with } \xi \neq \eta. \tag{1.7}$$

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of the Leray-Lions operators, but is nevertheless classical when we want to obtain a uniqueness result, even in the variational case (see [7]). Hypothesis (1.5) is not really demanding, since, for example, $a(x,s,\xi)=\tilde{a}(s)\xi$ (with $\tilde{a}\in L^\infty(\Omega)$) satisfies this hypothesis, but Hypothesis (1.6) is really strong and we would rather like to impose a weaker hypothesis, of the kind

$$\begin{gathered}\exists\delta>0 \text{ such that } |a(x,s,\xi)-a(x,t,\xi)|\le\delta|s-t|(1+|s|+|t|+|\xi|)\\ \text{for a.e. } x\in\Omega, \text{ for all } (s,t,\xi)\in\mathbb{R}\times\mathbb{R}\times\mathbb{R}^N\end{gathered}$$

to handle the case $a(x,s,\xi)=\tilde{a}(s)\xi$ with $\tilde{a}$ Lipschitz continuous.

Remark 1.2 *There are however many functions which satisfy Hypotheses (1.1)—(1.6). For example, for $M\ge 0$, $a(x,s,\xi)=(1+\inf(M,\ln(1+|s|+|\xi|)))\xi+\phi(x,s)$, with $\phi:\Omega\times\mathbb{R}\to\mathbb{R}^N$ a Caratheodory function, Lipschitz continuous with respect to $s\in\mathbb{R}$ (with a Lipschitz constant not depending on $x\in\Omega$) and such that $\sup_{s\in\mathbb{R}}|\phi(.,s)|\in L^2(\Omega)$.*

Consider the problem

$$\left\{\begin{array}{l}-\mathrm{div}(a(x,u,\nabla u))=f \quad \text{in} \quad \Omega,\\ u=0 \quad \text{on} \quad \partial\Omega.\end{array}\right. \tag{1.8}$$

It is well known (see [6]) that, when f is a bounded measure on Ω and a satisfies (1.1)—(1.3) and (1.7), we can find a solution to this problem (even when we consider an operator acting on $W_0^{1,p}(\Omega)$, $1<p<\infty$ — see also [2] when $p<1-\frac{2}{N}$ —, not only on $H_0^1(\Omega)$). The main idea of [6] is to approximate f by regular functions, find estimates on the corresponding solutions and pass to the limit.
Moreover, when a does not depend on s and f is a function in $L^1(\Omega)$, we can find (see [2]) a formulation (so-called "entropy formulation") for (1.8) which ensures the uniqueness of the solution (the existence is still obtained by approximation).

In [12], the author defines another sense of solution, the "solution by transposition", which gives an existence and uniqueness result when a still does not depend on s but f is a bounded measure. This definition requires the introduction of a particular matrix-valued function $M(.,.):\mathbb{R}^N\times\mathbb{R}^N\longrightarrow M_N(\mathbb{R})$ satisfying a few properties (general algebraic properties, completely independent of a); the formulation by transposition uses then the matrix $M(\nabla u-\nabla v, a(.,\nabla u)-a(.,\nabla v))$, where u is the solution by transposition and v is any function in $H_0^1(\Omega)$. There can be many different possible choices of the matrix $M(.,.)$ (the matrix chosen by the author depends on a parameter λ, which is any real number in $]0,\alpha[$, where α is given by (1.4)). The solution by transposition seems thus to depend on the particular choice of M; however, an additional work allows to see that, with the methods of [12], we can prove the uniqueness of the solution obtained as the limit of approximations (when a is independent of s).

When f is a bounded measure, a satisfies (1.1)—(1.5) but does not depend on s and is of class $\mathcal{C}^1$ with respect to ξ, the uniqueness of the solution obtained as the limit of approximations of Problem (1.8) is proven in [3].
We will prove here that the ideas of [3] can lead to a uniqueness result when f is a bounded measure, a depends on s (but satifies (1.6)) and is only Lipschitz continuous with respect to ξ. The main difficulty brought by the dependence of a on s is in the resolution of the "dual equation" (2.3) in which the operator is not coercive (because of the convection term). We will also consider more general boundary conditions; they bring a few more difficulties (in particular the regularity result we need on the solution of (2.3)) which are solved by the results of [10].

The boundary conditions we consider are of the mixed or Fourier kind (that is to say a condition on u on Γ_d and a condition on $a(x,u,\nabla u)\cdot\mathbf{n}+\lambda u$ on Γ_f).
To get the coercivity that will ensure the existence of a solution, we add the assumption

$$\begin{gathered}\sigma(\Gamma_d)>0 \text{ and } \lambda\in L^\infty(\partial\Omega)\,,\ \lambda\ge 0\ \sigma\text{-a.e. on } \partial\Omega\\ \text{or}\\ \Gamma_d=\emptyset \text{ and } \lambda\in L^\infty(\partial\Omega)\,,\ \lambda\ge 0\ \sigma\text{-a.e. on } \partial\Omega,\ \sigma(\{x\in\partial\Omega \mid \lambda(x)>0\})\ne 0.\end{gathered} \tag{1.9}$$

Remark 1.3 *Under Hypothesis (1.9), a classical reasoning shows that, for all $q \in [1,+\infty[$, $\overline{q} \in [1,q]$ and $\rho > 0$, there exists $\mathcal{K}_{q,\overline{q}}(\rho,\Omega,\Gamma_d,\lambda) > 0$ such that, for all $v \in W^{1,q}_{\Gamma_d}(\Omega)$, we have*

$$\rho \int_\Omega |\nabla v|^q + \left(\int_{\Gamma_f} \lambda |v|^{\overline{q}}\, d\sigma \right)^{q/\overline{q}} \geq \mathcal{K}_{q,\overline{q}}(\rho,\Omega,\Gamma_d,\lambda) ||v||^q_{W^{1,q}(\Omega)}. \tag{1.10}$$

The proof of uniqueness we present here uses an existence and regularity result of a solution to a dual problem. To obtain the required regularity result, we need some hypotheses on the way Γ_d and Γ_f are distributed along $\partial\Omega$.
Let us introduce two kinds of mapping of $\partial\Omega$:

$$\begin{array}{c} O \text{ is an open subset of } \mathbb{R}^N, \\ h : O \to B := \{x \in \mathbb{R}^N \mid |x| < 1\} \text{ is a Lipschitz continuous} \\ \text{homeomorphism with a Lipschitz continuous inverse mapping,} \\ h(O \cap \Omega) = B_+ := \{x \in B \mid x_N > 0\}, \\ h(O \cap \partial\Omega) = B^{N-1} := \{x \in \partial B_+ \mid x_N = 0\} \end{array} \tag{1.11}$$

(since Ω has a Lipschitz continuous boundary, there exists a finite number of $(O_i,h_i)_{i\in[1,m]}$, such that, for all $i \in [1,m]$, (O_i,h_i) satisfies (1.11) and $\partial\Omega \subset \cup_{i=1}^m O_i$) and

$$\begin{array}{c} O \text{ is an open subset of } \mathbb{R}^N, \\ h : O \to B \text{ is a Lipschitz continuous homeomorphism} \\ \text{with a Lipschitz continuous inverse mapping,} \\ h(O \cap \Omega) = B_{++} := \{x \in B \mid x_N > 0\,,\ x_{N-1} > 0\}, \\ h(O \cap \Gamma_f) = \Gamma_1 := \{x \in \partial B_{++} \mid x_{N-1} = 0\}, \\ h(O \cap \Gamma_d) = \Gamma_2 := \{x \in \partial B_{++} \mid x_N = 0\}. \end{array} \tag{1.12}$$

The additional assumption we make on Γ_d and Γ_f is the following:

$$\begin{array}{c} \text{There exists a finite number of } (O_i,h_i)_{i\in[1,m]} \text{ such that} \\ \partial\Omega \subset \cup_{i=1}^m O_i \text{ and, for all } i \in [1,m],\ (O_i,h_i) \text{ is of one of the following types:} \\ \left| \begin{array}{ll} (D) & O_i \cap \partial\Omega = O_i \cap \Gamma_d \text{ and } (O_i,h_i) \text{ satisfies (1.11)} \\ (F) & O_i \cap \partial\Omega = O_i \cap \Gamma_f \text{ and } (O_i,h_i) \text{ satisfies (1.11)} \\ (DF) & (O_i,h_i) \text{ satisfies (1.12).} \end{array} \right. \end{array} \tag{1.13}$$

1.2 The SOLA and the main result

We recall here some facts about the solutions obtained as the limit of approximations for quasilinear elliptic equations with measures as data.

We denote by $\mathcal{M}(\Omega)$ the space of bounded measures on Ω and $\mathcal{M}(\partial\Omega)$ the space of bounded measures on $\partial\Omega$.
If $\mu \in \mathcal{M}(\Omega)$ and $\mu^\partial \in \mathcal{M}(\partial\Omega)$, we consider the problem

$$\left\{ \begin{array}{l} -\text{div}(a(x,u,\nabla u)) = \mu \quad \text{in} \quad \Omega, \\ u = 0 \quad \text{on} \quad \Gamma_d, \\ a(x,u,\nabla u)\cdot \mathbf{n} + \lambda u = \mu^\partial \quad \text{on} \quad \Gamma_f. \end{array} \right. \tag{1.14}$$

The technique of approximation introduced in [6] is the following: let $(\mu_n)_{n\geq 1}\in L^1(\Omega)\cap(H^1_{\Gamma_d}(\Omega))'$([1]) such that $\mu_n\to\mu$ for the weak-$*$ topology of $(\mathcal{C}(\overline{\Omega}))'$, $(\mu_n^\partial)_{n\geq 1}\in L^1(\partial\Omega)\cap(H^{1/2}_{\Gamma_d}(\partial\Omega))'$ such that $\mu_n^\partial\to\mu^\partial$ for the weak-$*$ topology of $\mathcal{M}(\partial\Omega)$ and take u_n a solution to

$$\left\{\begin{array}{l} u_n\in H^1_{\Gamma_d}(\Omega)\,, \\ \displaystyle\int_\Omega a(x,u_n,\nabla u_n)\cdot\nabla\varphi+\int_{\Gamma_f}\lambda u_n\varphi\,d\sigma=\langle\mu_n,\varphi\rangle_{(H^1_{\Gamma_d}(\Omega))',H^1_{\Gamma_d}(\Omega)} \\ \qquad\qquad +\langle\mu_n^\partial,\varphi\rangle_{(H^{1/2}_{\Gamma_d}(\partial\Omega))',H^{1/2}_{\Gamma_d}(\partial\Omega)}\,,\ \forall\varphi\in H^1_{\Gamma_d}(\Omega). \end{array}\right. \tag{1.15}$$

We can prove that the sequence $(u_n)_{n\geq 1}$ is bounded in $W^{1,q}_{\Gamma_d}(\Omega)$ for all $q<N/(N-1)$; thus, up to a subsequence, $u_n\to u$ strongly in $L^q(\Omega)$ and weakly in $W^{1,q}_{\Gamma_d}(\Omega)$; it is then possible to prove that, up to a subsequence, $\nabla u_n\to\nabla u$ a.e. on Ω, which allows us to pass to the limit in the equation of (1.15) to see that u satisfies

$$\left\{\begin{array}{l} u\in\displaystyle\bigcap_{q<N/(N-1)}W^{1,q}_{\Gamma_d}(\Omega)\,, \\ \displaystyle\int_\Omega a(x,u,\nabla u)\cdot\nabla\varphi+\int_{\Gamma_f}\lambda u\varphi=\int_\Omega\varphi\,d\mu+\int_{\partial\Omega}\varphi\,d\mu^\partial\,,\ \forall\varphi\in\bigcup_{r>N}W^{1,r}_{\Gamma_d}(\Omega). \end{array}\right. \tag{1.16}$$

A Solution Obtained as the Limit of Approximations (a SOLA) for (1.14) is any u obtained by the method detailed above.

Remark 1.4 *In [6], where the SOLA (without this name, used for the first time in [8]) have been introduced, the authors study the pure homogeneous Dirichlet case (with $\Theta=0$). But the adaptation of their methods to the non-homogeneous mixed or Fourier case is quite straightforward (see [14] for the Fourier case with $\Theta\equiv 0$), even with a non-null $\Theta\in L^1(\Omega)$.*

When $N\geq 3$, the solution of (1.16) is not always unique; indeed, a counter-example by J. Serrin [15] modified by A. Prignet [13] gives a non-null solution of (1.16) in the linear ($a(x,s,\xi)=A(x)\xi$) Dirichlet case when $\mu=\mu^\partial=0$ (see also [10] for the adaptation of this counter-example to the mixed case). However, there is uniqueness of the SOLA for this problem, and this is the main result of this paper:

Theorem 1.1 *Under Hypotheses (1.1)—(1.6), (1.9) and (1.13), Problem (1.14) has one and only one SOLA.*

Remark 1.5 *In fact, the proof of the existence of a SOLA to (1.14) does not use all our hypotheses on a (it only uses (1.1)—(1.3), (1.7) and (1.9)). Our proof of the uniqueness of the SOLA does not use all the Hypotheses we put on a too; indeed, we will see that we do not use (1.2) and (1.3) in this paper, we only use the fact that a SOLA exists. Thus, this result of uniqueness can be extented to other equations for which we know a SOLA exists. For example, in [4], L. Boccardo proves a wide existence result (for a pure Dirichlet problem — this is quite important — with a right-hand side in L^1) that entails the existence of a SOLA for an operator defined by a function of the kind*

$$a(x,s,\xi)=a_0(x,s,\xi)+\phi(s),$$

where a_0 satisfies (1.1)—(1.6) and $\phi:\mathbb{R}\to\mathbb{R}^N$ is a Lipschitz continuous function; the hypotheses on ϕ in [4] are in fact much weaker and require thus $f\in L^1(\Omega)$, but our stronger hypotheses allow us to take a right-hand side in $\mathcal{M}(\Omega)$. Thus, a satisfies (1.1), (1.4)—(1.6) and the existence and uniqueness result of Theorem 1.1 is still valid for such an operator in the pure Dirichlet case.

[1]This means that μ_n is a function of $L^1(\Omega)$ such that there exists $C>0$ satisfying, for all $\varphi\in L^\infty(\Omega)\cap H^1_{\Gamma_d}(\Omega)$, $|\int\mu_n\varphi|\leq C||\varphi||_{H^1(\Omega)}$; by density of $L^\infty(\Omega)\cap H^1_{\Gamma_d}(\Omega)$ in $H^1_{\Gamma_d}(\Omega)$, there exists then a unique extension of μ_n as an element of $(H^1_{\Gamma_d}(\Omega))'$. The same kind of definition and consideration apply to $\mu_n^\partial\in L^1(\partial\Omega)\cap(H^{1/2}_{\Gamma_d}(\partial\Omega))'$.

We will also see that this uniqueness result implies the following (very simple) stability result.

Theorem 1.2 *Let $(\mu_n)_{n\geq 1}\in\mathcal{M}(\Omega)$ converges to μ in $(\mathcal{C}(\overline{\Omega}))'$ weak-$*$ and $(\mu_n^\partial)_{n\geq 1}\in\mathcal{M}(\partial\Omega)$ converges to μ^∂ in $\mathcal{M}(\partial\Omega)$ weak-$*$. Under Hypotheses (1.1)—(1.6), (1.9) and (1.13), if u_n is the SOLA of (1.14) with (μ_n,μ_n^∂) instead of (μ,μ^∂) and u is the SOLA of (1.14), then $u_n\to u$ strongly in $W^{1,q}_{\Gamma_d}(\Omega)$ for all $q<\frac{N}{N-1}$.*

Remark 1.6 *In fact, we will prove the following more general result: under Hypotheses (1.1)—(1.3), (1.7) and (1.9), if u_n is a SOLA — of a slightly particular kind, see in the proof of Theorem 1.2 — of (1.14) with (μ_n,μ_n^∂) instead of (μ,μ^∂), there exists a subsequence $(u_{n_k})_{k\geq 1}$ and a SOLA u of (1.14) such that $u_{n_k}\overset{k\to\infty}{\longrightarrow} u$ strongly in $W^{1,q}_{\Gamma_d}(\Omega)$ for all $q<N/(N-1)$. The fact that we can, with stronger hypotheses, get rid of the subsequence is of course due to the uniqueness of the SOLA in this case.*

Remark 1.7 *Once again, the proof of this stability result only uses the existence and uniqueness of the SOLA, not all the hypotheses on a (especially, we do not use (1.2) and (1.3)); thus Theorem 1.2 is also valid for other kinds of quasilinear equations for which we know a SOLA exists, such as the example given in Remark 1.5.*

A uniqueness result for a linear equation is very often linked to an existence result for a dual equation. It is also the case here, although (1.14) is not a linear problem; so, before the proof of Theorem 1.1, we study in Section 2 an equation which will appear as the dual equation of a problem coming from (1.14).

2 The "dual" equation

We make the following hypotheses:

$$\begin{array}{c}A:\Omega\to M_N(\mathbb{R})\text{ is a measurable matrix valued function which satisfies:}\\ \exists\alpha>0\text{ such that }A(x)\xi\cdot\xi\geq\alpha|\xi|^2\text{ for a.e. }x\in\Omega\,,\text{ for all }\xi\in\mathbb{R}^N,\\ \exists M>0\text{ such that }||A(x)||:=\sup\left\{|A(x)\xi|\,,\;\xi\in\mathbb{R}^N\,,\;|\xi|=1\right\}\leq M\text{ for a.e. }x\in\Omega,\end{array}\tag{2.1}$$

$$\mathbf{v}\in(L^\infty(\Omega))^N\,,\tag{2.2}$$

and we take α_A a coercivity constant for A, Λ_A an essential upper bound of $||A(.)||$ on Ω and $\Lambda_{\mathbf{v}}$ an upper bound of $||\,|\mathbf{v}|\,||_{L^\infty(\Omega)}$.

We will prove the following existence result:

Theorem 2.1 *Under Hypotheses (2.1), (2.2), (1.9) and (1.13), if $\theta\in L^\infty(\Omega)$ then, by denoting by Λ_θ an upper bound of $||\theta||_{L^\infty(\Omega)}$, there exists $\kappa\in]0,1[$ depending on $(\Omega,\alpha_A,\Lambda_A,\Lambda_{\mathbf{v}},\lambda)$, C_0 depending on $(\Omega,\Gamma_d,\alpha_A,\Lambda_A,\Lambda_{\mathbf{v}},\lambda,\Lambda_\theta)$ and C_1 depending on $(\Omega,\Gamma_d,\alpha_A,\Lambda_{\mathbf{v}},\Lambda_\theta)$ such that there exists a solution to*

$$\left\{\begin{array}{l} f\in H^1_{\Gamma_d}(\Omega)\cap\mathcal{C}^{0,\kappa}(\Omega)\,,\\ \displaystyle\int_\Omega A\nabla f\cdot\nabla\varphi+\int_\Omega\mathbf{v}\cdot\nabla f\varphi+\int_{\Gamma_f}\lambda f\varphi\,d\sigma=\int_\Omega\theta\varphi\,,\;\forall\varphi\in H^1_{\Gamma_d}(\Omega)\end{array}\right.\tag{2.3}$$

satisfying $||f||_{\mathcal{C}^{0,\kappa}(\Omega)}\leq C_0$ and $||f||_{H^1(\Omega)}\leq C_1$.

Remark 2.1 *We have denoted by $\mathcal{C}^{0,\kappa}(\Omega)$ the space of κ-Hölder continuous functions on Ω, endowed with the norm*

$$||f||_{\mathcal{C}^{0,\kappa}(\Omega)}=||f||_{L^\infty(\Omega)}+\sup_{x\neq y}\frac{|f(x)-f(y)|}{|x-y|^\kappa}.$$

Remark 2.2 *Without Hypothesis (1.13), we obtain a solution of the equation in Problem (2.3) in the space* $H^1_{\Gamma_d}(\Omega)\cap L^\infty(\Omega)$*, with the same kind of estimates (we will notice it in the course of the proof); Hypothesis (1.13) is only useful to apply the results of [10] in order to obtain the Hölder continuity of the solution.*

To prove the existence result of Theorem 2.1, we need an *a priori* estimate on the solutions of (2.3) (an L^1 estimate is enough). This is the aim of Lemma 2.1 for the proof of which the authors wish to thank Lucio Boccardo (for having given them the key estimate of Step 2).

Lemma 2.1 *Let* A *satisfy (2.1),* $\mathbf{w}\in(L^\infty(\Omega))^N$ *and* $\tau\in L^\infty(\Omega)$*; we denote by* $\Lambda_{\mathbf{w}}$ *an upper bound of* $||\,|\mathbf{w}|\,||_{L^\infty(\Omega)}$ *and* Λ_τ *an upper bound of* $||\tau||_{L^\infty(\Omega)}$*. Under Hypothesis (1.9), there exists* C_0 *depending on* $(\Omega,\Gamma_d,\alpha_A,\Lambda_{\mathbf{w}},\lambda,\Lambda_\tau)$ *and a solution to*

$$\left\{\begin{array}{l} g\in H^1_{\Gamma_d}(\Omega)\cap L^\infty(\Omega)\,,\\ \displaystyle\int_\Omega A^T\nabla g\cdot\nabla\varphi+\int_\Omega g\mathbf{w}\cdot\nabla\varphi+\int_{\Gamma_f}\lambda g\varphi\,d\sigma=\int_\Omega\tau\varphi\,,\ \forall\varphi\in H^1_{\Gamma_d}(\Omega)\end{array}\right. \tag{2.4}$$

such that $||g||_{H^1(\Omega)}+||g||_{L^\infty(\Omega)}\le C_0$.

Remark 2.3 *Once we know that* g *satisfies (2.4), since* $\varphi\to\int_\Omega g\mathbf{v}\cdot\nabla\varphi$ *is in* $(W^{1,1}_{\Gamma_d}(\Omega))'$ *(because* g *is essentially bounded), the results of [10] show that, under Hypothesis (1.13),* g *is in fact Hölder continuous on* $\overline{\Omega}$*.*

Remark 2.4 *The conclusions of Theorem 2.1 and Lemma 2.1 also hold when* θ *or* τ *only belong to* $\bigcup_{p>N}(W^{1,p'}_{\Gamma_d}(\Omega))'$ *(the proof of this uses the same ideas we present here; see [16] or [10] for the details concerning the treatment of right-hand sides of this kind).*

Remark 2.5 *(Lucio Boccardo [5]) A close examination of the second step of the proof of Lemma 2.1 shows that the bound we obtain on* $||\ln(1+|g_n|)||_{H^1_{\Gamma_d}(\Omega)}$ *depends on the* L^1*-norm of the right-hand side* τ*. Thus, we can easily prove (by approximation) an existence result for*

$$\left\{\begin{array}{l} -\mathrm{div}(A^T\nabla g)-\mathrm{div}(g\mathbf{v})=\tau\quad in\quad\Omega\,,\\ g=0\quad on\quad\Gamma_d\,,\\ A^T\nabla g\cdot\mathbf{n}+\lambda g=0\quad on\quad\Gamma_f\,,\end{array}\right. \tag{2.5}$$

(this problem has, when τ *is regular, (2.7) as variational formulation) when* τ *is a bounded measure on* Ω*; we must however be careful with the formulation of (2.5) since we only obtain a "solution"* g *such that, for all* $k\ge 0$*,* $T_k(g)\in H^1_{\Gamma_d}(\Omega)$ *(where* $T_k(s)=\min(k,\max(s,-k))$*).*

Remark 2.6 *Using the results of Theorem 2.1 and Lemma 2.1 and the ideas of their proofs, we can prove, when* $L\in(H^1(\Omega))'$*, the existence and uniqueness of solutions to*

$$\left\{\begin{array}{l} f\in H^1_{\Gamma_d}(\Omega)\,,\\ \displaystyle\int_\Omega A\nabla f\cdot\nabla\varphi+\int_\Omega\mathbf{v}\cdot\nabla f\varphi+\int_{\Gamma_f}\lambda f\varphi\,d\sigma=\langle L,\varphi\rangle_{(H^1_{\Gamma_d}(\Omega))',H^1_{\Gamma_d}(\Omega)}\,,\ \forall\varphi\in H^1_{\Gamma_d}(\Omega)\end{array}\right. \tag{2.6}$$

and

$$\left\{\begin{array}{l} g\in H^1_{\Gamma_d}(\Omega)\,,\\ \displaystyle\int_\Omega A^T\nabla g\cdot\nabla\varphi+\int_\Omega g\mathbf{v}\cdot\nabla\varphi+\int_{\Gamma_f}\lambda g\varphi\,d\sigma=\langle L,\varphi\rangle_{(H^1_{\Gamma_d}(\Omega))',H^1_{\Gamma_d}(\Omega)}\,,\ \forall\varphi\in H^1_{\Gamma_d}(\Omega).\end{array}\right. \tag{2.7}$$

Remark 2.7 *In fact, to prove Lemma 2.1 and Theorem 2.1 (as well as the results of Remark 2.6), we only need $\mathbf{v} \in (L^r(\Omega))^N$ with a $r > N$. But since such an hypothesis on $\mathbf{v}$ would not allow us to consider really better conditions in Theorem 1.1 (using the result of Theorem 2.1 with $\mathbf{v} \in (L^r(\Omega))^N$ for a $r > N$ would allow us to weaken Hypothesis (1.6), but not enough to handle the case of functions of the form $a(s,\xi) = \widetilde{a}(s)\xi$), we prefer to consider the stronger Hypothesis (2.2), which is sufficient to our purpose here.*

Proof of Lemma 2.1
We will approximate Problem (2.4) by problems for which we have, thanks to the Schauder fixed point theorem, a solution; then, by proving estimates on the solutions of these approximate problems, we will obtain a solution to (2.4) (without passing to the limit!).

Step 1: the approximate problems.
For $t \geq 0$, define $T_t(s) = \min(t, \max(-t, s))$. Let n be an integer and, if $\overline{g} \in L^2(\Omega)$, define $F(\overline{g}) = g$ as the unique solution to

$$\left\{ \begin{array}{l} g \in H^1_{\Gamma_d}(\Omega)\,, \\ \displaystyle\int_\Omega A^T \nabla g \cdot \nabla\varphi + \int_{\Gamma_f} \lambda g \varphi \, d\sigma = \int_\Omega \tau\varphi - \int_\Omega T_n(\overline{g})\mathbf{w} \cdot \nabla\varphi\,,\ \forall \varphi \in H^1_{\Gamma_d}(\Omega) \end{array} \right. \tag{2.8}$$

(the bilinear form is coercive on $H^1_{\Gamma_d}(\Omega)$ thanks to (1.10) applied to $q = \overline{q} = 2$ and $\rho = \alpha_A$).
We notice that $F : L^2(\Omega) \to L^2(\Omega)$ is continuous; indeed, if $\overline{g}_m \to \overline{g}_\infty$ in $L^2(\Omega)$, and if (for $m \in \mathbb{N}$ or $m = \infty$) L_m is the linear form

$$\langle L_m, \varphi \rangle_{(H^1_{\Gamma_d}(\Omega))', H^1_{\Gamma_d}(\Omega)} = \int_\Omega \tau\varphi - \int_\Omega T_n(\overline{g}_m)\mathbf{w} \cdot \nabla\varphi,$$

then $L_m \to L_\infty$ in $(H^1_{\Gamma_d}(\Omega))'$, so that $g_m = F(\overline{g}_m) \to g_\infty = F(\overline{g}_\infty)$ in $H^1_{\Gamma_d}(\Omega)$, thus in $L^2(\Omega)$.
Moreover, there exists $R > 0$ such that, for all $\overline{g} \in L^2(\Omega)$, $||F(\overline{g})||_{H^1(\Omega)} \leq R$; indeed, by taking g as a test function in (2.8), we get

$$\alpha_A || \, |\nabla g| \, ||^2_{L^2(\Omega)} + \int_{\Gamma_f} \lambda |g|^2 \, d\sigma \leq ||\tau||_{(H^1_{\Gamma_d}(\Omega))'} ||g||_{H^1(\Omega)} + n|| \, |\mathbf{w}| \, ||_{L^2(\Omega)} ||g||_{H^1(\Omega)},$$

which gives, thanks to (1.10),

$$\mathcal{K}_{2,2}(\alpha_A, \Omega, \Gamma_d, \lambda) ||g||_{H^1(\Omega)} \leq ||\tau||_{(H^1_{\Gamma_d}(\Omega))'} + n|| \, |\mathbf{w}| \, ||_{L^2(\Omega)};$$

thus, $R = \mathcal{K}_{2,2}(\alpha_A, \Omega, \Gamma_d, \lambda)^{-1}(||\tau||_{(H^1_{\Gamma_d}(\Omega))'} + n|| \, |\mathbf{w}| \, ||_{L^2(\Omega)})$ satisfies the property.
$F : L^2(\Omega) \to L^2(\Omega)$ is thus a compact application (thanks to the Rellich theorem) which sends the whole space $L^2(\Omega)$ in the ball of center 0 and radius R in $L^2(\Omega)$.
By the Schauder fixed point theorem, F has a fixed point in the ball of center 0 and radius R; we have thus proven that there exists g_n solution to

$$\left\{ \begin{array}{l} g_n \in H^1_{\Gamma_d}(\Omega)\,, \\ \displaystyle\int_\Omega A^T \nabla g_n \cdot \nabla\varphi + \int_\Omega T_n(g_n)\mathbf{w} \cdot \nabla\varphi + \int_{\Gamma_f} \lambda g_n \varphi \, d\sigma = \int_\Omega \tau\varphi\,,\ \forall \varphi \in H^1_{\Gamma_d}(\Omega) \end{array} \right. \tag{2.9}$$

satisfying

$$\begin{aligned} ||g_n||_{H^1(\Omega)} &\leq \mathcal{K}_{2,2}(\alpha_A, \Omega, \Gamma_d, \lambda)^{-1}(||\tau||_{(H^1_{\Gamma_d}(\Omega))'} + n|| \, |\mathbf{w}| \, ||_{L^2(\Omega)}) \\ &\leq \mathcal{K}_{2,2}(\alpha_A, \Omega, \Gamma_d, \lambda)^{-1}(\Lambda_\tau |\Omega|^{\frac{1}{2}} + n\Lambda_{\mathbf{w}} |\Omega|^{\frac{1}{2}}). \end{aligned}$$

Step 2: we prove that $(\ln(1+|g_n|))_{n\geq 1}$ is bounded in $H^1_{\Gamma_d}(\Omega)$, using the technique introduced in [6]. Let us first prove an estimate on $\int_{\Gamma_f}\lambda|g_n|\,d\sigma$. Take $\varphi = T_k(g_n)/k \in H^1_{\Gamma_d}(\Omega)$ as a test function in (2.9). We obtain, since $|T_k(s)/k|\leq 1$ for all $s\in\mathbb{R}$ and $\nabla(T_k(g_n)) = \mathbf{1}_{\{0<|g_n|<k\}}\nabla g_n$ a.e. on Ω (where $\mathbf{1}_E$ is the characteristic function of a set E),

$$\begin{aligned}
\int_{\Gamma_f}\lambda\frac{T_k(g_n)}{k}g_n\,d\sigma &\leq \frac{1}{k}\int_\Omega A^T\nabla g_n\nabla(T_k(g_n)) + \int_{\Gamma_f}\lambda\frac{T_k(g_n)}{k}g_n\,d\sigma \\
&\leq \int_\Omega|\tau| + \int_{\{0<|g_n|<k\}}|\mathbf{w}||g_n|\frac{|\nabla g_n|}{k} \\
&\leq \int_\Omega|\tau| + ||\,|\mathbf{w}|\,||_{L^2(\Omega)}\left(\int_{\{0<|g_n|<k\}}|\nabla g_n|^2\right)^{\frac{1}{2}}. \qquad (2.10)
\end{aligned}$$

But $g_nT_k(g_n)/k \to |g_n|$ on $\partial\Omega$ as $k\to 0$ (if $g_n(x)=0$, $g_n(x)T_k(g_n(x))/k = 0$ and, if $g_n(x)\neq 0$, $T_k(g_n(x))/k \to \mathrm{sgn}(g_n(x))$) and $|g_nT_k(g_n)/k|\leq |g_n|\in L^1(\partial\Omega)$; thus, by the dominated convergence theorem, $\int_{\Gamma_f}\lambda g_n(T_k(g_n)/k)\,d\sigma \to \int_{\Gamma_f}\lambda|g_n|$. Moreover, since $\nabla g_n\in L^2(\Omega)$ and $|\{0<|g_n|<k\}|\to 0$ as $k\to 0$ (this is the non-increasing continuity of the measure, associated to the fact that $\cap_{k>0}\{0<|g_n|<k\}=\emptyset$), we obtain $\int_{\{0<|g_n|<k\}}|\nabla g_n|^2\to 0$ as $k\to\infty$. Thus, passing to the limit $k\to 0$ in (2.10), we obtain

$$\int_{\Gamma_f}\lambda\ln(1+|g_n|)\,d\sigma \leq \int_{\Gamma_f}\lambda|g_n|\,d\sigma \leq \int_\Omega|\tau|\leq|\Omega|\Lambda_\tau. \qquad (2.11)$$

Let us now prove an estimate on the derivatives of g_n. Let $k\in\mathbb{N}$ and denote $r_k(s) = T_1(s-T_k(s))$, that is to say

$$\begin{cases}
r_k(s) = -1 & \text{if } s<-k-1\\
r_k(s) = s+k & \text{if } -k-1\leq s\leq -k\\
r_k(s) = 0 & \text{if } -k<s<k\\
r_k(s) = s-k & \text{if } k\leq s\leq k+1\\
r_k(s) = 1 & \text{if } k+1<s.
\end{cases}$$

We know that $r_k(g_n)\in H^1_{\Gamma_d}(\Omega)$ with $\nabla(r_k(g_n)) = \mathbf{1}_{B_k^n}\nabla g_n$, where $B_k^n = \{x\in\Omega\mid k\leq|g_n|<k+1\}$. Using $r_k(g_n)$ as a test function in (2.9), we get thus, since $|g_n|\leq k+1$ on B_k^n and $g_nr_k(g_n)\geq 0$ on $\partial\Omega$,

$$\begin{aligned}
\alpha_A||\,|\nabla(r_k(g_n))|\,||^2_{L^2(\Omega)} &\leq \int_\Omega A^T\nabla(r_k(g_n))\cdot\nabla(r_k(g_n)) + \int_{\Gamma_f}\lambda g_nr_k(g_n)\,d\sigma \\
&= \int_\Omega A^T\nabla g_n\cdot\nabla(r_k(g_n)) + \int_{\Gamma_f}\lambda g_nr_k(g_n)\,d\sigma \\
&= \int_\Omega\tau r_k(g_n) - \int_\Omega T_n(g_n)\mathbf{w}\cdot\nabla(r_k(g_n)) \\
&\leq ||\tau||_{L^1(\Omega)} + \int_{B_k^n}|\mathbf{w}||g_n||\nabla(r_k(g_n))| \\
&\leq \Lambda_\tau|\Omega| + (k+1)||\,|\mathbf{w}|\,||_{L^2(B_k^n)}||\,|\nabla(r_k(g_n))|\,||_{L^2(\Omega)} \\
&\leq \Lambda_\tau|\Omega| + \frac{\alpha_A}{2}||\,|\nabla(r_k(g_n))|\,||^2_{L^2(\Omega)} + \frac{||\,|\mathbf{w}|\,||^2_{L^2(B_k^n)}}{2\alpha_A}(k+1)^2.
\end{aligned}$$

Thus, we obtain

$$||\,|\nabla(r_k(g_n))|\,||^2_{L^2(\Omega)} \leq \frac{2\Lambda_\tau|\Omega|}{\alpha_A} + \frac{||\,|\mathbf{w}|\,||^2_{L^2(B_k^n)}}{\alpha_A^2}(k+1)^2. \qquad (2.12)$$

We will use this to show that $(\nabla(\ln(1+|g_n|)))_{n\geq 1}$ is bounded in $L^2(\Omega)$.

We have, since Ω is the disjoint union of $(B_k^n)_{k\geq 0}$, and $|g_n|\geq k$ on B_k^n,

$$
\begin{aligned}
\int_\Omega |\nabla(\ln(1+|g_n|))|^2 &= \int_\Omega \frac{|\nabla(|g_n|)|^2}{(1+|g_n|)^2}\\
&= \sum_{k\geq 0}\int_{B_k^n}\frac{|\nabla g_n|^2}{(1+|g_n|)^2}\\
&\leq \sum_{k\geq 0}\int_\Omega \frac{|\nabla(r_k(g_n))|^2}{(1+k)^2}.
\end{aligned}
$$

Using (2.12), this gives

$$
\begin{aligned}
\int_\Omega |\nabla(\ln(1+|g_n|))|^2 &\leq \frac{2\Lambda_\tau|\Omega|}{\alpha_A}\sum_{k\geq 0}\frac{1}{(1+k)^2}+\frac{1}{\alpha_A^2}\sum_{k\geq 0}\int_{B_k^n}|\mathbf{w}|^2\\
&\leq \frac{\pi^2\Lambda_\tau|\Omega|}{3\alpha_A}+\frac{||\,|\mathbf{w}|\,||^2_{L^2(\Omega)}}{\alpha_A^2}.
\end{aligned}
$$

This last estimate, associated to (2.11) and to (1.10) (with $q=2$ and $\overline{q}=1$) gives

$$
||\ln(1+|g_n|)||^2_{H^1(\Omega)}\leq \frac{1}{\mathcal{K}_{2,1}(1,\Omega,\Gamma_d,\lambda)}\left(\frac{\pi^2\Lambda_\tau|\Omega|}{3\alpha_A}+\frac{\Lambda_{\mathbf{w}}^2|\Omega|}{\alpha_A^2}+|\Omega|^2\Lambda_\tau^2\right):=C_1
$$

(C_1 depends on $(\Omega,\Gamma_d,\alpha_A,\Lambda_{\mathbf{w}},\lambda,\Lambda_\tau)$).

Step 3: we conclude by proving that $(g_n)_{n\geq 1}$ is bounded in $L^\infty(\Omega)$.
Let $S_k(s)=s-T_k(s)$; we have $S_k(g_n)\in H^1_{\Gamma_d}(\Omega)$ with $\nabla(S_k(g_n))=\mathbf{1}_{E_k^n}\nabla g_n$ (where $E_k^n=\{x\in\Omega\mid |g_n(x)|>k\}$). Since $S_k(g_n)=0$ outside E_k^n and since $g_nS_k(g_n)=|g_n||S_k(g_n)|\geq |S_k(g_n)|^2$, we have, using $S_k(g_n)$ as a test function in (2.9),

$$
\begin{aligned}
&\alpha_A||\,|\nabla(S_k(g_n))|\,||^2_{L^2(\Omega)}+\int_{\Gamma_f}\lambda|S_k(g_n)|^2\,d\sigma\\
&\leq \int_\Omega A^T\nabla g_n\cdot\nabla(S_k(g_n))+\int_{\Gamma_f}\lambda g_nS_k(g_n)\,d\sigma\\
&\leq \Lambda_\tau\int_\Omega |S_k(g_n)|+\int_\Omega |\mathbf{w}||g_n||\nabla(S_k(g_n))|\\
&\leq \Lambda_\tau||S_k(g_n)||_{L^2(\Omega)}|E_k^n|^{\frac12}+\int_{E_k^n}|\mathbf{w}|(|S_k(g_n)|+k)|\nabla(S_k(g_n))|\\
&\leq \Lambda_\tau||S_k(g_n)||_{L^2(\Omega)}|E_k^n|^{\frac12}+||\,|\nabla(S_k(g_n))|\,||_{L^2(\Omega)}(k||\,|\mathbf{w}|\,||_{L^2(E_k^n)}+||\,|\mathbf{w}|S_k(g_n)||_{L^2(E_k^n)})\\
&\leq \Lambda_\tau||S_k(g_n)||_{H^1(\Omega)}|E_k^n|^{\frac12}+k\Lambda_{\mathbf{w}}||\,|\nabla(S_k(g_n))|\,||_{L^2(\Omega)}|E_k^n|^{\frac12}\\
&\quad +||\,|\nabla(S_k(g_n))|\,||_{L^2(\Omega)}\Lambda_{\mathbf{w}}||S_k(g_n)||_{L^2(E_k^n)}.
\end{aligned}
\tag{2.13}
$$

Thanks to the Hölder inequality we have, when $p>2$,

$$
||S_k(g_n)||_{L^2(E_k^n)}\leq ||S_k(g_n)||_{L^p(\Omega)}|E_k^n|^{\frac12-\frac1p}.
$$

Since $2<2N/(N-2)$, there exists, by the Sobolev injection, $p>2$ and C_2 only depending on Ω such that

$$
||S_k(g_n)||_{L^p(\Omega)}\leq C_2||S_k(g_n)||_{H^1(\Omega)}.
$$

Thus, with (2.13) and (1.10), we get

$$\begin{aligned}\mathcal{K}_{2,2}(\alpha_A,\Omega,\Gamma_d,\lambda)||S_k(g_n)||^2_{H^1(\Omega)} &\leq \Lambda_\tau||S_k(g_n)||_{H^1(\Omega)}|E_k^n|^{\frac12}+\Lambda_{\mathbf{w}}k||S_k(g_n)||_{H^1(\Omega)}|E_k^n|^{\frac12}\\ &\quad +C_2\Lambda_{\mathbf{w}}|E_k^n|^{\frac12-\frac1p}||S_k(g_n)||^2_{H^1(\Omega)}. \qquad (2.14)\end{aligned}$$

The Tchebycheff inequality reads

$$\begin{aligned}|E_k^n|=|\{\ln(1+|g_n|)>\ln(1+k)\}| &\leq \frac{1}{(\ln(1+k))^2}||\ln(1+|g_n|)||^2_{L^2(\Omega)}\\ &\leq \frac{1}{(\ln(1+k))^2}||\ln(1+|g_n|)||^2_{H^1(\Omega)}\\ &\leq \frac{C_1^2}{(\ln(1+k))^2}\end{aligned}$$

where C_1 is the constant given by Step 2. Since $1/2>1/p$, there exists thus k_0 depending on C_2, $\Lambda_{\mathbf{w}}$, p, C_1 and $\mathcal{K}_{2,2}(\alpha_A,\Omega,\Lambda_d,\lambda)$, i.e. depending on $(\Omega,\Gamma_d,\alpha_A,\Lambda_{\mathbf{w}},\lambda,\Lambda_\tau)$, such that, for all $k\geq k_0$ and all $n\geq 1$, $C_2\Lambda_{\mathbf{w}}|E_k^n|^{\frac12-\frac1p}\leq \mathcal{K}_{2,2}(\alpha_A,\Omega,\Gamma_d,\lambda)/2$.
We obtain thus, for all $k\geq k_0$, thanks to (2.14),

$$||S_k(g_n)||_{H^1(\Omega)}\leq\left(\frac{2\Lambda_\tau}{\mathcal{K}_{2,2}(\alpha_A,\Omega,\Gamma_d,\lambda)}+\frac{2\Lambda_{\mathbf{w}}k}{\mathcal{K}_{2,2}(\alpha_A,\Omega,\Gamma_d,\lambda)}\right)|E_k^n|^{\frac12}\leq C_3(1+k)|E_k^n|^{\frac12},$$

where C_3 depends on $(\Omega,\Gamma_d,\alpha_A,\Lambda_{\mathbf{w}},\lambda,\Lambda_\tau)$.
By noticing that, when $h>k$, $|S_k(g_n)|\geq(h-k)$ on E_h^n, we get, thanks to the Sobolev injection $W^{1,1}(\Omega)\hookrightarrow L^{N/(N-1)}(\Omega)$ (the norm of which, denoted by C_4, only depends on Ω),

$$\begin{aligned}(h-k)|E_h^n|^{(N-1)/N} &\leq ||S_k(g_n)||_{L^{N/(N-1)}(\Omega)}\\ &\leq C_4||S_k(g_n)||_{W^{1,1}(\Omega)}\\ &\leq C_4|E_k^n|^{\frac12}||S_k(g_n)||_{H^1(\Omega)}\\ &\leq C_3C_4(1+k)|E_k^n|.\end{aligned}$$

Thus, as soon as $h>k\geq k_0$, we have, with $\beta=N/(N-1)>1$,

$$|E_h^n|\leq\frac{(C_3C_4)^\beta(1+k)^\beta}{(h-k)^\beta}|E_k^n|^\beta\leq\frac{(C_3C_4(1+k_0))^\beta(1+k-k_0)^\beta}{(h-k)^\beta}|E_k^n|^\beta$$

(because, when $k\geq k_0$, $(1+k_0)(1+k-k_0)\geq 1+k$). Lemma 2.2 given just after the end of this proof, and applied to the non-increasing function $G_n(k)=|E^n_{k+k_0}|$, allows us to see that, if

$$H_0=\exp\left(\sum_{m\geq 0}\frac{2^{\frac1\beta}C_3C_4(1+k_0)|\Omega|^{\frac{\beta-1}{\beta}}}{\left(2^{\frac{\beta-1}{\beta}}\right)^m}\right)\geq\exp\left(\sum_{m\geq 0}\frac{2^{\frac1\beta}C_3C_4(1+k_0)G_n(0)^{\frac{\beta-1}{\beta}}}{\left(2^{\frac{\beta-1}{\beta}}\right)^m}\right),$$

(notice that $H_0<+\infty$ depends on $(\Omega,\Gamma_d,\alpha_A,\Lambda_{\mathbf{w}},\lambda,\Lambda_\tau)$), then $G_n(H_0)=0$, that is to say $|g_n|\leq H_0+k_0$ a.e. on Ω for all $n\geq 1$.

Thus, by taking n_0 an integer greater than H_0+k_0 (n_0 depends on $(\Omega,\Gamma_d,\alpha_A,\Lambda_{\mathbf{w}},\lambda,\Lambda_\tau)$) and letting $g=g_{n_0}$, we have a solution to (2.4) (because $T_{n_0}(g_{n_0})=g_{n_0}=g$) which satisfies $||g||_{L^\infty(\Omega)}\leq H_0+k_0$ and $||g||_{H^1(\Omega)}=||g_{n_0}||_{H^1(\Omega)}\leq\mathcal{K}_{2,2}(\alpha_A,\Omega,\Gamma_d,\lambda)^{-1}(\Lambda_\tau|\Omega|^{\frac12}+n_0\Lambda_{\mathbf{w}}|\Omega|^{\frac12})$. This completes the proof of Lemma 2.1. ■

Lemma 2.2 *Let $G : \mathbb{R}^+ \to \mathbb{R}^+$ be a non-increasing function. If there exist $\beta > 1$ and $C > 0$ such that*

$$\forall h > k \geq 0\,,\ G(h) \leq \frac{C^\beta (1+k)^\beta}{(h-k)^\beta} G(k)^\beta$$

then, with

$$H = \exp\left(\sum_{m \geq 0} \frac{2^{\frac{1}{\beta}} C G(0)^{\frac{\beta-1}{\beta}}}{\left(2^{\frac{\beta-1}{\beta}}\right)^m} \right) < +\infty,$$

we have $G(H) = 0$.

For the proof of this lemma, which is a slight generalization of a lemma by G. Stampacchia ([16] Lemma 4.1, i)), we refer the reader to Lemma 2.2 in [10].

Proof of Theorem 2.1
The proof of the existence of a solution to

$$\left\{ \begin{array}{l} f \in H^1_{\Gamma_d}(\Omega)\,, \\ \displaystyle\int_\Omega A\nabla f \cdot \nabla\varphi + \int_\Omega \mathbf{v} \cdot \nabla f \varphi + \int_{\Gamma_f} \lambda f \varphi \, d\sigma = \int_\Omega \theta\varphi\,,\ \forall \varphi \in H^1_{\Gamma_d}(\Omega), \end{array} \right. \tag{2.15}$$

(i.e. Problem (2.3) without the regularity $f \in \mathcal{C}^{0,\kappa}(\Omega)$) uses the topological degree (see [9]); the proof of the Hölder continuity of the solution, as well as the estimates in the Hölder space, uses a result of [10].

Step 1: on a cut-off problem.
Let n be an integer. Recall that $T_n(s) = \min(n, \max(-n, s))$. We know that, for all $\varphi \in H^1_{\Gamma_d}(\Omega)$, $T_n(\varphi) \in H^1_{\Gamma_d}(\Omega)$ with $\nabla(T_n(\varphi)) = \mathbf{1}_{\{|\varphi|<n\}} \nabla\varphi$.
Let $\overline{f} \in H^1_{\Gamma_d}(\Omega)$; since $\mathbf{v} \cdot \nabla(T_n(\overline{f})) \in L^2(\Omega) \subset (H^1_{\Gamma_d}(\Omega))'$, there exists a unique solution $f = F(\overline{f})$ to

$$\left\{ \begin{array}{l} f \in H^1_{\Gamma_d}(\Omega)\,, \\ \displaystyle\int_\Omega A\nabla f \cdot \nabla\varphi + \int_{\Gamma_f} \lambda f \varphi \, d\sigma = \int_\Omega \theta\varphi - \int_\Omega \mathbf{v} \cdot \nabla(T_n(\overline{f}))\varphi\,,\ \forall \varphi \in H^1_{\Gamma_d}(\Omega). \end{array} \right. \tag{2.16}$$

This defines an application $F : H^1_{\Gamma_d}(\Omega) \to H^1_{\Gamma_d}(\Omega)$.
We will prove, using the topological degree, that F has a fixed point (conversely to the proof of Lemma 2.1, the Schauder fixed point theorem seems not applicable here).

Notice first that F is continuous; indeed, if $\overline{f}_m \to \overline{f}$ in $H^1_{\Gamma_d}(\Omega)$, then $T_n(\overline{f}_m) \to T_n(\overline{f})$ in $H^1_{\Gamma_d}(\Omega)$, so that $\mathbf{v} \cdot \nabla(T_n(\overline{f}_m)) \to \mathbf{v} \cdot \nabla(T_n(\overline{f}))$ in $L^2(\Omega)$, thus also in $(H^1_{\Gamma_d}(\Omega))'$ and the solution $F(\overline{f}_m)$ of (2.16) when $\overline{f}$ is replaced by $\overline{f}_m$ tends thus in $H^1_{\Gamma_d}(\Omega)$ to the solution $F(\overline{f})$ of (2.16).

We will now prove that, if $(\overline{f}_m)_{m\geq 1}$ is a bounded sequence in $H^1_{\Gamma_d}(\Omega)$, then there exists a subsequence (still denoted $(\overline{f}_m)_{m\geq 1}$) such that $(F(\overline{f}_m))_{m\geq 1}$ converges in $H^1_{\Gamma_d}(\Omega)$. Since $(\overline{f}_m)_{m\geq 1}$ is bounded in $H^1_{\Gamma_d}(\Omega)$, $(\mathbf{v} \cdot \nabla(T_n(\overline{f}_m)))_{m\geq 1}$ is bounded in $L^2(\Omega)$ and there exists thus a subsequence, still denoted $(\overline{f}_m)_{m\geq 1}$, such that $\mathbf{v} \cdot \nabla(T_n(\overline{f}_m)) \to \Phi$ weakly in $L^2(\Omega)$.
Since $(F(\overline{f}_m))_{m\geq 1}$ is bounded in $H^1_{\Gamma_d}(\Omega)$ (because of the coercivity of the operator in (2.16) and of the fact that $(\mathbf{v} \cdot \nabla(T_n(\overline{f}_m)))_{m\geq 1}$ is bounded in $L^2(\Omega)$), its trace is bounded in $L^2(\partial\Omega)$ and we can also suppose that, up to a subsequence, $(F(\overline{f}_m))_{m\geq 1}$ converges to F_0, weakly in $H^1_{\Gamma_d}(\Omega)$, strongly in $L^2(\Omega)$ and its trace weakly in $L^2(\partial\Omega)$; we see then that F_0 is the solution to

$$\left\{ \begin{array}{l} F_0 \in H^1_{\Gamma_d}(\Omega)\,, \\ \displaystyle\int_\Omega A\nabla F_0 \cdot \nabla\varphi + \int_{\Gamma_f} \lambda F_0 \varphi \, d\sigma = \int_\Omega \theta\varphi - \int_\Omega \Phi\varphi\,,\ \forall \varphi \in H^1_{\Gamma_d}(\Omega). \end{array} \right. \tag{2.17}$$

We have now to prove that the convergence of $(F(\overline{f}_m))_{m\geq 1}$ to F_0 is strong in $H^1_{\Gamma_d}(\Omega)$; to see this, we subtract the equation satisfied by F_0 from the equation satisfied by $F(\overline{f}_m)$ and we use the test function $\varphi = F(\overline{f}_m) - F_0 \in H^1_{\Gamma_d}(\Omega)$ to find

$$\begin{aligned}
&\alpha_A || \, |\nabla(F(\overline{f}_m) - F_0)| \, ||^2_{L^2(\Omega)} + \int_{\Gamma_f} \lambda |F(\overline{f}_m) - F_0|^2 \, d\sigma \\
&\leq \int_\Omega A\nabla(F(\overline{f}_m) - F_0) \cdot \nabla(F(\overline{f}_m) - F_0) + \int_{\Gamma_f} \lambda(F(\overline{f}_m) - F_0)(F(\overline{f}_m) - F_0) \, d\sigma \\
&= \int_\Omega (\Phi - \mathbf{v} \cdot \nabla(T_n(\overline{f}_m)))(F(\overline{f}_m) - F_0) \\
&\leq ||\Phi - \mathbf{v} \cdot \nabla(T_n(\overline{f}_m))||_{L^2(\Omega)} ||F(\overline{f}_m) - F_0||_{L^2(\Omega)}.
\end{aligned}$$

Since $(\mathbf{v} \cdot \nabla(T_n(\overline{f}_m)))_{m\geq 1}$ is bounded in $L^2(\Omega)$ and $F(\overline{f}_m) \to F_0$ in $L^2(\Omega)$, this inequality, associated to (1.10), gives

$$||F(\overline{f}_m) - F_0||_{H^1(\Omega)} \to 0.$$

Thus, $F : H^1_{\Gamma_d}(\Omega) \to H^1_{\Gamma_d}(\Omega)$ is a compact operator. To prove that F has a fixed point by an application of the Leray-Schauder topological degree, it remains to find $R > 0$ such that, if $t \in [0,1]$ and $\overline{f} \in H^1_{\Gamma_d}(\Omega)$ satisfies $\overline{f} - tF(\overline{f}) = 0$, then $||\overline{f}||_{H^1_{\Gamma_d}(\Omega)} \neq R$.
Suppose we have such a $t \in [0,1]$ and such a $\overline{f} \in H^1_{\Gamma_d}(\Omega)$; then $\overline{f}$ satisfies

$$\int_\Omega A\nabla\overline{f} \cdot \nabla\varphi + \int_{\Gamma_f} \lambda \overline{f} \varphi \, d\sigma = t\int_\Omega \theta\varphi - t\int_\Omega \mathbf{v} \cdot \nabla(T_n(\overline{f}))\varphi \qquad \text{for all } \varphi \in H^1_{\Gamma_d}(\Omega).$$

Take $\varphi = \overline{f}$; since $\nabla(T_n(\overline{f})) = \mathbf{1}_{\{|\overline{f}|<n\}}\nabla\overline{f}$, we find, with (1.10),

$$\begin{aligned}
\mathcal{K}_{2,2}(\alpha_A, \Omega, \Gamma_d, \lambda)||\overline{f}||^2_{H^1(\Omega)} &\leq \alpha_A \int_\Omega |\nabla\overline{f}|^2 + \int_{\Gamma_f} \lambda|\overline{f}|^2 \, d\sigma \\
&\leq ||\theta||_{(H^1_{\Gamma_d}(\Omega))'} ||\overline{f}||_{H^1(\Omega)} + n|| \, |\mathbf{v}| \, ||_{L^2(\Omega)} || \, |\nabla\overline{f}| \, ||_{L^2(\Omega)} \\
&\leq \left(||\theta||_{(H^1_{\Gamma_d}(\Omega))'} + n|| \, |\mathbf{v}| \, ||_{L^2(\Omega)}\right) ||\overline{f}||_{H^1(\Omega)},
\end{aligned}$$

which gives

$$||\overline{f}||_{H^1(\Omega)} \leq \frac{||\theta||_{(H^1_{\Gamma_d}(\Omega))'}}{\mathcal{K}_{2,2}(\alpha_A, \Omega, \Gamma_d, \lambda)} + \frac{n|| \, |\mathbf{v}| \, ||_{L^2(\Omega)}}{\mathcal{K}_{2,2}(\alpha_A, \Omega, \Gamma_d, \lambda)}.$$

Thus, by taking $R = 1 + (||\theta||_{(H^1_{\Gamma_d}(\Omega))'} + n|| \, |\mathbf{v}| \, ||_{L^2(\Omega)})/\mathcal{K}_{2,2}(\alpha_A, \Omega, \Gamma_d, \lambda)$, we deduce from the properties of the topological degree that F has a fixed point in the ball of center 0 and radius R in $H^1_{\Gamma_d}(\Omega)$.
We denote by f_n such a fixed point, which satisfies

$$\left\{ \begin{array}{l} f_n \in H^1_{\Gamma_d}(\Omega)\,, \\ \displaystyle\int_\Omega A\nabla f_n \cdot \nabla\varphi + \int_\Omega \mathbf{v} \cdot \nabla(T_n(f_n))\varphi + \int_{\Gamma_f} \lambda f_n \varphi \, d\sigma = \int_\Omega \theta\varphi\,,\ \forall \varphi \in H^1_{\Gamma_d}(\Omega) \end{array} \right. \tag{2.18}$$

and $||f_n||_{H^1(\Omega)} \leq 1 + (||\theta||_{(H^1_{\Gamma_d}(\Omega))'} + n\Lambda_{\mathbf{v}}|\Omega|^{\frac{1}{2}})/\mathcal{K}_{2,2}(\alpha_A, \Omega, \Gamma_d, \lambda)$.

Step 2: we prove an L^1 estimate for the sequence $(f_n)_{n\geq 1}$ constructed in Step 1.
Let $\mathbf{w}_n = \mathbf{1}_{\{|f_n|<n\}}\mathbf{v}$; we have, for all $\varphi \in H^1_{\Gamma_d}(\Omega)$,

$$\int_\Omega A\nabla f_n \cdot \nabla\varphi + \int_\Omega \mathbf{w}_n \cdot \nabla f_n \varphi + \int_{\Gamma_f} \lambda f_n \varphi \, d\sigma = \int_\Omega \theta\varphi. \tag{2.19}$$

Since $\Lambda_{\mathbf{v}}$ is an upper bound for $|| \, |\mathbf{w}_n| \, ||_{L^\infty(\Omega)}$, we can find, thanks to Lemma 2.1, a $g_n \in H^1_{\Gamma_d}(\Omega)$ satisfying, for all $\varphi \in H^1_{\Gamma_d}(\Omega)$,

$$\int_\Omega A^T \nabla g_n \cdot \nabla\varphi + \int_\Omega g_n \mathbf{w}_n \cdot \nabla\varphi + \int_{\Gamma_f} \lambda g_n \varphi \, d\sigma = \int_\Omega \mathrm{sgn}(f_n)\varphi, \tag{2.20}$$

and such that $||g_n||_{L^\infty(\Omega)} \le K_0$, where K_0 depends on $(\Omega, \Gamma_d, \alpha_A, \Lambda_{\mathbf{v}}, \lambda)$ but not n (sgn denotes the sign function, and we have thus $||\mathrm{sgn}(f_n)||_{L^\infty(\Omega)} \le 1$).
By putting $\varphi = f_n$ in (2.20) and $\varphi = g_n$ in (2.19), we get

$$||f_n||_{L^1(\Omega)} = \int_\Omega \mathrm{sgn}(f_n) f_n = \int_\Omega \theta g_n \le K_0 ||\theta||_{L^1(\Omega)}. \tag{2.21}$$

Step 3: with the same methods as in Step 3 of the proof of Lemma 2.1, we prove an L^∞ estimate on $(f_n)_{n\ge 1}$.
Define S_k as in Step 3 of the proof of Lemma 2.1, and use $S_k(f_n)$ as a test function in (2.18): we get, by denoting $E_k^n = \{x \in \Omega \mid |f_n(x)| > k\}$, and since $f_n S_k(f_n) \ge |S_k(f_n)|^2$,

$$\begin{aligned}
&\alpha_A || \, |\nabla(S_k(f_n))| \, ||^2_{L^2(\Omega)} + \int_{\Gamma_f} \lambda |S_k(f_n)|^2 \, d\sigma && (2.22)\\
&\le \int_\Omega A\nabla f_n \cdot \nabla(S_k(f_n)) + \int_{\Gamma_f} \lambda f_n S_k(f_n) \, d\sigma \\
&\le \Lambda_\theta ||S_k(f_n)||_{L^2(\Omega)} |E_k^n|^{\frac12} + \int_{\{|f_n|<n\}} |\mathbf{v}||\nabla f_n||S_k(f_n)| \\
&\le \Lambda_\theta ||S_k(f_n)||_{L^2(\Omega)} |E_k^n|^{\frac12} + \int_\Omega |\mathbf{v}||\nabla(S_k(f_n))||S_k(f_n)| \\
&\le \Lambda_\theta ||S_k(f_n)||_{H^1(\Omega)} |E_k^n|^{\frac12} + || \, |\nabla(S_k(f_n))| \, ||_{L^2(\Omega)} || \, |\mathbf{v}| S_k(f_n) ||_{L^2(\Omega)} \\
&\le \Lambda_\theta ||S_k(f_n)||_{H^1(\Omega)} |E_k^n|^{\frac12} + || \, |\nabla(S_k(f_n))| \, ||_{L^2(\Omega)} \Lambda_{\mathbf{v}} ||S_k(f_n)||_{L^2(\Omega)}, && (2.23)
\end{aligned}$$

because $\nabla f_n = \nabla(S_k(f_n))$ where $S_k(f_n) \neq 0$.
As before, we notice that, thanks to the Sobolev injection of H^1, there exists $p > 2$ and K_1 only depending on Ω such that

$$\begin{aligned}
||S_k(f_n)||_{L^2(\Omega)} &\le ||S_k(f_n)||_{L^p(\Omega)} |E_k^n|^{\frac12 - \frac1p} \\
&\le K_1 ||S_k(f_n)||_{H^1(\Omega)} |E_k^n|^{\frac12 - \frac1p},
\end{aligned}$$

which gives, introduced in (2.23) and thanks to (1.10),

$$\begin{aligned}
&\mathcal{K}_{2,2}(\alpha_A, \Omega, \Gamma_d, \lambda) ||S_k(f_n)||^2_{H^1(\Omega)} \\
&\le \Lambda_\theta ||S_k(f_n)||_{H^1(\Omega)} |E_k^n|^{\frac12} + K_1 \Lambda_{\mathbf{v}} |E_k^n|^{\frac12 - \frac1p} ||S_k(f_n)||^2_{H^1(\Omega)}. && (2.24)
\end{aligned}$$

But, with (2.21) and the Tchebycheff inequality, we see that

$$|E_k^n| \le \frac{1}{k} ||f_n||_{L^1(\Omega)} \le \frac{K_0 |\Omega| \Lambda_\theta}{k};$$

there exists thus k_0 depending on $(K_1, \Lambda_{\mathbf{v}}, p, K_0, \Omega, \Lambda_\theta, \mathcal{K}_{2,2}(\alpha_A, \Omega, \Gamma_d, \lambda))$ (i.e. on $(\Omega, \Gamma_d, \alpha_A, \Lambda_{\mathbf{v}}, \lambda, \Lambda_\theta)$), such that, for all $n \ge 1$ and all $k \ge k_0$, $K_1 \Lambda_{\mathbf{v}} |E_k^n|^{\frac12 - \frac1p} \le \mathcal{K}_{2,2}(\alpha_A, \Omega, \Gamma_d, \lambda)/2$.
Returning to (2.24), we have then, for all $k \ge k_0$,

$$||S_k(f_n)||_{H^1(\Omega)} \le \frac{2\Lambda_\theta}{\mathcal{K}_{2,2}(\alpha_A, \Omega, \Gamma_d, \lambda)} |E_k^n|^{\frac12} = K_2 |E_k^n|^{\frac12}$$

where K_2 depends on $(\Omega, \Gamma_d, \alpha_A, \lambda, \Lambda_\theta)$.
Then, reasoning as in the end of Step 3 of the proof of Lemma 2.1, we get, for all $h > k \geq k_0$,

$$|E_h^n| \leq \frac{MK_2^\beta}{(h-k)^\beta}|E_k^n|^\beta,$$

with $\beta = N/(N-1) > 1$ and M depending on Ω.
Using Lemma 2.2 (or, more directly, Lemma 4.1 i) in [16]), we see thus that there exists H_0 depending on (Ω, β, M, K_2), i.e. depending on $(\Omega, \Gamma_d, \alpha_A, \lambda, \Lambda_\theta)$ [notice that a dependence on Ω takes into account a dependence on N] such that, for all $n \geq 1$, $|E_{H_0+k_0}^n| = 0$, that is to say $||f_n||_{L^\infty(\Omega)} \leq K_3 = H_0 + k_0$, where K_3 depends on $(\Omega, \Gamma_d, \alpha_A, \Lambda_{\mathbf{v}}, \lambda, \Lambda_\theta)$.

By taking any integer $n_0 \geq K_3$ (such a choice of n_0 depends on K_3, thus on $(\Omega, \Gamma_d, \alpha_A, \Lambda_{\mathbf{v}}, \lambda, \Lambda_\theta)$) and letting $f = f_{n_0}$, we get a solution to

$$\left\{ \begin{array}{l} f \in H^1_{\Gamma_d}(\Omega) \\ \displaystyle\int_\Omega A\nabla f \cdot \nabla\varphi + \int_\Omega \mathbf{v} \cdot \nabla f \varphi + \int_{\Gamma_f} \lambda f \varphi \, d\sigma = \int_\Omega \theta\varphi \,,\ \forall \varphi \in H^1_{\Gamma_d}(\Omega), \end{array} \right. \tag{2.25}$$

(because, since $n_0 \geq K_3 \geq ||f_{n_0}||_{L^\infty(\Omega)}$, $T_{n_0}(f_{n_0}) = f_{n_0} = f$) such that

$$||f||_{H^1(\Omega)} \leq 1 + \frac{||\theta||_{(H^1_{\Gamma_d}(\Omega))'} + n_0\Lambda_{\mathbf{v}}|\Omega|^{\frac{1}{2}}}{\mathcal{K}_{2,2}(\alpha_A, \Omega, \Gamma_d, \lambda)} \leq 1 + \frac{\Lambda_\theta|\Omega|^{\frac{1}{2}} + n_0\Lambda_{\mathbf{v}}|\Omega|^{\frac{1}{2}}}{\mathcal{K}_{2,2}(\alpha_A, \Omega, \Gamma_d, \lambda)} := C_1,$$

where C_1 depends on $(\Omega, \Gamma_d, \alpha_A, \Lambda_{\mathbf{v}}, \lambda, \Lambda_\theta)$; notice also that

$$||f||_{L^\infty(\Omega)} \leq K_3. \tag{2.26}$$

Since, up to now, we have not used Hypothesis (1.13), this proves what we have claimed in Remark 2.2.

Step 4: conclusion.
It remains to prove that the solution $f \in H^1_{\Gamma_d}(\Omega)$ of (2.25) we found in the preceding section is in fact in $\mathcal{C}^{0,\kappa}(\Omega)$ for a $\kappa > 0$. This is the only part of the proof where we need Hypothesis (1.13).

We have, for all $\varphi \in H^1(\Omega)$,

$$\left|\int_\Omega \varphi\mathbf{v} \cdot \nabla\varphi\right| \leq ||\,|\nabla\varphi|\,||_{L^2(\Omega)}\Lambda_{\mathbf{v}}||\varphi||_{L^2(\Omega)} \leq \frac{\alpha_A}{2}||\,|\nabla\varphi|\,||^2_{L^2(\Omega)} + \frac{\Lambda_{\mathbf{v}}^2}{2\alpha_A}||\varphi||^2_{L^2(\Omega)}.$$

Thus, by taking $b = 1 + \frac{\Lambda_{\mathbf{v}}^2}{2\alpha_A}$, the bilinear continuous form

$$(\varphi, \psi) \in H^1(\Omega) \to \int_\Omega A\nabla\varphi \cdot \nabla\psi + \int_\Omega \mathbf{v} \cdot \nabla\varphi\psi + \int_\Omega b\varphi\psi$$

is coercive (notice that the choice of b depends on $(\Omega, \alpha_A, \Lambda_{\mathbf{v}})$).

f is the solution to

$$\left\{ \begin{array}{l} f \in H^1_{\Gamma_d}(\Omega)\,, \\ \displaystyle\int_\Omega A\nabla f \cdot \nabla\varphi + \int_\Omega \mathbf{v} \cdot \nabla f \varphi + \int_{\Gamma_f} \lambda f \varphi \, d\sigma + \int_\Omega bf\varphi = \int_\Omega \widetilde{\theta}\varphi \,,\ \forall \varphi \in H^1_{\Gamma_d}(\Omega), \end{array} \right. \tag{2.27}$$

where $\widetilde{\theta} = \theta + bf \in L^\infty(\Omega)$.
Thus, $\widetilde{\theta} \in (W^{1,1}_{\Gamma_d}(\Omega))'$ and, thanks to (2.26), the norm of $\widetilde{\theta}$ in $(W^{1,1}_{\Gamma_d}(\Omega))'$ is bounded by K_4 depending on $(\Omega, \Gamma_d, \alpha_A, \Lambda_{\mathbf{v}}, \lambda, \Lambda_\theta)$. With our choice of b, a slight adaptation of the methods of [16] and [10] shows then that (thanks to Hypothesis (1.13)), there exists $\kappa \in]0,1[$ depending on $(\Omega, \alpha_A, \Lambda_A, \Lambda_{\mathbf{v}}, \lambda, b)$, i.e. depending on $(\Omega, \alpha_A, \Lambda_A, \Lambda_{\mathbf{v}}, \lambda)$ and K_5 depending on $(\Omega, \alpha_A, \Lambda_A, \Lambda_{\mathbf{v}}, \lambda, b, K_4)$, i.e. depending on $(\Omega, \Gamma_d, \alpha_A, \Lambda_A, \Lambda_{\mathbf{v}}, \lambda, \Lambda_\theta)$, such that the solution f of (2.27) is in $\mathcal{C}^{0,\kappa}(\Omega)$ with $||f||_{\mathcal{C}^{0,\kappa}(\Omega)} \leq K_5$. ■

3 Proof of the uniqueness and stability theorems

We will use, in the course of this proof, the following result.

Lemma 3.1 *Let $f:\Omega\to\mathbb{R}$, $F:\Omega\to\mathbb{R}^N$ and $G:\Omega\to\mathbb{R}^N$ be measurable functions such that $|F-G|\in L^1(\Omega)$. Under Hypotheses (1.1), (1.4) and (1.5), there exists a measurable matrix-valued function $M:\Omega\to M_N(\mathbb{R})$ such that*

$$M(x)\tau\cdot\tau\geq\alpha|\tau|^2 \text{ for a.e. } x\in\Omega, \text{ for all } \tau\in\mathbb{R}^N, \tag{3.1}$$
$$||M(x)||\leq\Lambda \text{ for a.e. } x\in\Omega, \tag{3.2}$$
$$a(x,f(x),F(x))-a(x,f(x),G(x))=M(x)(F(x)-G(x)) \text{ for a.e. } x\in\Omega. \tag{3.3}$$

Remark 3.1 *Notice that α and Λ do not depend on f, F or G (only on a).*

Proof of Lemma 3.1.
When a is of class $\mathcal{C}^1$ with respect to ξ, it is very simple: just take

$$M(x)=\int_0^1\frac{\partial a}{\partial\xi}(x,f(x),F(x)+t(G(x)-F(x))\,dt$$

(where $\frac{\partial a}{\partial\xi}$, the partial derivative of a with respect to ξ, is identified to a $N\times N$ matrix; it is easy to see that this partial derivative satisfies (3.1) and (3.2)).
When a is only Lipschitz continuous with respect to ξ, it has a partial derivative for a.e. $\xi\in\mathbb{R}^N$, but we can not take the preceding expression since $F(.)+t(G(.)-F(.))$ could take (on the whole of Ω and for any $t\in[0,1]$) its values where a is not derivable with respect to ξ.
We solve this problem by the following trick: by denoting $(\rho_n)_{n\geq1}$ a sequence of mollifiers in $\mathbb{R}^N$, we take $a_n(x,s,\xi)=(a(x,s,.)*\rho_n)(\xi)$; a_n is a Caratheodory function which is of class $\mathcal{C}^1$ with respect to ξ. We have thus

$$a_n(x,f(x),F(x))-a_n(x,f(x),G(x))=M_n(x)(F(x)-G(x)), \tag{3.4}$$

where $M_n(x)=\int_0^1\frac{\partial a_n}{\partial\xi}(x,f(x),F(x)+t(G(x)-F(x))\,dt$; by noticing that $\frac{\partial a_n}{\partial\xi}(x,s,\xi)=(\frac{\partial a}{\partial\xi}(x,s,.)*\rho_n)(\xi)$, we see that $\frac{\partial a_n}{\partial\xi}$ — and thus M_n — satisfies (3.1) and (3.2) for all $n\geq1$.
Thus, $(M_n)_{n\geq1}$ being a bounded sequence in $(L^\infty(\Omega))^{N\times N}$, there exists a subsequence, still denoted $(M_n)_{n\geq1}$, which converges to M in $(L^\infty(\Omega))^{N\times N}$ weak-$*$; it is then quite clear that M satisfies (3.1) and (3.2). Moreover, since $|F-G|\in L^1(\Omega)$, $M_n(F-G)\to M(F-G)$ in the sense of distributions. Since $a_n(x,f(x),F(x))-a_n(x,f(x),G(x))\to a(x,f(x),F(x))-a(x,f(x),G(x))$ for a.e. $x\in\Omega$ (for all $x\in\Omega$ such that $a(x,.,.)$ is continuous) and is dominated by $\Lambda|F-G|\in L^1(\Omega)$, the convergence is also true in $(L^1(\Omega))^N$ (and thus in the sense of distributions). By passing to the limit (in the sense of distributions) in (3.4), and since the limits are functions, we get

$$a(x,f(x),F(x))-a(x,f(x),G(x))=M(x)(F(x)-G(x)) \quad \text{for a.e. } x\in\Omega,$$

and the measurable matrix valued function M is thus convenient. ■

Proof of Theorem 1.1
Let $\mu\in\mathcal{M}(\Omega)$, $\mu^\partial\in\mathcal{M}(\partial\Omega)$ and u, v two SOLA of (1.14).
By definition, there exists $(\mu_n,\nu_n)_{n\geq1}\in L^1(\Omega)\cap(H^1_{\Gamma_d}(\Omega))'$ satisfying $\mu_n\to\mu$ and $\nu_n\to\mu$ in $(\mathcal{C}(\overline{\Omega}))'$ weak-$*$, $(\mu_n^\partial,\nu_n^\partial)_{n\geq1}\in L^1(\partial\Omega)\cap(H^{1/2}_{\Gamma_d}(\Omega))'$ satisfying $\mu_n^\partial\to\mu^\partial$ and $\nu_n^\partial\to\mu^\partial$ in $\mathcal{M}(\partial\Omega)$ weak-$*$, u_n a solution of (1.15) and v_n a solution of (1.15) with (ν_n,ν_n^∂) instead of (μ_n,μ_n^∂) such that $u_n\to u$ and $v_n\to v$ in $L^1(\Omega)$ (in fact, the convergence is much stronger but we will not need it).

By substracting the equation satisfied by v_n from the equation satisfied by u_n, we have, for all $\varphi \in H^1_{\Gamma_d}(\Omega)$,

$$\begin{aligned} &\int_\Omega (a(x,u_n,\nabla u_n) - a(x,v_n,\nabla v_n)) \cdot \nabla\varphi + \int_{\Gamma_f} \lambda(u_n - v_n)\varphi \, d\sigma \\ &= \langle \mu_n - \nu_n, \varphi\rangle_{(H^1_{\Gamma_d}(\Omega))', H^1_{\Gamma_d}(\Omega)} + \langle \mu_n^\partial - \nu_n^\partial, \varphi\rangle_{(H^{1/2}_{\Gamma_d}(\Omega))', H^{1/2}_{\Gamma_d}(\Omega)}. \end{aligned} \tag{3.5}$$

Let $\mathcal{V} : \Omega\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}^N \to \mathbb{R}^N$ defined, for all $(x,s,t,\xi)\in\Omega\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}^N$, by

$$\begin{cases} \mathcal{V}(x,s,t,\xi) = \frac{a(x,s,\xi)-a(x,t,\xi)}{s-t} & \text{if } s\neq t, \\ \mathcal{V}(x,s,t,\xi) = 0 & \text{if } s = t. \end{cases}$$

Thanks to Hypothesis (1.1), $\mathcal{V}$ is Borel-measurable (it is Borel-measurable on the Borel set $\{s\neq t\}$ and on the Borel set $\{s=t\}$) and, by (1.6), $|\mathcal{V}(x,s,t,\xi)|\le\delta$ for a.e. $x\in\Omega$, for all $(s,t,\xi)\in\mathbb{R}\times\mathbb{R}\times\mathbb{R}^N$; we also have, for all $(x,s,t,\xi)\in\Omega\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}^N$,

$$a(x,s,\xi) - a(x,t,\xi) = (s-t)\mathcal{V}(x,s,t,\xi).$$

$\mathcal{V}$ being Borel-measurable and u_n, v_n, ∇v_n being measurable, $\mathbf{v}_n(.) = \mathcal{V}(.,u_n(.),v_n(.),\nabla v_n(.))$ is measurable on Ω and, for a.e. $x\in\Omega$, we have $|\mathbf{v}_n(x)|\le\delta$.
By denoting $M_n : \Omega \to M_N(\mathbb{R})$ the measurable matrix-valued function given by Lemma 3.1 applied to $f = u_n$, $F = \nabla u_n$ and $G = \nabla v_n$ (notice that $|F-G|\in L^2(\Omega)\subset L^1(\Omega)$), we obtain, for a.e. $x\in\Omega$,

$$\begin{aligned} &a(x,u_n(x),\nabla u_n(x)) - a(x,v_n(x),\nabla v_n(x)) \\ &= a(x,u_n(x),\nabla u_n(x)) - a(x,u_n(x),\nabla v_n(x)) + a(x,u_n(x),\nabla v_n(x)) - a(x,v_n(x),\nabla v_n(x)) \\ &= M_n(x)(\nabla u_n(x) - \nabla v_n(x)) + (u_n(x) - v_n(x))\mathbf{v}_n(x). \end{aligned}$$

By (3.5), $w_n = u_n - v_n$ is thus a solution to

$$\begin{cases} w_n \in H^1_{\Gamma_d}(\Omega)\,, \\ \displaystyle\int_\Omega M_n\nabla w_n\cdot\nabla\varphi + \int_\Omega w_n\mathbf{v}_n\cdot\nabla\varphi + \int_{\Gamma_f}\lambda w_n\varphi\,d\sigma = \langle\mu_n-\nu_n,\varphi\rangle_{(H^1_{\Gamma_d}(\Omega))',H^1_{\Gamma_d}(\Omega)} \\ \qquad\qquad + \langle\mu_n^\partial - \nu_n^\partial,\varphi\rangle_{(H^{1/2}_{\Gamma_d}(\partial\Omega))',H^{1/2}_{\Gamma_d}(\partial\Omega)}\,,\ \forall\varphi\in H^1_{\Gamma_d}(\Omega). \end{cases} \tag{3.6}$$

M_n^T is a measurable matrix-valued function which satisfies Properties (3.1) and (3.2) (notice that α and Λ do not depend on n) and we have $\mathbf{v}_n\in L^\infty(\Omega)$ with $\delta\ge||\,|\mathbf{v}_n|\,||_{L^\infty(\Omega)}$ (notice that δ does not depend on n).
Thanks to Theorem 2.1, since $\mathrm{sgn}(u-v)\in L^\infty(\Omega)$, there exists $\kappa>0$ and $C>0$ depending on $(\Omega,\Gamma_d,\alpha,\Lambda,\delta,\lambda)$ (i.e. κ and C do not depend on n) and, for all $n\ge1$, a solution to

$$\begin{cases} f_n\in H^1_{\Gamma_d}(\Omega)\cap\mathcal{C}^{0,\kappa}(\Omega)\,, \\ \displaystyle\int_\Omega M_n^T\nabla f_n\cdot\nabla\varphi + \int_\Omega \mathbf{v}_n\cdot\nabla f_n\varphi + \int_{\Gamma_f}\lambda f_n\varphi\,d\sigma = \int_\Omega \mathrm{sgn}(u-v)\varphi\,,\ \forall\varphi\in H^1_{\Gamma_d}(\Omega) \end{cases} \tag{3.7}$$

such that $||f_n||_{\mathcal{C}^{0,\kappa}(\Omega)}\le C$.
Using f_n as a test function in (3.6) and w_n as a test function in (3.7), we obtain

$$\begin{aligned} \int_\Omega w_n\mathrm{sgn}(u-v) &= \int_\Omega M_n\nabla w_n\cdot\nabla f_n + \int_\Omega w_n\mathbf{v}_n\cdot\nabla f_n + \int_{\Gamma_f}\lambda w_n f_n\,d\sigma \\ &= \int_\Omega f_n(\mu_n-\nu_n) + \int_{\partial\Omega} f_n(\mu_n^\partial - \nu_n^\partial)\,d\sigma. \end{aligned} \tag{3.8}$$

Since $(f_n)_{n\geq 1}$ is bounded in $\mathcal{C}^{0,\kappa}(\Omega)$, it is relatively compact in $\mathcal{C}(\overline{\Omega})$ (thanks to the Ascoli-Arzelà theorem) and we can thus suppose that, up to a subsequence still denoted $(f_n)_{n\geq 1}$, we have $f_n \to f$ in $\mathcal{C}(\overline{\Omega})$. Since $\mu_n - \nu_n \to 0$ in $(\mathcal{C}(\overline{\Omega}))'$ weak-$*$ and $\mu_n^\partial - \nu_n^\partial \to 0$ in $\mathcal{M}(\partial\Omega)$ weak-$*$, we get

$$\int_\Omega f_n(\mu_n - \nu_n) + \int_{\partial\Omega} f_n(\mu_n^\partial - \nu_n^\partial) \to 0.$$

Using the fact that $w_n \to u - v$ in $L^1(\Omega)$, we deduce then from (3.8), by passing to the limit $n \to \infty$, that

$$0 = \int_\Omega \operatorname{sgn}(u-v)(u-v) = \int_\Omega |u - v|,$$

which gives $u = v$ a.e. on Ω and concludes the proof. ■

Proof of Theorem 1.2.
We first prove the more general result stated in Remark 1.6. We suppose thus, to begin, only Hypotheses (1.1)—(1.3), (1.7) and (1.9) and we take $(u_n)_{n\geq 1}$ satisfying: for all $n \geq 1$, there exists three sequences $(\mu_{n,m})_{m\geq 1} \in L^1(\Omega) \cap (H^1_{\Gamma_d}(\Omega))'$, $(\mu^\partial_{n,m})_{m\geq 1} \in L^1(\partial\Omega) \cap (H^{1/2}_{\Gamma_d}(\partial\Omega))'$ and $(u_{n,m})_{m\geq 1} \in H^1_{\Gamma_d}(\Omega)$ such that

$$\begin{array}{l}
\mu_{n,m} \stackrel{m\to\infty}{\longrightarrow} \mu_n \text{ in } (\mathcal{C}(\overline{\Omega}))' \text{ weak-}*,\ \mu^\partial_{n,m} \stackrel{m\to\infty}{\longrightarrow} \mu^\partial_n \text{ in } \mathcal{M}(\partial\Omega) \text{ weak-}*,\\
\exists C > 0 \text{ such that } ||\mu_{n,m}||_{\mathcal{M}(\Omega)} + ||\mu^\partial_{n,m}||_{\mathcal{M}(\partial\Omega)} \leq C \text{ for all } n \geq 1 \text{ and } m \geq 1,\\
\forall m \geq 1\,,\ u_{n,m} \text{ is a solution of (1.15) with } (\mu_{n,m}, \mu^\partial_{n,m}) \text{ instead of } (\mu_n, \mu^\partial_n),\\
u_{n,m} \stackrel{m\to\infty}{\longrightarrow} u_n \text{ in } W^{1,q}_{\Gamma_d}(\Omega) \text{ for all } q \in [1, N/(N-1)[
\end{array} \tag{3.9}$$

((3.9) is the additional hypothesis we must make — see below for the reason).

Let $\{\varphi_k\,,\ k \geq 1\}$ (respectively $\{\psi_k\,,\ k \geq 1\}$) be a countable dense subset of $\mathcal{C}(\overline{\Omega})$ (respectively $\mathcal{C}(\partial\Omega)$). For all $n \geq 1$, there exists $m_n \geq 1$ such that

- $\left|\int_\Omega \varphi_k\, d\mu_{n,m_n} - \int_\Omega \varphi_k\, d\mu_n\right| \leq 1/n$ for all $k \in [1,n]$,
- $\left|\int_{\partial\Omega} \psi_k\, d\mu^\partial_{n,m_n} - \int_{\partial\Omega} \psi_k\, d\mu^\partial_n\right| \leq 1/n$ for all $k \in [1,n]$,
- $||u_{n,m_n} - u_n||_{W^{1,N/(N-1)-1/n}_{\Gamma_d}(\Omega)} \leq 1/n$.

It is then quite clear that $(\nu_n)_{n\geq 1} = (\mu_{n,m_n})_{n\geq 1} \in L^1(\Omega) \cap (H^1_{\Gamma_d}(\Omega))'$ and $(\nu^\partial_n)_{n\geq 1} = (\mu^\partial_{n,m_n})_{n\geq 1} \in L^1(\partial\Omega) \cap (H^{1/2}_{\Gamma_d}(\partial\Omega))'$ converge respectively to μ in $(\mathcal{C}(\overline{\Omega}))'$ weak-$*$ and to μ^∂ in $\mathcal{M}(\partial\Omega)$ weak-$*$. Indeed, $(\nu_n)_{n\geq 1} = (\mu_{n,m_n})_{n\geq 1}$ is bounded in $\mathcal{M}(\Omega)$ by C (this is where we need (3.9)) and, for all $k \geq 1$, if $n \geq k$,

$$\left|\int_\Omega \varphi_k\, d\nu_n - \int_\Omega \varphi_k\, d\mu\right| \leq \frac{1}{n} + \left|\int_\Omega \varphi_k\, d\mu_n - \int_\Omega \varphi_k\, d\mu\right| \to 0 \text{ as } n \to \infty.$$

The bound of $(\nu_n)_{n\geq 1}$ and this convergence on a dense subset of $\mathcal{C}(\overline{\Omega})$ gives the weak-$*$ convergence. We can do the same for $(\nu^\partial_n)_{n\geq 1}$.
Thus, by definition of a SOLA, since $v_n = u_{n,m_n}$ is a solution of (1.15) with (ν_n, ν^∂_n) instead of (μ_n, μ^∂_n), there exists a subsequence $(v_{n_k})_{k\geq 1}$ and a SOLA u of (1.14) such that $v_{n_k} \to u$ in $W^{1,q}_{\Gamma_d}(\Omega)$ for all $q \in [1, N/(N-1)[$. Let $q \in [1, N/(N-1)[$; for all $k \geq (N/(N-1) - q)^{-1}$, since $n_k \geq k$, we have then (with $r_k = N/(N-1) - 1/n_k > q$),

$$\begin{aligned}
||u_{n_k} - u||_{W^{1,q}(\Omega)} &\leq ||u_{n_k} - v_{n_k}||_{W^{1,q}(\Omega)} + ||v_{n_k} - u||_{W^{1,q}(\Omega)}\\
&\leq |\Omega|^{1/q - 1/r_k} ||u_{n_k} - v_{n_k}||_{W^{1,r_k}(\Omega)} + ||v_{n_k} - u||_{W^{1,q}(\Omega)}\\
&\leq \frac{\sup(1, |\Omega|)}{n_k} + ||v_{n_k} - u||_{W^{1,q}(\Omega)} \to 0 \text{ as } k \to \infty,
\end{aligned}$$

which gives the convergence of $(u_{n_k})_{k\geq 1}$ to u in $W^{1,q}_{\Gamma_d}(\Omega)$, for all $q\in[1,N/(N-1)[$.

Suppose now that we add the hypotheses of Theorem 1.1, or that we are in the case of Remark 1.5. We have then the uniqueness of the SOLA.
The SOLA u_n thus does not depend on the way we approximate (μ_n,μ_n^∂), and we can always take $(\mu_{n,m},\mu_{n,m}^\partial)_{m\geq 1}$ which approximate these measures and satisfy moreover $||\mu_{n,m}||_{\mathcal{M}(\Omega)}\leq||\mu_n||_{\mathcal{M}(\Omega)}$ and $||\mu_{n,m}^\partial||_{\mathcal{M}(\partial\Omega)}\leq||\mu_n^\partial||_{\mathcal{M}(\partial\Omega)}$ for all $m\geq 1$; in this case, since $(\mu_n)_{n\geq 1}$ is bounded in $\mathcal{M}(\Omega)$ and $(\mu_n^\partial)_{n\geq 1}$ is bounded in $\mathcal{M}(\partial\Omega)$ (they converge for the weak-$*$ topology), we see that $(\mu_{n,m},\mu_{n,m}^\partial)_{n\geq 1,m\geq 1}$ satisfy (3.9).
By supposing that $(u_n)_{n\geq 1}$ does not converge to the SOLA u of (1.14), we would take $\varepsilon>0$ and a subsequence, still denoted $(u_n)_{n\geq 1}$, such that, for a $q_0\in[1,N/(N-1)[$, $||u_n-u||_{W^{1,q_0}(\Omega)}>\varepsilon$ for all n. Applying the preceding reasoning, we get a subsequence $(u_{n_k})_{k\geq 1}$ which converges in $W^{1,q_0}_{\Gamma_d}(\Omega)$ to a SOLA v of (1.14). The SOLA being unique, we have in fact $u=v$ and this leads to a contradiction, thus proving Theorem 1.2 and Remark 1.7. ■

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