Convergence of an Upstream Finite Volume Scheme for a Nonlinear Hyperbolic Equation on a Triangular Mesh

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Abstract
We study here the discretisation of the nonlinear hyperbolic equation $u_t + div(\mathbf{v}f(u)) = 0$ in $\mathbb{R}^2 \times \mathbb{R}_+$, with given initial condition $u(.,0) = u_0(.)$ in $\mathbb{R}^2$, where $\mathbf{v}$ is a function from $\mathbb{R}^2 \times \mathbb{R}_+$ to $\mathbb{R}^2$ such that $\mathrm{div}\mathbf{v} = 0$ and f is a given nondecreasing function from $\mathbb{R}$ to $\mathbb{R}$. An explicit Euler scheme is used for the time discretisation of the equation, and a triangular mesh for the spatial discretisation. Under a usual stability condition, we prove the convergence of the solution given by an upstream finite volume scheme towards the unique entropy weak solution to the equation.

1 Introduction

We consider here the following nonlinear hyperbolic equation in two space dimensions, with initial condition:

$$\left\{ \begin{array}{rlcll} u_t(x,t) + \mathrm{div}(\mathbf{v}(x,t)f(u(x,t))) & = 0, & x & \in & \mathbb{R}^2, \ t \text{ in } \mathbb{R}_+, \\ u(x,0) & = u_0(x), & x & \in & \mathbb{R}^2, \end{array} \right. \tag{1}$$

where u_t denotes the derivative of u with respect to t, $\mathrm{div} = \sum_{i=1}^{2} \partial_i$, where ∂_i denotes the partial derivative w.r.t. the i-th component, x_i, of $x \in \mathbb{R}^2$; $\mathbf{v}$ is a function from $\mathbb{R}^2 \times \mathbb{R}_+$ to $\mathbb{R}^2$, of class C^1, such that $\mathrm{div}\mathbf{v} = 0$ and $\sup_{(x,t)\in\mathbb{R}^2\times\mathbb{R}_+} |\mathbf{v}(x,t)| = V \in \mathbb{R}$, where $|.|$ denotes the euclidian norm on $\mathbb{R}^p$; f a given nondecreasing function of class C^1 from $\mathbb{R}$ to $\mathbb{R}$, and u_0 a given bounded function with compact support. Let be a triangular mesh and k the constant time step (the generalisation to variable time steps is straightforward). Let us discretise equation (1) using the explicit Euler scheme for the time discretisation and a finite volume scheme with upstream weighting for the spatial discretisation. The discrete unknowns are the values u_K^n, $n \in \mathbb{N}$, $K \in$, given by the numerical scheme (2) (described below) ; the value u_K^n is expected to be an approximation of the mean value of u in the triangle $K \in$ at time $t_n = nk$. In order to describe the scheme, some notations are required. We denote by $x.y = (x^t)y$ the scalar product of $x \in \mathbb{R}^2$ with $y \in \mathbb{R}^2$. Let $K \in$, we denote by $S(K)$ the area of the triangle K, by $c_i(K)$, $i = 1,2,3$ the sides of K, and $\mathbf{n}_{K,i}$ the normal to side $c_i(K)$, outward to K. Let $\mathcal{A}$ denote the set of edges of the mesh ; for $a \in \mathcal{A}$, $l(a)$ is the length of the edge a, $\mathbf{n}_a^n$ a unit normal vector to a such that $\overline{\mathbf{v}^n.\mathbf{n}_a^n} \geq 0$, where $\overline{\mathbf{v}^n.\mathbf{n}_a^n}$ denotes the mean value of $\mathbf{v}(.,t_n).\mathbf{n}_a^n$ on the edge a, $K_a^{n,+}$ (resp. $K_a^{n,-}$) the upstream (resp. downstream) triangle to a at time t_n, i.e. such that there exists $i \in \{1,2,3\}$ such that $a = c_i(K_a^{n,+})$ (resp. $a = c_i(K_a^{n,-})$), and $\mathbf{n}_{K_a^{n,+},i} = \mathbf{n}_a^n$ (resp. $\mathbf{n}_{K_a^{n,-},i} = -\mathbf{n}_a^n$). The numerical scheme requires an approximate value, $\{u^n\}_a$, of u on any edge a at time t_n ; since $f' \geq 0$, $\{u^n\}_a$ is taken to be the upstream value, i.e. $\{u^n\}_a = u^n_{K_a^{n,+}}$. We may then define the upstream finite volume scheme by :

$$\left\{ \begin{array}{ll} \dfrac{u_K^{n+1} - u_K^n}{k} S(K) + \displaystyle\sum_{i=1}^{3} l(c_i(K)) \ \overline{\mathbf{v}^n.\mathbf{n}_{K,i}} \ f(\{u^n\}_{c_i(K)}) = 0, & \forall \ K \text{ in } , \ \forall \ n \text{ in } \mathbb{N}, \\ u_K^0 = \dfrac{1}{S(K)} \displaystyle\int_K u_0(x)dx, & \forall \ K \text{ in } , \end{array} \right. \tag{2}$$

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We recall that $\overline{\mathbf{v}^n.\mathbf{n}_{K,i}}$ denotes the mean value of $\mathbf{v}(.,t_n).\mathbf{n}_{K,i}$ on side $c_i(K)$. Let the approximate solution at time t_n be defined by : $u^n(x) = u^n_K$ if $x \in K$, so that, for a given mesh and a time step k, the approximate solution is defined everywhere by :

$$u_{,k}(x,t) = u^n(x), \quad \text{for} \;\; t \in [nk,(n+1)k[, \quad x \in \mathbb{R}^2. \tag{3}$$

The aim of the present paper is to show the convergence (in a convenient topology) of $u_{,k}$ towards the (unique) entropy weak solution u to (1), when the time step and the "space step" go to zero. To this purpose, we shall need some nondegenerescence assumptions on the mesh (very much like those used within the framework of finite element methods) and a classical stability assumption on the time step. More precisely, let α, β be two fixed positive real numbers, we consider meshes such that there exists $h \in \mathbb{R}^*_+$, (the "space step") satisfying :

$$\begin{cases} \alpha h^2 & \leq & S(K) & \leq \beta h^2, & \forall \;\; K \in , \\ \alpha h & \leq & l(a) & \leq \beta h, & \forall \;\; a \in \mathcal{A}. \end{cases} \tag{4}$$

We assume the following stability condition on the time step :

$$k \leq \frac{S(K_a^{n,-})}{2l(a) \;\; |\overline{\mathbf{v}^n.\mathbf{n}_a^n}| \;\; M}(1-\zeta), \;\; \forall \;\; a \in \mathcal{A}, \;\; \forall \;\; n \in \mathbb{N}, \tag{5}$$

where $\zeta > 0$ is a given number, $M = \;\; \sup \;\; (f'(s), s \in [-U,U])$, and $U = \|u_0\|_\infty$. We denote by $\|.\|_p$ the norm on $L^p(\mathbb{R}^2)$ or $L^p(\mathbb{R}^2 \times \mathbb{R}_+)$, for $1 \leq p \leq \infty$. Note that, when the nondegenerescence assumptions (4) are satisfied, the global linear condition between h and k : $k \leq \dfrac{\alpha h}{2\beta \;\; V \;\; M}(1-\zeta)$ implies condition (5).

We then prove the following theorem :

THEOREM 1

Let $\mathbf{v}$ be a function from $\mathbb{R}^2 \times \mathbb{R}_+$ to $\mathbb{R}^2$, of class C^1, such that $\operatorname{div}\mathbf{v} = 0$ *and* $\sup_{(x,t)\in\mathbb{R}^2\times\mathbb{R}_+} |\mathbf{v}(x,t)| = V \in \mathbb{R}$, *$f$ a given nondecreasing function from $\mathbb{R}$ to $\mathbb{R}$, of class C^1, and u_0 a given bounded function with compact support ; consider the space and time discretisations and k satisfying assumptions* (4) *and* (5)*, let u be the entropy weak solution to* (1) *and $u_{,k}$ be the approximate solution given by* (2), (3)*; then :*

(stability of the numerical scheme)

$$\|u_{,k}\|_\infty \leq \|u_0\|_\infty, \tag{6}$$

(convergence of the numerical scheme)

$$u_{,k} \to u \;\; as \;\; h \to 0, \;\; in \;\; L^p_{loc}(\mathbb{R}^2 \times \mathbb{R}_+) \;\; for \;\; any \;\; finite \;\; p. \tag{7}$$

The originality of this result lies in the two-dimensional space dimension and the absence of assumptions on the mesh (other than the classical nondegenerescence assumptions (4)). For the one space dimension problem, convergence results have been obtained by several authors for the upstream weighting scheme, and also for some more general equations and schemes (the Godunov scheme for instance, and higher order schemes). Most of these results use some kind of L^∞ stability and "BV-stability" ("TVD" schemes, for instance), see Godunov (1976), Osher (1984), Harten (1983). In particular, the BV stability is used to obtain the relative compactness in L^1_{loc} of a family of approximate solutions. This result allows the passage to the limit in the nonlinear term of the equation. These methods may be generalised for two space dimensions problems for first order schemes (such as Godunov) on rectangular meshes, at least for constant $\mathbf{v}$ (see Crandall, Majda (1980), Sanders (1983)). Technical difficulties appear for higher order shemes or when nonrectangular meshes are used (i.e. in the case of triangles or irregular quadrangles). Indeed, the L^∞ stability result still holds, but it seems that the BV-stability result is no longer valid. The L^∞ stability is not sufficient to prove convergence (even in the linear case, see Champier, Gallouët (1992)). Another estimate on the spatial derivatives of the approximate solution (weaker than the BV stability) will be used to prove convergence. It is obtained by using the numerical diffusion of the scheme (see Section 3). This technique was developped for the nonlinear equation (1), but may be easily extended to more general equations and schemes. It may also be used for other types of finite volume methods, for instance those using nodal values as discrete unknowns (sometimes called "cell vertex" methods) instead of cell values ("cell centered" methods), see Eymard, Gallouët (accepted). Other authors have also obtained convergence results with

a different nonlinearity Cockburn et al. (submitted) (and for higher order schemes on rectangles). In Cockburn et al. (submitted), the nonlinear term $\mathbf{v}(x,t)f(u)$ is replaced by a function $\mathbf{f}(u)$, where $\mathbf{f}$ is a function from $\mathbb{R}$ to $\mathbb{R}^2$. Their convergence result, however, is proven via a weak estimate on the spatial derivatives which requires restrictive assumptions on the mesh (and either an additional assumption on the nonlinearity, or a uniform bound by below of the numerical viscosity of the scheme). Note that, even if $\mathbf{v}$ is constant, these assumptions cannot hold in our case since, in particular, $\overline{\mathbf{v}^n.\mathbf{n}_a^n}$ may be equal to 0, and f' may be equal to 0 (even on whole intervals of $\mathbb{R}$).
We shall prove the above theorem in the following sections. The L^∞ stability is given in Section 2. In Section 3, we introduce a "weak BV stability" estimate and prove it. Section 4 is devoted to the equation which is satisfied by a possible limit of a family of approximate solutions. In Section 5, we prove that this same possible limit satisfies some entropy inequalities. Finally, we use the two above sections and a generalisation of a result by DiPerna to prove the convergence of the numerical scheme (in Section 6).

2 Stability estimates

In this section, we prove the following stability result :

PROPOSITION 1. *Let $\mathbf{v}$ be a function from $\mathbb{R}^2 \times \mathbb{R}_+$ to $\mathbb{R}^2$, of class C^1, such that* $\mathrm{div}\mathbf{v} = 0$ *and* $\sup_{(x,t)\in\mathbb{R}^2\times\mathbb{R}_+} |\mathbf{v}(x,t)| = V \in \mathbb{R}$, *$f$ a given nondecreasing function from $\mathbb{R}$ to $\mathbb{R}$, of class C^1, and u_0 a given bounded function with compact support ; consider the space and time discretisations and k satisfying the following stability condition :*

$$k \le \frac{S(K_a^{n,-})}{2l(a)\ \ |\overline{\mathbf{v}^n.\mathbf{n}_a^n}|\ \ M} \quad \forall \quad a \in \mathcal{A}, \quad \forall \quad n \in \mathbb{N}, \tag{8}$$

where $M = \sup\limits_{s\in[-U,U]} f'(s)$, *where* $U = \|u_0\|_\infty$ *; let u_K^n and $u_{,k}$ be defined by (2) and (3), then :*

$$\min_{K\in} u_K^n \le \min_{K\in} u_K^{n+1} \le \max_{K\in} u_K^{n+1} \le \max_{K\in} u_K^n, \quad \forall \quad n \in \mathbb{N}, \tag{9}$$

and therefore :

$$\|u_{,k}\|_\infty \le \|u_0\|_\infty. \tag{10}$$

PROOF. Integrating the equation $\mathrm{div}\mathbf{v} = 0$ at time t_n over any cell $K \in$, we obtain :

$$\sum_{i=1}^{3} l(c_i(K))\ \ \overline{\mathbf{v}^n.\mathbf{n}_{K,i}} = 0, \quad \forall \quad K \in .$$

Hence, the numerical scheme (2) leads to :

$$u_K^{n+1} = u_K^n + \sum_{i=1}^{3} \quad a_{K,i}^n \quad (\{u^n\}_{c_i(K)} - u_K^n),$$

where

$$a_{K,i}^n = \begin{cases} 0 & \text{if } \overline{\mathbf{v}^n.\mathbf{n}_{K,i}} \ge 0, \\ \dfrac{k}{S(K)} l(c_i(K))\ \ |\overline{\mathbf{v}^n.\mathbf{n}_{K,i}}|\ \ \dfrac{f(\{u^n\}_{c_i(K)}) - f(u_K^n)}{\{u^n\}_{c_i(K)} - u_K^n} & \text{if } \overline{\mathbf{v}^n.\mathbf{n}_{K,i}} < 0, \end{cases}$$

(With the convention that $a_{K,i}^n = 0$ if $\{u^n\}_{c_i(K)} = u_K^n$.) Therefore,

$$u_K^{n+1} = (1 - \sum_{i=1}^{3} a_{K,i}^n) u_K^n + \sum_{i=1}^{3} a_{K,i}^n \{u^n\}_{c_i(K)}.$$

From this it is easily seen that (9) and (10) hold, and that u_K^{n+1} lies in the convex hull of u_K^n and $\{u^n\}_{c_i(K)}$, $i=1,2,3$.

REMARK 1. Under the assumptions of Proposition 1, we can also prove the following L^1 estimate :

$$\sum_{K\in} S(K)|u_K^{n+1}| \leq \sum_{K\in} S(K)|u_K^n| \ \ \forall \ \ n \in \mathbb{N}, \tag{11}$$

and therefore :

$$\|u_{,k}(.,t)\|_1 \leq \|u_0\|_1 \ \ \forall \ \ t \in \mathbb{R}_+. \tag{12}$$

The proof of Assertions (11) and (12) requires some further notations and is taken care of in Remark 5 of Section 5. Actually, these estimates will not be needed in the sequel.

REMARK 2. Note that Condition (8) is weaker than Condition (5) which we assumed in Theorem 1.

3 A Weak Estimate on the Spatial Derivatives

PROPOSITION 2. *Let* $\mathbf{v}$ *be a function from* $\mathbb{R}^2 \times \mathbb{R}_+$ *to* $\mathbb{R}^2$*, of class* C^1*, such that* $\mathrm{div}\mathbf{v} = 0$ *and* $\sup_{(x,t)\in\mathbb{R}^2\times\mathbb{R}_+} |\mathbf{v}(x,t)| = V \in \mathbb{R}$*,* f *a given nondecreasing function from* $\mathbb{R}$ *to* $\mathbb{R}$*, of class* C^1*, and* u_0 *a given bounded function with compact support ; let* $\zeta > 0$ *and consider the space and time discretisations and* k *satisfying condition* (5)*, let* u_K^n *be defined by* (2) *; then there exists* $C > 0$*, depending only on* f, u_0 *and* ζ*, such that :*

$$\sum_{n=0}^{+\infty} k \sum_{a\in\mathcal{A}} |\overline{\mathbf{v}^n.\mathbf{n}_a^n}| \ \ l(a) \ \ |f(u^n_{K_a^{n,+}}) - f(u^n_{K_a^{n,-}})|^2 \leq C. \tag{13}$$

PROOF. Multiplying the numerical scheme (2) by $f(u_K^n)$ and remarking that

$$\sum_{i=1}^{3} l(c_i(K)) \ \ \overline{\mathbf{v}^n.\mathbf{n}_{K,i}} = 0,$$

we obtain :

$$S(K)\frac{u_K^{n+1} - u_K^n}{k} f(u_K^n) + \sum_{i=1}^{3} l(c_i(K)) \ \ \overline{\mathbf{v}^n.\mathbf{n}_{K,i}} \ \ \left[f(\{u^n\}_{c_i(K)}) f(u_K^n) - \frac{(f(u_K^n))^2}{2} \right] = 0. \tag{14}$$

We set $F(\xi) = \displaystyle\int_0^\xi f(s)ds, \ \ \forall \ \ \xi \in \mathbb{R}$. We may assume, without loss of generality, that $f(0) = 0$, so that, since f is nondecreasing, $F \geq 0$. Then :

$$(u_K^{n+1} - u_K^n) f(u_K^n) = F(u_K^{n+1}) - F(u_K^n) - \int_{u_K^n}^{u_K^{n+1}} \big(f(\xi) - f(u_K^n)\big) d\xi. \tag{15}$$

Using (14) and (15), and after summation on n and K, we obtain, for any $N \in \mathbb{N}$:

$$\begin{aligned} \sum_{K\in} \frac{S(K)}{k} F(u_K^{N+1}) \quad & - \sum_{K\in} \frac{S(K)}{k} F(u_K^0) - \sum_{n=0}^{N} \sum_{K\in} \frac{S(K)}{k} \int_{u_K^n}^{u_K^{n+1}} \big(f(\xi) - f(u_K^n)\big) d\xi \\ & + \sum_{n=0}^{N} \sum_{K\in} l(c_i(K)) \ \ \overline{\mathbf{v}^n.\mathbf{n}_{K,i}} \ \ \left[f(\{u^n\}_{c_i(K)}) f(u_K^n) - \frac{(f(u_K^n))^2}{2} \right] = 0. \end{aligned} \tag{16}$$

In (16), all summations are finite because u_0 and therefore all u^n have compact supports. Replacing the sum on the triangles by a sum on the edges, the last term of the left hand side of (16) becomes :

$$\begin{aligned}\sum_{n=0}^{N}\sum_{K\in} l(c_i(K))\ \overline{\mathbf{v}^n.\mathbf{n}_{K,i}} \quad & \left[f(\{u^n\}_{c_i(K)})f(u_K^n)-\frac{\big(f(u_K^n)\big)^2}{2}\right] \\ &= \frac{1}{2}\sum_{n=0}^{N}\sum_{a\in\mathcal{A}} l(a)\ \ |\overline{\mathbf{v}^n.\mathbf{n}_a^n}|\ \ \left[f(u^n_{K_a^{n},+})-f(u^n_{K_a^{n},-})\right]^2.\end{aligned} \tag{17}$$

We now prove that the third term of the left hand side of equation (16) is smaller than the term given in (17), which is nonnegative, and which is in fact the expression which we want to estimate in Proposition 2. In order to do so, using the numerical scheme (2) and the fact that $(a+b)^2 \leq 2(a^2+b^2)$, we obtain :

$$\begin{aligned}(0\leq)\int_{u_K^n}^{u_K^{n+1}}\big(f(\xi)-f(u_K^n)\big)d\xi \ &\leq M\frac{(u_K^{n+1}-u_K^n)^2}{2} \\ &\leq \frac{M\ \ k^2}{2\big(S(K)\big)^2}\Big[\sum_{i=1}^{3} l(c_i(K))\ \ \overline{\mathbf{v}^n.\mathbf{n}_{K,i}}\ \ \big(f(\{u^n\}_{c_i(K)})-f(u_K^n)\big)\Big]^2 \\ &\leq \frac{M\ \ k^2}{\big(S(K)\big)^2}\sum_{i=1}^{3}\ \Big[l(c_i(K))\ \ \overline{\mathbf{v}^n.\mathbf{n}_{K,i}}\Big(f(\{u^n\}_{c_i(K)})-f(u_K^n)\Big)\Big]^2.\end{aligned}$$

Therefore :

$$\begin{aligned}\sum_{n=0}^{N}\sum_{K\in}\frac{S(K)}{k} \quad & \int_{u_K^n}^{u_K^{n+1}}\big(f(\xi)-f(u_K^n)\big)d\xi \\ &\leq \sum_{n=0}^{N}\sum_{K\in}\frac{M\ \ k}{S(K)}\sum_{i=1}^{3}\ [l(c_i(K))\ \ \overline{\mathbf{v}^n.\mathbf{n}_{K,i}}]^2\ \ \left[f(\{u^n\}_{c_i(K)})-f(u_K^n)\right]^2 \\ &= \sum_{n=0}^{N}\sum_{a\in\mathcal{A}}\frac{M\ \ k}{S(K_a^{n,-})}l(a)^2\ \ |\overline{\mathbf{v}^n.\mathbf{n}_a^n}|^2\ \ \left[f(u^n_{K_a^{n},+})-f(u^n_{K_a^{n},-})\right]^2.\end{aligned} \tag{18}$$

Using the stability condition (5), (18) yields :

$$\begin{aligned}\sum_{n=0}^{N}\sum_{K\in}\frac{S(K)}{k} \quad & \int_{u_K^n}^{u_K^{n+1}}(f(\xi)-f(u_K^n))d\xi \\ &\leq \frac{1-\zeta}{2}\sum_{n=0}^{N}\sum_{a\in\mathcal{A}}|\overline{\mathbf{v}^n.\mathbf{n}_a^n}|\ \ l(a)\ \ |f(u^n_{K_a^{n},+})-f(u^n_{K_a^{n},-})|^2.\end{aligned} \tag{19}$$

Using (17) and (19) in (16), and the fact that $F\geq 0$, we obtain :

$$\begin{aligned}\frac{\zeta}{2}\sum_{n=0}^{N}\sum_{a\in\mathcal{A}}|\overline{\mathbf{v}^n.\mathbf{n}_a^n}|\ \ l(a)\ \ |f(u^n_{K_a^{n},+})-f(u^n_{K_a^{n},-})|^2 \quad &\leq \sum_{K\in}\frac{S(K)}{k}F(u_K^0) \\ &\leq \frac{1}{k}\int_{\mathbb{R}^2}F(u_0(x)dx.\end{aligned} \tag{20}$$

Since, in (20), N is arbitrary, Proposition 2 is hence proven with $C=\dfrac{2}{\zeta}\displaystyle\int_{\mathbb{R}^2}F(u_0(x)dx$.

In the following, we shall need a straightforward consequence of Proposition 2, which we now state.

Corollary.

Let $\mathbf{v}$ *be a function from* $\mathbb{R}^2\times\mathbb{R}_+$ *to* $\mathbb{R}^2$*, of class* C^1*, such that* $\operatorname{div}\mathbf{v}=0$ *and* $\sup_{(x,t)\in\mathbb{R}^2\times\mathbb{R}_+}|\mathbf{v}(x,t)|=V\in\mathbb{R}$*,* f *a given nondecreasing function from* $\mathbb{R}$ *to* $\mathbb{R}$*, of class* C^1*, and* u_0 *a given bounded function with compact support*

; *let $\zeta > 0$ and consider the space and time discretisations and k satisfying conditions* (4)*, with $h \le 1$, and* (5) *; let u_K^n be defined by* (2)*. Let r, $T \ge 0$; then there exists $C_{r,T} \ge 0$, depending only on $u_0, f, \mathbf{v}, \alpha, \beta, \zeta, r$ and T, such that :*

$$\sum_{n=0}^{N_T} k \sum_{a \in \mathcal{A}_r} |\overline{\mathbf{v}^n.\mathbf{n}_a^n}| \quad l(a) \quad |f(u^n_{K_a^n,+}) - f(u^n_{K_a^n,-})| \le C_{r,T} h^{-\frac{1}{2}}, \tag{21}$$

where $k(N_T - 1) < T \le kN_T$ and $\mathcal{A}_r = \{a \in \mathcal{A}; a \cap B(0, r+\beta) \neq \emptyset\}$.

Remark 3. The left hand side of (21) can be seen as the norm, in the space of Radon measures on $B(0,r) \times [0,T]$, of $\operatorname{div}(\mathbf{v} f(u_{,k}))$. Thus, the above corollary states that this norm may tend to infinity when h tends to 0, but no faster than $h^{-\frac{1}{2}}$. Also note that inequality (21) does not imply an inequality of the type

$$\sum_{n=0}^{N_T} k \sum_{a \in \mathcal{A}_r} l(a) \quad |u^n_{K_a^n,+} - u^n_{K_a^n,-}| \le C'_{r,T} h^{-\frac{1}{2}},$$

since f' or $\overline{\mathbf{v}^n.\mathbf{n}_a^n}$ may be equal to 0.

4 The Equation for a Limit of a Family of Approximate Solutions

When and k satisfy the stability condition (5), then, by Proposition 1, the family $(u_{,k})_{(,k)}$ is bounded in $L^\infty(\mathbb{R}^2 \times \mathbb{R}_+)$; we may therefore assume that $u_{,k}$ tends towards a limit u in $L^\infty(\mathbb{R}^2 \times \mathbb{R}_+)$ for the weak star topology, when h tends to 0 (in fact, as usual, from any sequence $((_i, k_i))_{i \in \mathbb{N}}$, with $(_i, k_i)$ satisfying (5), for all i, we may extract a subsequence $((_{s(i)}, k_{s(i)}))_{i \in \mathbb{N}}$, such that $(u_{_{s(i)}, k_{s(i)}})_{i \in \mathbb{N}}$ converges in L^∞ for the weak star topology). At this point, we have no strong convergence result on $u_{,k}$ (in $L^1_{loc}(\mathbb{R}^2 \times \mathbb{R}_+)$, for instance) ; therefore, since f may be nonlinear, we cannot assert that $f(u_{,k})$ tends to $f(u)$. Indeed, one of the major difficulties to be overcome here is that the weak estimate on the spatial derivatives, which we proved in Section 3, does not give the relative compactness of $(u_{,k})_{(,k)}$ in $L^1_{loc}(\mathbb{R}^2 \times \mathbb{R}_+)$; the compactness of $(u_{,k})_{(,k)}$ in $L^1_{loc}(\mathbb{R}^2 \times \mathbb{R}_+)$ was obtained in most previous works on the subject (in the case of one space dimension or two space dimensions with a rectangular mesh) by a BV estimate ; more precisely, the compactness of $(u_{,k})_{(,k)}$ in $L^1_{loc}(\mathbb{R}^2 \times \mathbb{R}_+)$ is a consequence of the L^∞ stability (cf (10)) and of an estimate of the type (with the notations of the above corollary) :

$$\sum_{n=0}^{N_T} k \sum_{a \in \mathcal{A}_r} l(a) \quad |u^n_{K_a^n,+} - u^n_{K_a^n,-}| \le C,$$

which does not seem to hold in our case. We may only assume, and we shall in the sequel, that, for any continuous function g, from $\mathbb{R}$ to $\mathbb{R}$, $g(u_{,k})$ converges to a limit μ_g in $L^\infty(\mathbb{R}^2 \times \mathbb{R}_+)$ for the weak star topology, as h tends to 0 (as stated above, we are in fact working with convenient subsequences). Assuming that $u_{,k}$ tends to u and $f(u_{,k})$ tends to μ_f when h tends to 0, $(,k)$ satisfying assumptions (4) and (5), we now prove that $u_t + \operatorname{div}(\mathbf{v}\mu_f) = 0$ and $u(.,0) = u_0(.)$ in a weak sense which we state in the following proposition :

Proposition 3. *Let $\mathbf{v}$ be a function from $\mathbb{R}^2 \times \mathbb{R}_+$ to $\mathbb{R}^2$, of class C^1, such that $\operatorname{div}\mathbf{v} = 0$ and $\sup_{(x,t) \in \mathbb{R}^2 \times \mathbb{R}_+} |\mathbf{v}(x,t)| = V \in \mathbb{R}$, f a given nondecreasing function from $\mathbb{R}$ to $\mathbb{R}$, of class C^1, and u_0 a given bounded function with compact support, let and k satisfy conditions* (4) *and* (5)*. Let $u_{,k}$ be given by* (2) *and* (3)*. Assume that $u_{,k} \to u$, $f(u_{,k}) \to \mu_f$, in $L^\infty(\mathbb{R}^2 \times \mathbb{R}_+)$ for the weak star topology, as $h \to 0$. Then :*

$$\int_{\mathbb{R}^2 \times \mathbb{R}_+} (u(x,t)\varphi_t(x,t) + \mu_f(x,t)\mathbf{v}(x,t).\mathbf{grad}\varphi(x,t))dxdt + \int_{\mathbb{R}^2} u_0(x)\varphi(0,x)dx = 0, \quad \forall \ \varphi \in C^1_c(\mathbb{R}^2 \times [0,+\infty[, \mathbb{R}). \tag{22}$$

PROOF. Let $u_{,k}$ be given by (2) and (3). Let $\varphi \in C_c^1(\mathbb{R}^2 \times [0,+\infty[,\mathbb{R})$. Let $n \in \mathbb{N}$, $x \in K$; we multiply (2) by $\varphi(x,t_n)$, and integrate the resulting equation over K. We then sum for $n \in \mathbb{N}$ and $K \in$:

$$\begin{aligned} & k\sum_{n=0}^{+\infty}\sum_{K\in}\int_K \frac{u_K^{n+1}-u_K^n}{k}\varphi(x,t_n)dx \\ & \qquad +k \sum_{n=0}^{+\infty}\sum_{K\in}\frac{1}{S(K)}\int_K\sum_{i=1}^{3} l(c_i(K))\ \ \overline{\mathbf{v}^n.\mathbf{n}_{K,i}}\ \ f(\{u^n\}_{c_i(K)})\ \ \varphi(x,t_n)dx = 0. \end{aligned} \tag{23}$$

The proof of (22) starts from (23) and is decomposed into two steps, where we study successively the first and second terms of the left hand side of (23).

Step 1. We study here the limit of the first term of the left hand side of (23) as h tends to 0.

$$\begin{aligned} & k\sum_{n=0}^{+\infty}\sum_{K\in}\int_K \frac{u_K^{n+1}-u_K^n}{k}\varphi(x,t_n)dx = \\ & \qquad k\sum_{n=1}^{+\infty}\sum_{K\in}\int_K u_K^n\frac{\varphi(x,t_{n-1})-\varphi(x,t_n)}{k}dx - \sum_{K\in}\int_K u_K^0\varphi(x,0)dx \end{aligned} \tag{24}$$

Using the convergence of $u_{,k}$ in $L^\infty(\mathbb{R}^2\times\mathbb{R}_+)$ for the weak star topology and the convergence of u^0 to u_0 in $L^1(\mathbb{R}^2)$, (24) yields :

$$\begin{aligned} & k\sum_{n=0}^{+\infty}\sum_{K\in}\int_K \frac{u_K^{n+1}-u_K^n}{k}\varphi(x,t_n)dx \to \\ & \qquad -\int_{\mathbb{R}^2\times\mathbb{R}_+} u(x,t)\ \ \varphi_t(x,t)dxdt - \int_{\mathbb{R}^2} u_0(x)\varphi(x,0)dx,\ \text{ as } h\to 0. \end{aligned} \tag{25}$$

This gives the limit of the first term of the left hand side of (23) as $h \to 0$.

Step 2. Let us now study the second term of the left hand side of (23) and prove that :

$$\begin{aligned} & k\sum_{n=0}^{+\infty}\sum_{K\in}\frac{1}{S(K)}\int_K\sum_{i=1}^{3} l(c_i(K))\ \ \overline{\mathbf{v}^n.\mathbf{n}_{K,i}}\ \ f(\{u^n\}_{c_i(K)})\ \ \varphi(x,t_n)dx \to \\ & \qquad -\int_{\mathbb{R}^2\times\mathbb{R}_+} \mu_f(x,t)\ \ \mathbf{v}(x,t).\mathbf{grad}\varphi(x,t)\ \ dxdt,\ \text{ as } h\to 0. \end{aligned} \tag{26}$$

Once (26) is proven, we deduce from (25) that

$$\int_{\mathbb{R}^2\times\mathbb{R}_+}\Big(u(x,t)\varphi_t(x,t)+\mu_f(x,t)\ \ \mathbf{v}(x,t).\mathbf{grad}\varphi(x,t)\Big)dxdt + \int_{\mathbb{R}^2} u_0(x)\ \ \varphi(0,x)dx = 0,$$

so that the proof of Proposition 3 is complete. In order to prove (26), we first write that, since $f(u_{,k}) \to \mu_f$ in $L^\infty(\mathbb{R}^2\times\mathbb{R}_+)$ for the weak star topology (and $\sum_{n=0}^{+\infty}\mathbf{v}(.,t_n)\mathbf{grad}\varphi(.,t_n)1_{[t_n,t_{n+1}[} \to \mathbf{v}\mathbf{grad}\varphi$ in $L^1(\mathbb{R}^2\times\mathbb{R}_+)$) as h goes to 0,

$$k\sum_{n=0}^{+\infty}\sum_{K\in}\int_K f(u_K^n)\ \ (\mathbf{v}.\mathbf{grad}\varphi)(x,t_n)dx \to \int_{\mathbb{R}^2\times\mathbb{R}_+}\mu_f(x,t)\ \ (\mathbf{v}.\mathbf{grad}\varphi)(x,t)dxdt \tag{27}$$

Thus, if we prove that :

$$R = k\sum_{n=0}^{+\infty}\sum_{K\in}\frac{1}{S(K)}\quad\int_K\sum_{i=1}^{3} l(c_i(K))\ \ \overline{\mathbf{v}^n.\mathbf{n}_{K,i}}\ \ f(\{u^n\}_{c_i(K)})\ \ \varphi(x,t_n)dx \tag{28}$$
$$+k\sum_{n=0}^{+\infty}\sum_{K\in}\int_K f(u_K^n)\ \ (\mathbf{v}.\mathbf{grad}\varphi)(x,t_n)dx \to 0,\ \text{ as } h\to 0,$$

the proof of (26) is complete. In order to prove (28), we introduce the following functions : let $\mathbf{p}_{K,i}$ be the affine function defined from K to $\mathbb{R}^2$ by :

$$\mathbf{p}_{K,i}(x).\mathbf{n}_{K,j} = \delta_{i,j}\quad\forall\quad x\in c_j(K),\quad i,j=1,2,3, \tag{29}$$

where $\delta_{i,j}$ is the Krönecker symbol (note that these functions are the shape functions of the mixed finite element method). We set :

$$\mathbf{P}_e^n(x) = f(u_K^n)\mathbf{v}(x,t_n),\quad\forall\quad x\in K,\quad\forall\quad n\in\mathbb{N}. \tag{30}$$

Since $\sum_{i=1}^{3}\mathbf{p}_{K,i}(x)\mathbf{n}_{K,i}^t = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$ for all $x\in K$,

$$\mathbf{P}_e^n(x) = \sum_{i=1}^{3} f(u_K^n)\ \ \mathbf{v}(x,t_n).\mathbf{n}_{K,i}\ \ \mathbf{p}_{K,i}(x),\quad\forall\quad x\in K,\quad\forall\quad n\in\mathbb{N}. \tag{31}$$

We also set :

$$\mathbf{P}^n(x) = \sum_{i=1}^{3} f(\{u^n\}_{c_i(K)})\ \ \overline{\mathbf{v}^n.\mathbf{n}_{K,i}}\ \ \mathbf{p}_{K,i}(x),\quad\forall\quad x\in K,\quad\forall\quad n\in\mathbb{N}. \tag{32}$$

Since $\mathbf{P}^n|_{K_a^n,+}.\mathbf{n}_a^n = \mathbf{P}^n|_{K_a^n,-}.\mathbf{n}_a^n,\quad\forall\quad a\in\mathcal{A}$, and $\operatorname{div}(\mathbf{p}_{K,i}) = \dfrac{l\big(c_i(K)\big)}{S(K)}$, we obtain :

$$\begin{aligned}
\int_{\mathbb{R}^2}\mathbf{P}^n(x)\ \ \mathbf{grad}\varphi(x,t_n)dx\quad &= -\int_{\mathbb{R}^2}\operatorname{div}(\mathbf{P}^n(x))\ \ \varphi(x,t_n)dx\\
&= -\sum_{K\in}\int_K\sum_{i=1}^{3} f(\{u^n\}_{c_i(K)})\ \ \overline{\mathbf{v}^n.\mathbf{n}_{K,i}}\ \ \operatorname{div}(\mathbf{p}_{K,i}(x))\ \ \varphi(x,t_n)dx\\
&= -\sum_{K\in}\int_K\frac{1}{S(K)}\sum_{i=1}^{3} l(c_i(K))\ \ f(\{u^n\}_{c_i(K)})\ \ \overline{\mathbf{v}^n.\mathbf{n}_{K,i}}\ \ \varphi(x,t_n)dx
\end{aligned}$$

We may then write R defined in (28) in terms of $\mathbf{P}_e^n$ and $\mathbf{P}^n$:

$$R = k\sum_{n=0}^{+\infty}\sum_{K\in}\int_K(\mathbf{P}_e^n(x)-\mathbf{P}^n(x))\ \ \mathbf{grad}\varphi(x,t_n)dx. \tag{33}$$

Under the assumptions (4) of nondegeneresence of the mesh, there exists C_1 depending only on α and β, such that $|\mathbf{p}_{K,i}(x)|\le C_1,\ \forall\quad x\in K,\ \forall\quad K\in\ ,\ \forall\quad i\in\{1,2,3\}$. Therefore :

$$\begin{aligned}
|R| \le C_1 k\sum_{n=0}^{+\infty}\sum_{K\in}\int_K\sum_{i=1}^{3}\Big(&|f(u_K^n)-f(\{u^n\}_{c_i(K)})|\ \ |\overline{\mathbf{v}^n.\mathbf{n}_{K,i}}|\ \ +\\
&|\mathbf{v}(x,t_n).\mathbf{n}_{K,i}-\overline{\mathbf{v}^n.\mathbf{n}_{K,i}}||f(u_K^n)|\Big)|\mathbf{grad}\varphi(x,t_n)|dx.
\end{aligned}$$

Let r and T such that supp $\varphi\subset B(0,r)\times[0,T[$, N_T such that : $k(N_T-1)<T\le kN_T$ and $\mathcal{A}_r=\{a\in\mathcal{A};\ a\cap B(0,r+\beta)\neq\emptyset\}$. There exists C_2, depending only on φ, α, β, $\mathbf{v}$, u_0 and f, such that for $h\le 1$,

$$\begin{aligned}
|R|\quad &\le C_2 k\beta h^2\sum_{n=0}^{N_T}\sum_{a\in\mathcal{A}_r}|\overline{\mathbf{v}^n.\mathbf{n}_a^n}|\ \ |f(u^n_{K_a^n,+})-f(u^n_{K_a^n,-})|+C_2h\\
&\le C_2\frac{\beta}{\alpha}kh\sum_{n=0}^{N_T}\sum_{a\in\mathcal{A}_r} l(a)\ \ |\overline{\mathbf{v}^n.\mathbf{n}_a^n}|\ \ |f(u^n_{K_a^n,+})-f(u^n_{K_a^n,-})|+C_2h\\
&\le C_{r,T}C_2\frac{\beta}{\alpha}h^{\frac12}+C_2h,
\end{aligned}$$

where $C_{r,T}$ is defined in the corollary of Proposition 2. Hence, R goes to 0 with h, which proves Proposition 3.

REMARK 4. In order to prove the convergence of the numerical scheme, there remains to show that $\mu_f = f(u)$ and that u is the entropy weak solution of (1). The only possible limit of a sequence of approximate solutions (with conditions (4) and (5) satisfied) is then the unique entropy weak solution u to (1), and, by a classical argument, we deduce that $u_{,k}$ tends to u as h tends to 0, in $L^\infty(\mathbb{R}^2 \times \mathbb{R}_+)$ for the weak star topology. Of course, in the linear case (i.e. $f(u) = u$), Proposition 3 proves the convergence of $u_{,k}$ to u in $L^\infty(\mathbb{R}^2 \times \mathbb{R}_+)$ for the weak star topology, as h goes to 0 (with $,k$ satisfying Conditions (4) and (5)), and u is the weak (entropy) solution to (1).

5 The Entropy Inequalities

We assume here that, when h tends to 0 (and k satisfying conditions (4) and (5)), $g(u_{,k})$ tends to μ_g for any continuous function g from $\mathbb{R}$ to $\mathbb{R}$ (which is, in fact, true for convenient subsequences of sequences of approximate solutions). We already know, by the preceding section, that $u_t + \mathrm{div}(\mathbf{v}\mu_f) = 0$ and $u(.,0) = u_0(.)$ in a weak sense (see Proposition 3). (Recall that $u = \mu_{Id}$.) In this section, we prove the following entropy inequalities :

PROPOSITION 4
Let $\mathbf{v}$ be a function from $\mathbb{R}^2 \times \mathbb{R}_+$ to $\mathbb{R}^2$, of class C^1, such that $\mathrm{div}\mathbf{v} = 0$ and $\sup_{(x,t)\in\mathbb{R}^2\times\mathbb{R}_+} |\mathbf{v}(x,t)| = V \in \mathbb{R}$, f a given nondecreasing function from $\mathbb{R}$ to $\mathbb{R}$, of class C^1, and u_0 a given bounded function with compact support, let and k satisfy conditions (4) and (5), and $u_{,k}$ be given by (2) and (3). Assume that, for any continuous function g from $\mathbb{R}$ to $\mathbb{R}$, $g(u_{,k}) \to \mu_g$, in $L^\infty(\mathbb{R}^2 \times \mathbb{R}_+)$ for the weak star topology, as $h \to 0$. Then, for any convex function η from $\mathbb{R}$ to $\mathbb{R}$ of class C^1 and Φ such that $\Phi' = \eta' f'$:

$$\begin{aligned} &\int_{\mathbb{R}^2\times\mathbb{R}_+} \Big[\mu_\eta(x,t)\varphi_t(x,t) + \mu_\Phi(x,t)\mathbf{v}(x,t).\mathbf{grad}\varphi(x,t))\Big]dxdt + \int_{\mathbb{R}^2} \eta\big(u_0(x)\big)\varphi(0,x)dx \geq 0, \\ &\forall \ \ \varphi \in C_c^1(\mathbb{R}^2 \times [0,+\infty[, \mathbb{R}_+). \end{aligned} \tag{34}$$

PROOF. Let $u_{,k}$ be given by (2) and (3), and k satisfying conditions (4) and (5). Let η be a C^1 convex function, $\Phi(s) = \displaystyle\int_0^s \eta'(\sigma)f'(\sigma)d\sigma$. Let $\varphi \in C_c^1(\mathbb{R}^2 \times [0,+\infty[, \mathbb{R}_+)$. Let $n \in \mathbb{N}$, multiplying (2) by $\eta'(u_K^n)$, we obtain :

$$\frac{u_K^{n+1} - u_K^n}{k}\eta'(u_K^n) + \frac{1}{S(K)}\sum_{i=1}^3 l(c_i(K)) \ \ \overline{\mathbf{v}^n.\mathbf{n}_{K,i}} \ \ f(\{u^n\}_{c_i(K)}) \ \ \eta'(u_K^n) = 0. \tag{35}$$

Note that :

$$\begin{aligned} \eta(u_K^{n+1}) - \eta(u_K^n) &= \int_{u_K^n}^{u_K^{n+1}} \eta'(\xi)d\xi \\ &= \eta'(u_K^n)(u_K^{n+1} - u_K^n) + \int_{u_K^n}^{u_K^{n+1}} \big(\eta'(\xi) - \eta'(u_K^n)\big)d\xi. \end{aligned}$$

Thus, we may write (35) as :

$$Z_K^n - E_K^n + X_K^n + H_K^n = 0, \ \ \forall \ \ K \in , \ \ \forall \ \ n \in \mathbb{N}, \tag{36}$$

where:

$$\begin{aligned} Z_K^n &= \frac{1}{k}(\eta(u_K^{n+1}) - \eta(u_K^n)), \\ E_K^n &= \frac{1}{k}\int_{u_K^n}^{u_K^{n+1}} \big(\eta'(\xi) - \eta'(u_K^n)\big)d\xi, \\ X_K^n &= \frac{1}{S(K)}\sum_{i=1}^3 l(c_i(K)) \ \ \overline{\mathbf{v}^n.\mathbf{n}_{K,i}} \ \ \Phi(\{u^n\}_{c_i(K)}), \\ H_K^n &= \frac{1}{S(K)}\sum_{i=1}^3 l(c_i(K)) \ \ \overline{\mathbf{v}^n.\mathbf{n}_{K,i}} \ \ \Big(f(\{u^n\}_{c_i(K)}) \ \ \eta'(u_K^n) - \Phi(\{u^n\}_{c_i(K)})\Big). \end{aligned}$$

In a similar way as in the proof of Proposition 3, using the fact that $g(u_{,k}) \rightarrow \mu_g$ (in $L^\infty(\mathbb{R}^2 \times \mathbb{R}_+)$ for the weak star topology) when $h \rightarrow 0$ for any continuous function g, it can be proven that :

$$\begin{aligned} k\sum_{n=0}^{+\infty}\sum_{K\in}\int_K \quad & Z_K^n \varphi(x,t_n)dxdt \rightarrow \\ & -\int_{\mathbb{R}^2\times\mathbb{R}_+} \mu_\eta(x,t)\varphi_t(x,t)dxdt - \int_{\mathbb{R}^2}\eta\big(u_0(x)\big)\varphi(x,0)dx \text{ as } h \rightarrow 0. \end{aligned} \tag{37}$$

We prove thereafter that :

$$k\sum_{n=0}^{+\infty}\sum_{K\in}\int_K X_K^n\varphi(x,t_n) \ \ dxdt \rightarrow -\int_{\mathbb{R}^2\times\mathbb{R}_+}\mu_\Phi(x,t) \ \ \mathbf{v}(x,t).\mathbf{grad}\varphi(x,t) \ \ dxdt, \tag{38}$$

and that

$$H_K^n \geq E_K^n, \ \ \forall \ \ K \in, \ \ \forall \ \ n \in \mathbb{N}. \tag{39}$$

Assertions $(36) - (39)$ yield :

$$\int_{\mathbb{R}^2\times\mathbb{R}_+}\Big[\mu_\eta(x,t) \ \ \varphi_t(x,t) + \mu_\Phi(x,t) \ \ \mathbf{v}(x,t).\mathbf{grad}\varphi(x,t)\Big]dxdt + \int_{\mathbb{R}^2}\eta(u_0(x)) \ \ \varphi(0,x)dx \geq 0,$$

which completes the proof of Proposition 4. There remains to show assertions (38) and (39), which we now turn to in steps 1 and 2 below.

Step 1. (Proof of Assertion (38)*).* We introduce here similar entities to those we did in the proof of Proposition 3 :

$$\mathbf{Q}_e^n(x) = \Phi(u_K^n) \ \ \mathbf{v}(x,t_n), \ \ \forall \ \ x \in K, \ \ \forall \ \ n \in \mathbb{N}. \tag{40}$$

Note that :

$$\mathbf{Q}_e^n(x) = \sum_{i=1}^{3}\Phi(u_K^n) \ \ \mathbf{v}(x,t_n).\mathbf{n}_{K,i} \ \ \mathbf{p}_{K,i}(x), \ \ \forall \ \ x \in K, \ \ \forall \ \ n \in \mathbb{N}, \tag{41}$$

where $\mathbf{p}_{K,i}$ was defined in the proof of Proposition 3 (see (29)). Let :

$$\mathbf{Q}^n(x) = \sum_{i=1}^{3}\Phi(\{u^n\}_{c_i(K)}) \ \ \overline{\mathbf{v}^n.\mathbf{n}_{K,i}} \ \ \mathbf{p}_{K,i}(x), \ \ \forall \ \ x \in K, \ \ \forall \ \ n \in \mathbb{N}. \tag{42}$$

On one hand, we write that :

$$\begin{aligned} k\sum_{n=0}^{+\infty}\sum_{K\in}\int_K \mathbf{Q}^n(x) \quad \mathbf{grad}\varphi(x,t_n)dx &= -k\sum_{n=0}^{+\infty}\sum_{K\in}\int_K \operatorname{div}\big(\mathbf{Q}^n(x)\big) \ \ \varphi(x,t_n)dx \\ &= -k\sum_{n=0}^{+\infty}\sum_{K\in}\int_K \frac{1}{S(K)}\sum_{i=1}^{3} l\big(c_i(K)\big) \ \ \overline{\mathbf{v}^n.\mathbf{n}_{K,i}} \ \ \Phi(\{u^n\}_{c_i(K)})\varphi(x,t_n)dx \\ &= -k\sum_{n=0}^{+\infty}\sum_{K\in}\int_K X_K^n \ \ \varphi(x,t_n)dx. \end{aligned}$$

On the other hand, since $\Phi(u_{,k}) \rightarrow \mu_\Phi$ in $L^\infty(\mathbb{R}^2 \times \mathbb{R}_+)$ for the weak star topology,

$$k\sum_{n=0}^{+\infty}\sum_{K\in}\int_K \mathbf{Q}_e^n(x) \ \ \mathbf{grad}\varphi(x,t_n)dx \rightarrow \int_{\mathbb{R}^2\times\mathbb{R}_+}\mu_\Phi(x,t) \ \ \mathbf{v}(x,t).\mathbf{grad}\varphi(x,t)dxdt, \text{ as } h \rightarrow 0.$$

Therefore, Assertion (38) is proven provided that :

$$k\sum_{n=0}^{+\infty}\sum_{K\in}\int_K \big(\mathbf{Q}_e^n(x) - \mathbf{Q}^n(x)\big)\mathbf{grad}\varphi(x,t_n)dx \rightarrow 0, \text{ as } h \rightarrow 0. \tag{43}$$

In order to prove (43), we first notice that, since $f' \geq 0$, for any real numbers a, b such that $a \leq b$, there exists $c \in]a, b[$ such that : $\Phi(b) - \Phi(a) = \displaystyle\int_a^b \eta'(s) f'(s) ds = \eta'(c)(f(b) - f(a))$. Therefore, $|\Phi(u_K^n) - \Phi(\{u^n\}_{c_i(K)})| \leq \sup_{s \in [-U, U]} |\eta'(s)| \; |f(u_K^n) - f(\{u^n\}_{c_i(K)})|$, $\forall \; K \in$, $\forall \; n \in \mathbb{N}$, $\forall \; i \in \{1, 2, 3\}$ (where $U = \|u_0\|_\infty$). Using the weak estimate on the spatial derivatives (21), we derive (43), in a similar way to the proof of $R \to 0$ (using (33)) in the proof of Proposition 3.

Step 2 (Proof of Assertion (39)*).* We are now going to prove that

$$H_K^n \geq E_K^n, \quad \forall \quad K \in , \quad \forall \quad n \in \mathbb{N}. \tag{39}$$

(Note that $E_K^n \geq 0$.) Since $\displaystyle\sum_{i=1}^{3} l\big(c_i(K)\big) \;\; \overline{\mathbf{v}^n.\mathbf{n}_{K,i}} = 0$ (because $\mathrm{div}\mathbf{v} = 0$), we may write :

$$H_K^n = \frac{1}{S(K)} \sum_{i=1}^{3} l\big(c_i(K)\big) \;\; \overline{\mathbf{v}^n.\mathbf{n}_{K,i}} \;\; \Big[\Big(f(\{u^n\}_{c_i(K)}) - f(u_K^n) \Big) \eta'(u_K^n) - \Big(\Phi(\{u^n\}_{c_i(K)}) - \Phi(u_K^n) \Big) \Big].$$

Thus :

$$H_K^n = \frac{1}{S(K)} \sum_{i=1}^{3} l\big(c_i(K)\big) \;\; \overline{\mathbf{v}^n.\mathbf{n}_{K,i}} \;\; \int_{u_K^n}^{\{u^n\}_{c_i(K)}} f'(\xi) \Big(\eta'(u_K^n) - \eta'(\xi) \Big) d\xi. \tag{44}$$

(At this point, using $f' \geq 0$ and the fact that η' is nondecreasing, we may remark that $H_K^n \geq 0$.) Recall that :

$$E_K^n = \frac{1}{k} \quad \int_{u_K^n}^{u_K^{n+1}} \Big(\eta'(\xi) - \eta'(u_K^n) \Big) \quad d\xi, \tag{45}$$

and

$$u_K^{n+1} = u_K^n + \sum_{i=1}^{3} \quad b_{K,i}^n \quad \Big(f(\{u^n\}_{c_i(K)}) - f(u_K^n) \Big),$$

where

$$b_{K,i}^n = \begin{cases} 0, & \text{if } \overline{\mathbf{v}^n.\mathbf{n}_{K,i}} \geq 0, \\ \dfrac{k}{S(K)} \quad l\big(c_i(K)\big) \quad |\overline{\mathbf{v}^n.\mathbf{n}_{K,i}}|, & \text{if } \overline{\mathbf{v}^n.\mathbf{n}_{K,i}} < 0. \end{cases}$$

Condition (8) (which is implied by condition (5)), yields :

$$0 \leq b_{K,i}^n \leq \frac{1}{2M}, \quad \forall \quad K \in , \quad \forall \quad n \in \mathbb{N}, \quad \forall \quad i \in \{1, 2, 3\}. \tag{46}$$

(Recall that $M = \sup(|f'(s)|, \;\; s \in [-U, U])$, where $U = \|u_0\|_\infty$.) Assertion (39) is therefore a consequence of the following lemma :

Lemma. *Let $U \in \mathbb{R}_+^*$, $a, b, c \in [-U, U]$, f a nondecreasing C^1 function from $\mathbb{R}$ to $\mathbb{R}$, $M = sup(|f'(s)|, s \in [-U, U])$. Let γ, $\delta \geq 0$, such that :*

$$\gamma \leq \frac{1}{2M}, \qquad \delta \leq \frac{1}{2M},$$

and η be a C^1 convex function from $\mathbb{R}$ to $\mathbb{R}$. Then :

$$\begin{aligned} \int_a^{a+\gamma(f(b)-f(a))+\delta(f(c)-f(a))} (\eta'(\xi) - \eta'(a)) d\xi \leq \quad & \gamma \int_a^b f'(\xi)(\eta'(\xi) - \eta'(a)) d\xi \\ + \quad & \delta \int_a^c f'(\xi)(\eta'(\xi) - \eta'(a)) d\xi. \end{aligned} \tag{47}$$

Assertion (39) is deduced from the above lemma in the following way. For $n \in \mathbb{N}$ and $K \in$, there exists $i \in \{1, 2, 3\}$ such that $\overline{\mathbf{v}^n.\mathbf{n}_{K,i}} \geq 0$; we may therefore assume that, for instance, $\overline{\mathbf{v}^n.\mathbf{n}_{K,1}} \geq 0$, so that $b_{K,1}^n = 0$.

Assertion (39) is then a consequence of (46) and of the lemma, with : $a = u_K^n$, $b = \{u^n\}_{c_2(K)}$, $c = \{u^n\}_{c_3(K)}$, $U = \|u_0\|_\infty$, $\gamma = b_{K,2}^n$, $\delta = b_{K,3}^n$ (noting that, when $b_{K,i}^n \neq 0$, $\overline{\mathbf{v}^n.\mathbf{n}_{K,i}} < 0$). In order to complete the proof of Step 2, and thereby of Proposition 4, there remains to prove the lemma.

PROOF OF THE LEMMA : We first remark that (47) is equivalent to :

$$\int_a^{a+\gamma(f(b)-f(a))+\delta(f(c)-f(a))} \eta'(\xi)d\xi \leq \gamma \int_a^b f'(\xi)\eta'(\xi)d\xi + \delta \int_a^c f'(\xi)\eta'(\xi)d\xi. \tag{48}$$

For $t \in [0,1]$, we set :

$$\begin{aligned}\chi(t) = \eta(a + 2t\gamma(f(b) - f(a)) + 2(1-t)\delta(f(c) - f(a))) - \eta(a) \quad & -2t\gamma \int_a^b f'(\xi)\ \ \eta'(\xi)d\xi \\ & -2(1-t)\delta \int_a^c f'(\xi)\ \ \eta'(\xi)d\xi,\end{aligned}$$

so that (48) can be written $\chi(\frac{1}{2}) \leq 0$. Noting that $\chi'(t) = (2\gamma(f(b)-f(a))-2\delta(f(c)-f(a)))\,\eta'(a+2t\gamma(f(b)-f(a))+2(1-t)\delta(f(c)-f(a))) - D$, where D does not depend on t, and that η' is nondecreasing, we may assert that χ' is nondecreasing, i.e. χ is convex. Assertion (48) is therefore valid if $\chi(0) \leq 0$ and $\chi(1) \leq 0$. The proofs of these two inequalities are identical, and we shall prove the first one, that is :

$$\int_a^{a+2\delta(f(c)-f(a))} \eta'(\xi)d\xi \leq 2\delta \int_a^c f'(\xi)\eta'(\xi)d\xi. \tag{49}$$

Let $l = \dfrac{f(c)-f(a)}{M}$, and $g = M\ \ 1_{[a,a+l]}$, where $[a, a+l] = [a, a+l]$ if $l \geq 0$, and $[a, a+l] = [a+l, a]$ if $l < 0$. Note that $a+l$ lies between a and c. Now :

$$\int_a^c (f'(\xi) - g(\xi))\eta'(\xi)d\xi = \int_a^c (f'(\xi) - g(\xi))(\eta'(\xi) - \eta'(a+l))d\xi \geq 0,$$

so that :

$$\int_a^c f'(\xi)\ \ \eta'(\xi)d\xi \geq \int_a^c g(\xi))\ \ \eta'(\xi)d\xi = M \int_a^{a+l} \eta'(\xi)d\xi.$$

Since η' is nondecreasing, the application $l \to \dfrac{1}{l}\displaystyle\int_a^{a+l} \eta'(\xi)d\xi$ is also nondecreasing and, since $l = \dfrac{f(c)-f(a)}{M}$, we have $|l| \geq 2\delta|f(c) - f(a)|$ (because of the assumption $\delta \leq \dfrac{1}{2M}$) ; thus, we deduce that :

$$\int_a^c f'(\xi)\ \ \eta'(\xi)d\xi \geq \frac{1}{2\delta}\int_a^{a+2\delta(f(c)-f(a))} \eta'(\xi)d\xi.$$

The proofs of the lemma and of Proposition 4 are thereby complete.

REMARK 5. Proof of assertions (11) and (12) of Remark 1.
We show here that, under the assumptions of Proposition 1, the following inequality holds for any C^1 convex function η :

$$\sum_{K\in} S(K)\eta(u_K^{n+1}) \leq \sum_{K\in} S(K)\eta(u_K^n), \quad \forall \ \ n \in \mathbb{N}, \tag{50}$$

which yields (11) of Remark 1 by taking a family of C^1 convex functions $(\eta_\varepsilon)_{\varepsilon>0}$ converging uniformly on $\mathbb{R}$ towards the function $s \to |s|$ (recall that the number of nonzero terms in the sum over $K \in$ is always finite).
In order to show (50), using the notations of Proposition 4, we remark that :

$$\sum_{K\in} S(K)\Big(\eta(u_K^{n+1}) - \eta(u_K^n)\Big) = k\sum_{K\in} S(K)(E_K^n - H_K^n) - k\sum_{K\in} S(K)X_K^n.$$

In Step 2 of the proof of Proposition 4, we showed that $E_K^n \leq H_K^n$ under the assumptions of Proposition 1 (i.e. Assumption (8) was used, and not Assumption (5)). Since :

$$\sum_{K\in} S(K)X_K^n = \sum_{K\in}\sum_{i=1}^{3} l\big(c_i(K)\big)\overline{\mathbf{v}^n.\mathbf{n}_{K,i}}\Phi(\{u^n\}_{c_i(K)}) = 0,$$

Assertion (50) is proven.

6 Proof of Theorem 1

In this section, we generalise a uniqueness result of DiPerna (cf DiPerna (1985), see also Szepessy (1989)) to show that proposition 4 imply that $\mu_g = g(u)$ for all continuous function g from $\mathbb{R}$ to $\mathbb{R}$ (recall that μ_g is the limit of $g(u_{,k})$, in $L^\infty(\mathbb{R}^2\times\mathbb{R}_+)$ for the weak star topology, and $u=\mu_{Id}$), so that $u_{,k}$ tends to u in $L^p_{loc}(\mathbb{R}^2\times\mathbb{R}_+)$ for any finite p, and u is the unique entropy weak solution to (1).

THEOREM 2 *Let $r>0$, $\mathbf{v}$ be a function from $\mathbb{R}^2\times\mathbb{R}_+$ to $\mathbb{R}^2$, of class C^1, such that* $\mathrm{div}\mathbf{v}=0$ *and* $\sup_{(x,t)\in\mathbb{R}^2\times\mathbb{R}_+}|\mathbf{v}(x,t)| = V\in\mathbb{R}$, *$f$ a given nondecreasing function from $\mathbb{R}$ to $\mathbb{R}$, of class C^1, and u_0 a given bounded function. Let $(u^{(m)})_{m\in\mathbb{N}}$ be a sequence of bounded functions from $\mathbb{R}^2\times\mathbb{R}_+$ into $\mathbb{R}$ such that*

$$\|u^{(m)}\|_\infty \leq r,\ \forall\ \ m\in\mathbb{N}, \tag{51}$$

and such that, for any continuous function g from $\mathbb{R}$ to $\mathbb{R}$, $g(u^{(m)})$ converges to μ_g in $L^\infty(\mathbb{R}^2\times\mathbb{R}_+)$ for the weak star topology ; assume that for all C^1 convex function η from $\mathbb{R}$ to $\mathbb{R}$, and Φ such that $\Phi'=f'\eta'$,the following assumption holds :

$$\begin{aligned}&\int_{\mathbb{R}^2\times\mathbb{R}_+} \mu_\eta(x,t)\varphi_t(x,t)+\mu_\Phi(x,t)\mathbf{v}(x,t).\mathbf{grad}\varphi(x,t))dxdt + \int_{\mathbb{R}^2}\eta(u_0(x))\varphi(0,x)dx \geq 0,\\ &\forall\ \ \varphi\in C_c^1(\mathbb{R}^2\times[0,+\infty[,\mathbb{R}_+).\end{aligned} \tag{52}$$

Then $u^{(m)}$ tends to u in $L^p_{loc}(\mathbb{R}^2\times\mathbb{R}_+)$ for any finite p (and in $L^\infty(\mathbb{R}^2\times\mathbb{R}_+)$ for the weak star topology), as m tends to infinity and u is the entropy weak solution to :

$$\left\{\begin{array}{rll} u_t(x,t)+\mathrm{div}(\mathbf{v}(x,t)f(u(x,t))) & =0, & x\in\mathbb{R}^2, t\in\mathbb{R}_+,\\ u(x,0) & = u_0(x), & x\in\mathbb{R}^2.\end{array}\right. \tag{1}$$

PROOF OF THEOREM 2 : Throughout this proof, we denote by $\overline{u}$ the (unique) entropy weak solution to (1). Since $\|u^{(m)}\|_\infty\leq r$, and $g(u^{(m)})\to\mu_g$ in $L^\infty(\mathbb{R}^2\times\mathbb{R}_+)$ for the weak star topology, there exists, for all $y=(x,t)\in\mathbb{R}^2\times\mathbb{R}_+$ a probability ν_y on $\mathbb{R}$, supported inside $[-r,r]$, such that, for any continuous function g from $\mathbb{R}$ to $\mathbb{R}$, and for a.e. $y\in\mathbb{R}^2\times\mathbb{R}_+$, $\mu_g(y)=\int_{\mathbb{R}} g(\lambda)d\nu_y(\lambda)$. Remarking that bounded functions from $\mathbb{R}^2\times\mathbb{R}_+$ to $\mathbb{R}$ can be uniformly approached by linear combinations of characteristic functions of sets of $\mathbb{R}^2\times\mathbb{R}_+$, we may also assert that, for any $g\in C(\mathbb{R},\mathbb{R})$, and any $v\in L^\infty(\mathbb{R}^2\times\mathbb{R}_+)$, the sequence $(|g(u^{(m)})-v|)_m$ has a limit in $L^\infty(\mathbb{R}^2\times\mathbb{R}_+)$ for the weak star topology, denoted by $\mu_{|g-v|}$, which satisfies :

$$\mu_{|g-v|}(x,t)=\int_{\mathbb{R}}|g(\lambda)-v(x,t)|d\nu_{x,t}(\lambda),\ \text{for a.e. } (x,t)\in\mathbb{R}^2\times\mathbb{R}_+. \tag{53}$$

The application $\nu: y\mapsto\nu_y$ from $\mathbb{R}^2\times\mathbb{R}_+$ into the set of probabilities on $\mathbb{R}$ is an entropy measure valued solution to (1), i.e. satisfies (52) (see Gallouët, Herbin (accepted) ; note that DiPerna gave a definition of entropy measure valued solution in the case $\mathbf{v}$ constant, with a somewhat stronger formulation for the initial condition). We show below that this is sufficient to assert that $\nu_y=\delta_{\overline{u}(y)}$, for a.e. $y\in\mathbb{R}^2\times\mathbb{R}_+$, where $\overline{u}$ is the (unique) entropy weak solution to (1). We only give here a sketch of proof of this result (a detailed proof may be found in Gallouët, Herbin (accepted), this is essentially a generalisation of the uniqueness result of Di Perna, see Theorem 4.2 in

DiPerna (1985)). We show then that $u^{(m)} \to \overline{u}$ in $L^1_{loc}(\mathbb{R}^2 \times \mathbb{R}_+)$, as $m \to +\infty$. The proof of Theorem 2 will then be complete.

We decompose the proof into three steps. Let $a > 0$ and $\omega = VM_f$, with $M_f = \sup_{s \in [-b,b]} f'(s)$, where $b = \sup(r, U)$ and $U = \|u_0\|_\infty$. For $R > 0$, we set $B_R = \{x \in \mathbb{R}^2, |x| \leq R\}$. We define :

$$A(t) = \int_{B_{a-\omega t}} \mu_{|Id-\overline{u}|}(x,t)dx, \quad 0 < t < \frac{a}{\omega}.$$

In Step 1 below we prove that :

$$A(t_1) \leq A(t_2) \text{ for a.e. } t_1, t_2 \in [0, \frac{a}{\omega}], t_1 \geq t_2. \tag{54}$$

In Step 2 we prove that :

$$\lim_{T \to 0} \frac{1}{T} \int_0^T \int_{B_a} \mu_{|Id-u_0|}(x,t)dxdt = 0. \tag{55}$$

Finally, in Step 3, we deduce from (54) and (55) that $\mu_{|Id-\overline{u}|} = 0$ a.e. in $\mathbb{R}^2 \times \mathbb{R}_+$ and therefore that $u^{(m)} \to \overline{u}$ in $L^1_{loc}(\mathbb{R}^2 \times \mathbb{R}_+)$.

Step 1 (Proof of Assertion (54)*).*
In this step, we make use of the crucial following inequality, namely :

$$\Big(\int |\lambda - \overline{u}(x,t)| d\nu_{x,t}(\lambda)\Big)_t + \operatorname{div}\Big(\mathbf{v}(x,t) \int |f(\lambda) - f(\overline{u}(x,t))| d\nu_{x,t}(\lambda)\Big) \leq 0, \tag{56}$$

in the distribution sense in $\mathbb{R}^2 \times]0, +\infty[$. For constant $\mathbf{v}$, this assertion is proved in Theorem 4.1 of DiPerna (1985) (where (4.16) should be read instead of (4.17)) in the one dimensional case. In Remark 3 of DiPerna (1985), Theorem 4.1 is generalised to the multidimensional case which is of interest to us here. The proof may be found in Gallouët, Herbin (accepted) for the case where $\mathbf{v}$ depends on x and t. With (53), we can rewrite (56) as :

$$\begin{gathered}\int_{\mathbb{R}^2 \times \mathbb{R}_+} \mu_{|Id-\overline{u}|}(x,t)\varphi_t(x,t)dxdt + \int_{\mathbb{R}^2 \times \mathbb{R}_+} \mu_{|f-f(\overline{u})|}(x,t)\mathbf{v}(x,t).\mathbf{grad}\varphi(x,t)dxdt \geq 0, \\ \forall \ \varphi \in C^1_c(\mathbb{R}^2 \times]0, +\infty[, \mathbb{R}_+).\end{gathered} \tag{57}$$

Let $T = \frac{a}{\omega}$, $0 < t_1 < t_2 < T$, $0 < \varepsilon < \min(t_1, T - t_2)$, $\delta > 0$. Let $\psi \in C^1_c(\mathbb{R}_+, [0,1])$ with $\psi = 1$ on $[0, a]$, $\psi = 0$ on $[a + \delta, +\infty[$, and $\psi' \leq 0$. Let r_ε be defined by :

$$r_\varepsilon(t) = \begin{cases} 0, & \text{if } 0 \leq t \leq t_1 - \varepsilon, \\ \dfrac{t - (t_1 - \varepsilon)}{\varepsilon}, & \text{if } t_1 - \varepsilon \leq t \leq t_1, \\ 1, & \text{if } t_1 \leq t \leq t_2, \\ \dfrac{(t_2 + \varepsilon) - t}{\varepsilon}, & \text{if } t_2 \leq t \leq t_2 + \varepsilon, \\ 0, & \text{if } t_2 + \varepsilon \leq t < +\infty. \end{cases}$$

Taking $\varphi(x,t) = \psi(|x| + \omega t)\ r_\varepsilon(t)$ in (57) (this is possible by taking regularisations of the functions r_ε), we prove (see details in Gallouët, Herbin (accepted)) :

$$\frac{1}{\varepsilon} \int_{t_1-\varepsilon}^{t_1} A(t)dt - \frac{1}{\varepsilon} \int_{t_2}^{t_2+\varepsilon} A(t)dt \geq 0. \tag{58}$$

Since $A \in L^1(]0, T[)$ (in fact, $0 \leq A(t) \leq (r + \|u_0\|_\infty) meas(B_{a-\omega t})$), we deduce (54) from (58). (More precisely, if t_1 and t_2 are Lebesgue points of A, then : $0 < t_1 \leq t_2 < T \Rightarrow A(t_1) \geq A(t_2)$).

Step 2 (Proof of Assertion (55))

Let $\psi \in C_c^1(\mathbb{R}^2, \mathbb{R}_+)$ and $T > 0$. We define ρ by

$$\rho(t) = \begin{cases} \dfrac{T-t}{T} & \text{if } 0 \leq t \leq T \\ 0 & \text{if } t > T. \end{cases}$$

Let $(u_0^{(n)})_n \subset C^1(\mathbb{R}^2, \mathbb{R})$ be a sequence such that $\|u_0^{(n)}\|_\infty \leq \|u_0\|_\infty$, for every n, and $u_0^{(n)} \to u_0$ a.e. in $\mathbb{R}^2$ as $n \to +\infty$. Taking $\eta(s) = s$ in (52), (52) becomes an equality, which is valid for any $\varphi \in C_c^1(\mathbb{R}^2 \times [0, +\infty[, \mathbb{R})$. Then, taking regularisations of ρ, it is clear that we can take $\varphi(x,t) = \psi(x) u_0^{(n)}(x) \rho(t)$. This gives :

$$\begin{aligned} -\frac{1}{T} \int_0^T \int_{\mathbb{R}^2} \Big(\mu_{Id}(x,t) - u_0(x)\Big) \quad \psi(x) u_0^{(n)}(x) dx dt + \\ \int_0^T \int_{\mathbb{R}^2} \mu_f(x,t) \mathbf{v}(x,t) . \mathbf{grad}\Big(\psi(x) u_0^{(n)}(x)\Big) \rho(t) dx dt = 0. \end{aligned} \tag{59}$$

Letting n tend to infinity, we obtain (see details in Gallouët, Herbin (accepted)) :

$$\lim_{T \to 0} \frac{1}{T} \int_0^T \int_{\mathbb{R}^2} \Big(\mu_{Id}(x,t) - u_0(x)\Big) \psi(x) u_0(x) dx dt = 0. \tag{60}$$

Similarly, choosing $\varphi(x,t) = \psi(x)\rho(t)$ in (52) with $\eta(s) = s^2$, we obtain :

$$\limsup_{T \to 0} \frac{1}{T} \int_0^T \int_{\mathbb{R}^2} \Big(\mu_{|Id|^2}(x,t) - u_0(x)^2\Big) \psi(x) dx dt \leq 0. \tag{61}$$

Therefore, for all $\psi \in C_c^1(\mathbb{R}^2, \mathbb{R}_+)$, (60) and (61) imply :

$$\limsup_{T \to 0} \frac{1}{T} \int_0^T \int_{\mathbb{R}^2} \Big(\mu_{|Id|^2}(x,t) - 2\mu_{Id}(x,t) u_0(x) + u_0(x)^2\Big) \psi(x) dx dt \leq 0. \tag{62}$$

from which we deduce :

$$\lim_{T \to 0} \frac{1}{T} \int_0^T \int_{\mathbb{R}^2} \mu_{|Id - u_0|} \psi(x) dx dt = 0. \tag{63}$$

Assertion (63) yields (55) taking $\psi = 1$ on B_a.

Step 3 (conclusion of the proof of Theorem 2)

Using (55) and $\overline{u}(.,t) \to u_0$ in $L^1_{loc}(\mathbb{R}^2)$ as $t \to 0$, we deduce that $\dfrac{1}{T} \displaystyle\int_0^T A(t) dt \to 0$ as $t \to 0$. Using (54), we conclude that $A(t) = 0$ a.e. on]0,T[; since a is arbitrary, we conclude that $\mu_{|Id - \overline{u}|} = 0$ a.e. in $\mathbb{R}^2 \times \mathbb{R}_+$, so that $|u^{(m)} - \overline{u}| \to 0$ in $L^\infty(\mathbb{R}^2 \times \mathbb{R}_+)$ for the weak star topology ; therefore, $u^{(m)} \to \overline{u}$ in $L^1_{loc}(\mathbb{R}^2 \times \mathbb{R}_+)$, and in $L^p_{loc}(\mathbb{R}^2 \times \mathbb{R}_+)$ for any $p < +\infty$ (since $u^{(m)}$ is bounded in $L^\infty(\mathbb{R}^2 \times \mathbb{R}_+)$) as m tends to $+\infty$. The proof of Theorem 2 is thereby complete. Note that we have also proved that $\nu_{x,t} = \delta_{\overline{u}(x,t)}$ for a.e. $(x,t) \in \mathbb{R}^2 \times \mathbb{R}_+$.

Proof of theorem 1. Since the discretisations and k satisfy assumptions (4) and (5), Assertion (6) of Theorem 1 is proven by Proposition 1.
Let $\overline{u}$ be the entropy weak solution to (1) ; we prove Assertion (7) of Theorem 1 by contradiction : assume that there exist p_0, such that $1 \leq p_0 < +\infty$, $\varepsilon > 0$, K a compact subset of $\mathbb{R}^2 \times \mathbb{R}_+$, and a sequence $((_i, k_i))_{i \in \mathbb{N}}$ such that, for any $i \in \mathbb{N}$, $(_i, k_i)$ satisfies assumptions (4) and (5) (with h_i associated to $_i$ in (4), $h_i \to 0$, as $i \to \infty$), and

$$\int_K |u_{_i, k_i} - \overline{u}|^{p_0} dx dt \geq \varepsilon. \tag{64}$$

Using (6) of Theorem 1, we extract a subsequence of $((_i, k_i))_{i \in \mathbb{N}}$, which we still denote $((_i, k_i))_{i \in \mathbb{N}}$, such that, for any continuous function g from $\mathbb{R}$ to $\mathbb{R}$, $g(u_{_i, k_i}) \to \mu_g$ in $\mathrm{L}^\infty(\mathbb{R}^2 \times \mathbb{R}_+)$ for the weak star topology, as $i \to \infty$. By Proposition 1, $u_{_i, k_i}$ satisfies condition (51) with $r = \|u_0\|_\infty$, and, by proposition 4, μ_g satisfies (52) ; therefore, by Theorem 2, $u_{_i, k_i}$ tends to $\overline{u}$ in $\mathrm{L}^p_{loc}(\mathbb{R}^2 \times \mathbb{R}_+)$ for any finite p, which contradicts (64).

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