Existence and uniqueness of the entropy solution to a nonlinear hyperbolic equation

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Abstract
This work is concerned with the proof of the existence and uniqueness of the entropy weak solution to the following nonlinear hyperbolic equation: $u_t + div(vf(u)) = 0$ in $IR^N \times [0,T]$, with initial data $u(.,0) = u_0(.)$ in IR^N. where $u_0 \in L^\infty(IR^N)$ is a given function, v is a divergence-free bounded function of class C^1 from $IR^N \times [0,T]$ to IR^N, and f is a function of class C^1 from IR to IR. It also gives a result of convergence of a numerical scheme for the discretization of theis equation. We first show the existence of a "process" solution (which generalizes the concept of entropy weak solutions, and can be obtained by passing to the limit of solutions of the numerical scheme). The uniqueness of this entropy process solution is then proved; it is also proven that the entropy process solution is in fact a entropy weak solution, hence the existence and uniqueness of the entropy weak solution, and the convergence of the numerical scheme.

Keywords: Nonlinear hyperbolic equation, process solution, existence and uniqueness, convergence of finite volume schemes;
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1 Introduction.

The present work is concerned with the existence of an entropy weak solution $u \in L^\infty(IR^N \times]0,T[)$ to the following nonlinear hyperbolic equation with initial data:

$$u_t(x,t) + div(vf(u(x,t))) = 0, \quad x \in IR^N, \quad t \in [0,T] \tag{1}$$

$$u(x,0) = u_0(x), \quad x \in IR^N. \tag{2}$$

where $T > 0$, u_t denotes the partial derivative of u with respect to time variable t, div denotes the divergence operator with respect to the space variable $x = (x_1, \ldots, x_N)$, and under the following assumptions on the data T, v, f, u_0:

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$$v = (v_1, \ldots, v_N) \in C^1(IR^N \times [0,T], IR^N), \tag{3}$$

$$\sup_{(x,t)\in IR^N\times[0,T]} |v(x,t)| = V < +\infty, \tag{4}$$

$$divv(x,t) = \sum_{i=1}^{N} \partial_{x_i} v_i(x,t) = 0, \text{ for all } (x,t) \in IR^N \times [0,T], \tag{5}$$

$$f \in C^1(IR, IR), u_0 \in L^\infty(IR^N). \tag{6}$$

Remark 1 *Note that assumption (4) is crucial in the sequel: it ensures the property of "propagation in finite time" which is needed for the uniqueness of the solution. The assumption (5), on the other hand, is only considered for the sake of simplicity. The result of existence and uniqueness presented below is easily extendable to the case* $divv \neq 0$*.*

A function $u \in L^\infty(IR^N \times]0,T[)$ is said to be an entropy weak solution to Problem (1)-(2) if it satisfies:

$$\begin{aligned} &\int_{IR^N} \int_0^T \Big(\eta(u(x,t)) \varphi_t(x,t) + \Phi(u(x,t)) v(x,t).grad\varphi(x,t) \Big) dt dx \;\; + \\ &\int_{IR^N} \eta(u_0(x)) \varphi(x,0) dx \geq 0, \\ &\forall \;\; \varphi \in C_c^1(IR^N \times [0,T[, IR_+), \end{aligned} \tag{7}$$

for any convex function $\eta \in C^1(IR, IR)$, and $\Phi \in C^1(IR, IR)$ such that $\Phi' = f'\eta'$ (where $C_c^1(E,F)$ denotes the set of functions C^1 from E to F, with compact support in E).

Note that existence and uniqueness results are already known for the entropy weak solution of problem (1), under different assumptions than assumptions (3)-(6) (see e.g. [11], [15]). In particular, these results were obtained with a nonlinearity F (in our case $F = vf$) of class C^3. The methods which were used in [11] require a regularization of the function u_0, in order to take advantadge of compactness properties in spaces smaller than $L^\infty(IR^N \times]0,T[)$ (the "BV space" of functions having locally bounded variation in the sense of Tonelli-Cesaro; indeed bounded sets of $L^\infty \cap BV$ are compact in L^1_{loc}).

The existence of solutions to problems similar to (1)-(2) was already proven by passing to the limit on solutions of an appropriate numerical scheme. This is done in the work of [12] in the case of a single space variable, which was generalized to several space dimensions [4]. The work of [4] uses a finite difference scheme on a uniform rectangular grid, and requires that the initial condition u_0 (an thus, the solution to Problem (1)-(2)) have locally bounded variation in the sense of Tonelli-Cesaro. Here we only assume $u_0 \in L^\infty(IR^N \times]0,T[)$ and we may also work with more general meshes, for instance triangular mesh in the case $N = 2$. For each of these reasons, the BV framework may not be used. The lack of compactness forces us to work with the weak star convergence in L^∞ of a family of approximate solutions. Passing to the limit with the solutions given by a finite volume scheme gives the existence of a so-called "process solution" (which is defined below) to Problem (1)-(2). For an introduction to finite volume schemes, see e.g. [10], [7] and [1], [2], [14] for convergence results and error estimates.

Uniqueness results have recently been proven [5], [13], [9]. The proofs of these results rely on one hand on the concept of measure valued solutions and on the other hand on the existence of an entropy weak solution. The direct proof of the uniqueness of a measure valued solution (i.e. without assuming any existence result of entropy weak solutions) leads to a difficult problem involving the application of the theorem of continuity in means. This difficulty is overcome by the introduction of this new concept of solution (namely entropy process solution), which generalizes the concept of measure valued-solution. The proof of the uniqueness of the entropy process solution which is given here strongly uses the properties related to the weak star convergence in the space $L^\infty(IR^N\times]0,T[)$.

Let us now define the **entropy process solutions**:

Definition 1 *A function $\mu \in L^\infty(IR^N\times]0,T[\times]0,1[,IR)$ is an entropy process solution of problem (1) if μ satisfies:*

$$\begin{aligned}&\int_{IR^N}\int_0^T\int_0^1\Big(\eta(\mu(x,t,\alpha))\varphi_t(x,t)+\Phi(\mu(x,t,\alpha))v(x,t).grad\varphi(x,t)\Big)d\alpha dt dx \quad +\\&\int_{IR^N}\eta(u_0(x))\varphi(x,0)dx\geq 0,\\&\forall\quad \varphi\in C_c^1(IR^N\times[0,T[,IR_+),\end{aligned}\qquad(8)$$

for any convex function $\eta\in C^1(IR,IR)$, and $\Phi\in C^1(IR,IR)$ a function such that $\Phi' = f'\eta'$.

Remark 2 *From an entropy weak solution $u(x,t)$ to problem (1)-(2), one may easily construct an entropy process solution to problem (1)-(2) by setting $\mu(x,t,\alpha)=u(x,t)$ for any $a\in[0,1]$. Reciprocally, if μ is an entropy process solution to problem (1)-(2) such that there exists $u\in L^\infty(IR^N\times]0,T[)$ such that $\mu(x,t,\alpha)=u(x,t)$, for a.e. $(x,t,\alpha)\in IR^N\times]0,T[\times]0,1[$, then u is an entropy weak solution to problem (1)-(2).*

The name "entropy process solution" was derived from the notion of bounded measurable process, that is a measurable mapping from a probability space into a space of bounded measurable functions.
Here, the probability space consists in the interval $]0,1[$, with the borelian σ-algebra and the Lebesgue measure, and the set of bounded measurable functions is the bounded subset of $L^\infty(IR^N\times]0,T[)$ defined by $\{\mu(.,.,\alpha),\alpha\in]0,1[;\|\mu(.,.,\alpha)\|_\infty\leq C\}$, where $C>0$ is independent of α.

The first aim of this work is to prove the following result of existence and uniqueness of the entropy process solution to problem (1)-(2). We then also obtain existence and uniqueness of the entropy weak solution and also L^p_{loc} strong convergence for any finite p of the finite volume scheme.

Theorem 1 *Under the assumptions (3)-(6), there exists a unique entropy process solution μ of problem (1)-(2), as defined by relation (8). Moreover, there exists a function $u\in L^\infty(IR^N\times]0,T[)$ such that $u(x,t)=\mu(x,t,\alpha)$, for almost any $(x,t,\alpha)\in IR^N\times]0,T[\times]0,1[$. The function u is thus the unique entropy weak solution to Problem* (1).

The existence of an entropy process solution will be proven by the study of numerical schemes of finite volume type in section 2. The uniqueness of such a solution is proven in section 3, and a by-product of this proof is that the values of the entropy process solution $\mu(x,t,\alpha)$ do not depend on a α, so that finally the entropy process solution is therefore an entropy weak solution to problem (1)-(2).

Let us conclude this introduction by a characterization of an entropy process solution (which is an adaptation of the wellknown Kruzkov entropies to process solutions):

Proposition 1 *A function $\mu \in L^\infty(I\!R^N \times]0,T[\times]0,1[, I\!R)$ is an entropy process solution of problem (1) if and only if for any $k \in I\!R$, one has :*

$$\int_{I\!R^N} \int_0^T \int_0^1 \Big(|\mu(x,t,\alpha) - k|\varphi_t(x,t) + F(\mu(x,t,\alpha),k)v(x,t).grad\varphi(x,t)\Big)d\alpha dt dx \ + \int_{I\!R^N} |u_0(x) - k|\varphi(x,0)dx \geq 0 \quad \forall \quad \varphi \in C_c^1(I\!R^N \times [0,T[, I\!R_+), \forall k,l \in I\!R \tag{9}$$

where $F(k,l) = sign(l-k)(f(l) - f(k))$.

This is a wellknown result for the classical entropy weak solutions. The characterization (9) can be obtained from (8), by using regularizations of the function $|.-k|$. Reciprocally, (8) may be obtained from (9) by approximating any convex function $\eta \in C^1(I\!R, I\!R)$ by functions of the form: $\eta_n(.) = \sum_{i=1}^{n} \alpha_i^{(n)} |. - k_i^{(n)}|$, with $\alpha_i^{(n)} \geq 0$.

This characterization will be essential for the proof of the uniqueness of the entropy process solution of problem (1).

2 Existence of an entropy process solution.

This section is devoted to the proof of the existence of an entropy process solution of problem (1), i.e. of a function $\mu \in L^\infty(I\!R^N \times]0,T[\times]0,1[, I\!R)$ satisfying (8). In order to construct such a solution, we use a property of probability measures and a passage to the limit on the solutions given by a numerical scheme: indeed, passing to the limit for approximate solutions which are obtained by a numerical scheme yields the existence of measure valued solutions. This method is described in recent articles [1], [2], [14]. For the problem obtained here, one may for instance adapt the proof given in [1], using a decomposition $f(u) = f_1(u) + f_2(u)$, with $f_1'(u) \geq 0$ and $f_2'(u) \leq 0$. We omit the proof here (for a detailed proof, one should notice that the key estimate in [1] which is called "weak BV estimate" can be obtained here by multiplying the equation by u). One could also use the methods of [2] or [14]. Using a property of probability measures, we shall deduce below the existence of an entropy process solution of problem (1).

2.1 A property of probability measures.

Let m be a probability measure on $I\!R$ and define, for any function $g \in C_b(I\!R, I\!R)$: $m(g) = \int_{I\!R} g(k)dm(k)$, (where $C_b(I\!R, I\!R)$ is the space of bounded continuous functions from $I\!R$ to $I\!R$). Let $F_m : I\!R \to [0,1]$ be the repartition function of the measure m, defined for any $x \in I\!R$ by:

$$F_m(x) = \sup\{m(g),\ g \in C_b(I\!R, I\!R),\ g \le 1_{]-\infty,x[}\}. \tag{10}$$

(Recall that $1_{]-\infty,x[}(t) = 1$ if $t \in]-\infty, x[$, $1_{]-\infty,x[}(t) = 0$ otherwise.)
Let $M_m :]0,1[\to I\!R$ be the function defined by

$$M_m(\alpha) = \inf\{x \in I\!R,\ F_m(x) > \alpha\},\ \text{for any } \alpha \in]0,1[\tag{11}$$

We may then state the following result:

Proposition 2 *Let m be a probability measure on $I\!R$ and M_m defined by (10) and (11), one has:*

$$m(g) = \int_0^1 g(M_m(\alpha))d\alpha, \quad \forall g \in C_b(I\!R, I\!R). \tag{12}$$

PROOF
Since the function F_m is non-decreasing and left-continuous, it is easily seen that:

$$\sup\{\alpha \in]0,1[, M_m(\alpha) < x\} = F_m(x),\ \text{for any } x \in I\!R \text{ such that } F_m(x) > 0. \tag{13}$$

Hence the function M_m is non-decreasing, right-continuous, and it is the reciprocal of the function F_m if it is continuous. Hence:

$$\int_0^1 1_{]-\infty,x[}(M_m(\alpha))d\alpha = F_m(x), \tag{14}$$

for any $x \in I\!R$ such that $F_m(x) > 0$. Therefore, the repartition function of the measure defined by $g \to \int_0^1 g(M_m(\alpha))d\alpha$ is also the function F_m. Hence, the measure $g \to \int_0^1 g(M_m(\alpha))d\alpha$ is identical to the measure m, which ends the proof of relation (12).

2.2 A property of bounded sequences of $L^\infty(I\!R^N \times]0,T[)$.

Proposition 3 *Let $(u_n)_{n\in I\!N}$ be a bounded sequence of $L^\infty(I\!R^N \times]0,T[)$. There exists a subsequence of $(u_n)_{n\in I\!N}$ (which will still be denoted by $(u_n)_{n\in I\!N}$) such that, for any function $g \in C(I\!R, I\!R)$, the sequence $(g(u_n))_{n\in I\!N}$ converges in $L^\infty(I\!R^N \times]0,T[)$ for the weak star topology towards a function $U_g \in L^\infty(I\!R^N \times]0,T[)$.*
Furthermore, there exists $\mu \in L^\infty(I\!R^N \times]0,T[\times]0,1[)$ such that for any function $g \in C(I\!R, I\!R)$, $\int_{]0,1[} g(\mu(x,t,\alpha))d\alpha = U_g(x,t)$, for almost any $(x,t) \in I\!R^N \times]0,T[$, which is equivalent to :

$$\lim_{n\to\infty}\int_{I\!R^N\times]0,T[} g(u_n(x,t))\varphi(x,t)dxdt = \int_{I\!R^N\times]0,T[\times]0,1[} g(\mu(x,t,\alpha))\varphi(x,t)dxdtd\alpha, \quad (15)$$

for any function $\varphi \in L^1(I\!R^N\times]0,T[)$.

PROOF
Let $(u_n)_{n\in I\!N}$ be a bounded sequence of $L^\infty(I\!R^N\times]0,T[)$ and $r \geq 0$ such that $\|u_n\|_\infty \leq r, \forall n \in I\!N$.
Step 1 Thanks to the separability of the set of continuous functions defined from $[-r,r]$ into $I\!R$ (endowed with the uniform norm) and the sequential weak star relative compactness of the bounded sets of $L^\infty(I\!R^N\times]0,T[)$, there exists (using a diagonal process) a subsequence (which will still be denoted by $(u_n)_{n\in I\!N}$) such that, for any function $g \in C(I\!R, I\!R)$, the sequence $(g(u_n))_{n\in I\!N}$ converges in $L^\infty(I\!R^N\times]0,T[)$ for the weak star topology towards a function $U_g \in L^\infty(I\!R^N\times]0,T[)$.
Step 2 In this step, we prove the existence of a family of probabilities $(m_{(x,t)})_{(x,t)\in I\!R^N\times]0,T[}$ defined on $I\!R$ with support in $[-r,r]$, such that for any $g \in C(I\!R, I\!R)$, $m_{(x,t)}(g) = \int g dm_{(x,t)} = U_g(x,t)$ for a.e. $(x,t) \in I\!R^N\times]0,T[$. This is a classical result on Young measures and we only sketch the proof for the sake of completeness. Let $y = (x,t) \in I\!R^N\times]0,T[$ and $F_y = \{g \in C([-r,r], I\!R); \lim_{h\to 0} ⨍_{B(y,h)} U_g(z)dz$ exists in $I\!R\}$. If $g \in F_y$, we set $\overline{U}_g(y) = \lim_{h\to 0} ⨍_{B(y,h)} U_g(z)dz$. We define T_y from F_y in $I\!R$ by $T_y(g) = \overline{U}_g(y)$. It is easily seen that F_y is a vector space which contains the constants and T_y is a linear positive form over F_y. Hence using a modified version of Hahn-Banach's theorem, one can prolonge T_y into a positive linear form $\overline{T}_y$ defined over $C([-r,r], I\!R)$. By Riesz' theorem, there exists a (positive) measure ν_y on the borelian sets of $[-r,r]$ such that:

$$\overline{T}_y(g) = \overline{U}_g(y) = \int_{-r}^{r} g d\nu_y, \forall g \in C([-r,r], I\!R). \quad (16)$$

The function $g \equiv 1$ is in F_y, and for $g \equiv 1$, $\overline{U}_g(1) = 1$. Hence, from (16) ν_y is a probability over $[-r,r]$, and therefore a probability over $I\!R$ by prolonging it by 0 outside of $[-r,r]$. In order to complete this step, one should remark that if $g \in C(I\!R, I\!R)$, then for a.e. $y \in I\!R^N\times]0,T[$, $U_g(y) = \overline{U}_g(y)$.
Step 3 ¿From Relation (12), one has:

$$m_{(x,t)}(g) = \int_0^1 g(M_{m_{(x,t)}}(\alpha))d\alpha, \forall g \in C(I\!R, I\!R),$$

Defining μ by $\mu(x,t,\alpha) = M_{m_{(x,t)}}(\alpha)$ for $(x,t,\alpha) \in I\!R_N \times [0,T] \times [0,1]$, we obtain $\mu \in L^\infty(I\!R^N\times]0,T[\times]0,1[)$ and, for any function $g \in C(I\!R, I\!R)$, $\int_{]0,1[} g(\mu(x,t,\alpha))d\alpha = U_g(x,t)$, for almost any $(x,t) \in I\!R^N\times]0,T[$.

2.3 Existence of an entropy process solution.

We now prove the following existence result:

Theorem 2 *Under assumptions (3)-(6),there exists an entropy process solution to problem* $(1)-(2)$.

Proof Under assumptions (3)-(6), the results presented in [2,3,4,...] allow, by means of finite volume schemes, the construction of a sequence $(u_n)_{n\in IN} \subset L^\infty(IR^N\times]0,T[)$ such that :
- there exists $r>0$, such that:

$$\|u_n\|_\infty \le r, \forall \;\; n \in IN\;;$$

- for any function $g \in C(IR, IR)$, the sequence $(g(u_n))_{n\in IN}$ converges in $L^\infty(IR^N\times]0,T[)$ for the weak star topology towards a function $U_g \in L^\infty(IR^N\times]0,T[)$;
- for any convex function η of class C^1 from IR to IR, and Φ such that $\Phi' = f'\eta'$, and for any function $\varphi \in C^1_c(IR^N \times [0,T[, IR_+)$, one has:

$$\begin{array}{l}\displaystyle\int_{IR^N}\int_0^T \Big(U_\eta(x,t)\varphi_t(x,t) + U_\Phi(x,t)v(x,t).grad\varphi(x,t)\Big)dtdx \;\; + \\ \displaystyle\int_{IR^N}\eta(u_0(x))\varphi(x,0)dx \ge 0.\end{array} \tag{17}$$

Hence, from the results of the above section, a function $\mu \in L^\infty(IR^N\times]0,T[\times]0,1[)$ satisfying (15) may be constructed. ¿From (15) and (17) one can deduce:

$$\begin{array}{l}\displaystyle\int_{IR^N}\int_0^T\int_0^1 \Big(\eta(\mu(x,t,\alpha))\varphi_t(x,t) + \Phi(\mu(x,t,\alpha))v(x,t).grad\varphi(x,t)\Big)dtdxd\alpha \;\; + \\ \displaystyle\int_{IR^N}\eta(u_0(x))\varphi(x,0)dx \ge 0,\end{array} \tag{18}$$

for any convex function η of class C^1 from IR to IR, and Φ such that $\Phi' = f'\eta'$, and for any function $\varphi \in C^1_c(IR^N \times [0,+\infty[, IR_+)$. The function μ is therefore an entropy process solution, in the sense of relations (8) et (9).

3 Uniqueness of the entropy process solution.

In this section, the uniqueness of the entropy process solution to problem (1)-(2) is proven, under assumptions (H1). Here we follow the method which was introduced in [1]. Note, however, that in [1], the existence of an entropy weak solution was assumed; hence stronger assumptions were needed in order to use previous results of existence of an entropy weak solution, such as [11].

Let us now give the uniqueness result:

Theorem 3 *Under the assumptions (3)-(6), the entropy process solution μ of problem (1)-(2), as defined by relation (8), is unique. Moreover, there exists a function $u \in L^\infty(IR^N\times]0,T[)$ such that $u(x,t) = \mu(x,t,\alpha)$, for almost any $(x,t,\alpha)\in IR^N\times]0,T[\times]0,1[$. Hence, with Theorem 2 and Remark 2, there exists a unique entropy weak solution to Problem $(1)-(2)$.*

Proof.

The proof can be decomposed into four steps. Denote by μ and ν two entropy process solutions to Problem (1)-(2), and r_μ, r_ν their respective $L^\infty(IR^N\times]0,T[\times]0,1[)$ norms.
In step 1, the initial condition is proven to be satisfied in the following sense:

$$\lim_{\tau\to 0} \frac{1}{\tau}\int_0^\tau \int_{B_{N,a}} \int_0^1 |\mu(x,t,\alpha) - u_0(x)| d\alpha dx dt = 0,\ \forall a > 0, \tag{19}$$

where $B_{p,r} = \{x \in IR^p,\ |x| \le r\}$ for any $r > 0$ and any $p \in IN$.
The same property is clearly also verified if μ is replaced with ν.
In step 2, it is shown that:

$$\begin{aligned}&\int_0^T \int_{IR^N} \int_0^1 \int_0^1 \Big(|\mu(x,t,\alpha) - \nu(x,t,\beta)|\varphi_t(x,t) + \\ &F(\mu(x,t,\alpha),\nu(x,t,\beta))v(x,t).grad\varphi(x,t)\ \Big) d\alpha d\beta dx dt \ge 0 \\ &\forall\ \ \varphi \in C_c^1(IR^N\times]0,T[, IR_+).\end{aligned} \tag{20}$$

Then, for a given $a > 0$, define $A(t)$ for $0 < t < \inf(T, \frac{a}{\omega})$ by:

$$A(t) = \int_{B_{N,a-\omega t}} \int_0^1 \int_0^1 |\mu(x,t,\alpha) - \nu(x,t,\beta)| d\alpha d\beta\ \ dx. \tag{21}$$

where $\omega = VM_f$, avec $M_f = \sup_{s\in[-b,b]} |f'(s)|$ and $b = \sup(r_\nu, r_\mu)$,
It is shown in step 3 that A is a.e. non-increasing, i.e.:

$$A(t_1) \le A(t_2) \text{ for a.e. } t_1, t_2 \in [0, \inf(T, \frac{a}{\omega})], t_1 \ge t_2. \tag{22}$$

In step 4, it is deduced from (22) and (19), that $\int_0^1 \int_0^1 |\mu(x,t,\alpha) - \nu(x,t,\beta)| d\alpha d\beta = 0$, for a.e. $(x,t) \in IR^N\times]0,T[$, and we prove that one may define $u(x,t)$, such that $\mu(x,t,\alpha) = \nu(x,t,\beta) = u(x,t)$ for a.e. $(x,t) \in IR^N\times]0,T[$ and$(\alpha,\beta) \in]0,1[\times]0,1[$.

Step 1 : proof of relation (19).

In order to prove relation (19), a sequence of mollifiers is now introduced for the space dimension. This technique will also be used for step 2 below.

For $p \in IN$, define $\rho_p \in C_c^\infty(IR^p, IR)$ satisfying the following properties:

$$\mathrm{supp}(\rho_p) = \{x \in I\!R^p, \rho_p(x) \neq 0\} \subset B_{p,1} = \{x \in I\!R^p; |x| \leq 1\}; \tag{23}$$
$$\rho_p \geq 0; \tag{24}$$
$$\rho_p(-x) = \rho_p(x) \forall x \in B_{p,1}; \tag{25}$$
$$\int_{B_{p,1}} \rho_p(x)dx = 1. \tag{26}$$

For $n \in I\!N, n \geq 1$ define $\rho_{p,n} = n^p \rho_p(nx)$.In the present step, the value of p will be $p = N$. In the following step,the values $p = 1$ and $p = N$ will be used.

let $\tau \in I\!R$ such that $0 < \tau < T$ and ρ be defined by:

$$\rho(t) = \begin{cases} \dfrac{\tau - t}{\tau} & \text{if } 0 \leq t \leq \tau \\ 0 & \text{if } t > \tau. \end{cases} \tag{27}$$

Let $a > 0$ and $\psi \in C_c^\infty(I\!R^N, I\!R_+)$ such that $\psi(x) = 1 \; \forall x \in B_{N,a}$, and let $y \in I\!R^N$. Take $\varphi(x,t) = \psi(x)\rho_{N,n}(x-y)\rho(t)$ (this is possible by means of regularaizations of the function ρ) and $k = u_0(y)$ in (9); integrating the resulting relation over $I\!R^N$ with respect to y yields the following relation:

$$T_{1n\tau} + T_{2n\tau} + T_{3n} \geq 0, \tag{28}$$

with:

$$T_{1n\tau} = -\frac{1}{\tau}\int_0^\tau \int_{I\!R^N} \int_{I\!R^N} \int_0^1 |\mu(x,t,\alpha) - u_0(y)|\psi(x)\rho_{N,n}(x-y)d\alpha dx dy dt, \tag{29}$$

$$T_{2n\tau} = \int_0^\tau \int_{I\!R^N} \int_{I\!R^N} \int_0^1 F(\mu(x,t,\alpha), u_0(y))\rho(t) \; v(x,t).grad\Big(\psi(x)\rho_{N,n}(x-y)\Big)d\alpha dx dy dt, \tag{30}$$

and

$$T_{3n} = \int_{I\!R^N} \int_{I\!R^N} |u_0(x) - u_0(y)|\psi(x)\rho_{N,n}(x-y)dx dy. \tag{31}$$

Using the change of variables: $x = x'$, $y = x' - \frac{y'}{n}$ in (29), and denoting again by (x,y) the new variables (x', y') yields:

$$T_{1n\tau} = -\frac{1}{\tau}\int_0^\tau \int_{B_{N,1}} \int_{I\!R^N} \int_0^1 |\mu(x,t,\alpha) - u_0(x - \frac{y}{n})|\psi(x)\rho_{N,1}(y)d\alpha dx dy dt, \tag{32}$$

and therefore, denoting by K_ψ the support of ψ:

$$\begin{aligned} &|T_{1n\tau} + \frac{1}{\tau}\int_0^\tau \int_{I\!R^N} \int_0^1 |\mu(x,t,\alpha) - u_0(x)|\psi(x)d\alpha dx dt| \leq \\ &\int_{B_{N,1}} \|u_0(. - \frac{y}{n}) - u_0(.)\|_{L^1(K_\psi)} \;\; \|\psi\|_\infty \;\; \rho_{N,1}(y)dy. \end{aligned} \tag{33}$$

The same upper bound, independent of τ, also applies to T_{3n}.
Let $\varepsilon > 0$; by the theorem of continuity in mean, there exists n_0 such that $\int_{B_{N,1}} \|u_0(.-\frac{y}{n_0}) -u_0(.)\|_{L^1(K_\psi)} \, \|\psi\|_\infty \; \rho_{N,1}(y)dy \leq \varepsilon/3$. Since the integrand of the right-hand-side of (30) is bounded (for fixed $n = n_0$), one may choose $\tau_0 > 0$, such that for any $\tau \leq \tau_0$, one has $|T_2(\tau, n_0)| \leq \varepsilon/3$.

Hence, from relations (28) et (33) :

$$0 \leq \frac{1}{\tau}\int_0^\tau \int_{I\!R^N}\int_0^1 |\mu(x,t,\alpha) - u_0(x)|\psi(x)d\alpha dx dt \leq \varepsilon, \; \forall \tau \leq \tau_0.$$

Since $\psi(x) = 1 \; \forall x \in B_{N,a}$, the relation (19) is proven.

Step 2: Proof of relation (20).

Taking regularizations of φ, it is sufficient to prove (20) for $\varphi \in C_c^\infty(I\!R^N \times]0,T[, I\!R_+)$. The sequence of mollifiers $(\rho_{p,n})_{n\in I\!N}$ which was introduced in Step 1 will be now used with $p = 1$ and $p = N$.

Let $\psi \in C_c^\infty(I\!R^N \times]0,T[, I\!R_+)$ and $c > 0$, such that for any $t \in]0,c[\cup]T-c,T[$, and for any $x \in I\!R^N$, one has: $\psi(x,t) = 0$; Let $n \in I\!N$ such that $1/n < c$ (n is chosen such that the support of the test functions does not require the use of the initial condition).

Define, for $(x,t) \in I\!R^N \times]0,T[$ and $(y,s) \in I\!R^N \times]0,T[$,

$$\varphi(x,t,y,s) = \psi(\frac{x+y}{2}, \frac{t+s}{2})\rho_{N,n}(x-y)\rho_{1,n}(t-s). \tag{34}$$

The function φ hence defined satisfies $\varphi(.,.,y,s) \in C_c^\infty(I\!R^N \times]0,T[, I\!R_+)$ and $\varphi(x,t,.,.) \in C_c^\infty(I\!R^N \times]0,T[, I\!R_+)$.

Let ν be an entropy process solution ; hence ν satisfies:

$$\begin{aligned} &\int_{I\!R^N}\int_0^T\int_0^1 \Big(|\nu(x,t,\alpha) - k|\varphi_t(x,t) + F(\nu(x,t,\alpha),k)v(x,t).grad\varphi(x,t)\Big)d\alpha dt dx \\ &+\int_{I\!R^N} |u_0(x) - k|\varphi(x,0)dx \geq 0 \;\; \forall \;\; \varphi \in C_c^1(I\!R^N \times [0,T[, I\!R_+), \end{aligned} \tag{35}$$

Let $(y,s) \in I\!R^N \times]0,T[$and $\beta \in]0,1[$. Taking $\varphi = \varphi(.,.,y,s)$ where $\varphi(x,t,y,s)$ is defined in (34) and $k = \mu(y,s,\beta)$ in (35) and integrating over $\mathcal{E}_3 = I\!R^N \times]0,T[\times]0,1[$ for the Lebesgue measure $dy\; ds\; d\beta$ (note that the integrand is integrable for this measure) yields:

$$\begin{aligned} \int_{\mathcal{E}_3\times\mathcal{E}_3} \;\; &\Big[|\nu(x,t,\alpha) - \mu(y,s,\beta)|\varphi_t(x,t,y,s) + \\ &F(\nu(x,t,\alpha),\mu(y,s,\beta))v(x,t).grad_x\varphi(x,t,y,s)\Big]dx dt d\alpha dy ds d\beta \geq 0. \end{aligned} \tag{36}$$

One may then swap μ with ν and (x,t) with (y,s) in (36), hence:

$$\int_{\mathcal{E}_3\times\mathcal{E}_3} \Big[|\nu(x,t,\alpha)-\mu(y,s,\beta)|\varphi_s(x,t,y,s)+ F(\nu(x,t,\alpha),\mu(y,s,\beta))v(y,s).grad_y\varphi(x,t,y,s)\Big]dxdtd\alpha dydsd\beta \geq 0. \tag{37}$$

Let us now compute the various derivatives of the function φ. For any $(x,t)\in IR^N\times]0,T[$ and $(y,s)\in IR^N\times]0,T[$, one has:

$$\varphi_t(x,t,y,s) = \rho_{N,n}(x-y) \Big(\frac{1}{2}\psi_t(\frac{x+y}{2},\frac{t+s}{2})\rho_{1,n}(t-s)+\psi(\frac{x+y}{2},\frac{t+s}{2})\rho'_{1,n}(t-s)\Big), \tag{38}$$

$$\varphi_s(x,t,y,s) = \rho_{N,n}(x-y) \Big(\frac{1}{2}\psi_t(\frac{x+y}{2},\frac{t+s}{2})\rho_{1,n}(t-s)-\psi(\frac{x+y}{2},\frac{t+s}{2})\rho'_{1,n}(t-s)\Big), \tag{39}$$

$$grad_x\varphi(x,t,y,s) = \rho_{1,n}(t-s) \Big(\frac{1}{2}grad\psi(\frac{x+y}{2},\frac{t+s}{2})\rho_{N,n}(x-y)+ \psi(\frac{x+y}{2},\frac{t+s}{2})grad\rho_{N,n}(x-y)\Big), \tag{40}$$

and

$$grad_y\varphi(x,t,y,s) = \rho_{1,n}(t-s) \Big(\frac{1}{2}grad\psi(\frac{x+y}{2},\frac{t+s}{2})\rho_{N,n}(x-y)- \psi(\frac{x+y}{2},\frac{t+s}{2})grad\rho_{N,n}(x-y)\Big). \tag{41}$$

Using these relations and summing (36)and (37):

$$X_{1n}+X_{2n}+X_{3n}\geq 0, \tag{42}$$

with:

$$X_{1n} = \int_{\mathcal{E}_3\times\mathcal{E}_3} |\nu(x,t,\alpha)-\mu(y,s,\beta)| \;\psi_t(\frac{x+y}{2},\frac{t+s}{2}) \rho_{N,n}(x-y)\rho_{1,n}(t-s)dxdtd\alpha dydsd\beta, \tag{43}$$

$$X_{2n} = \int_{\mathcal{E}_3\times\mathcal{E}_3} F(\nu(x,t,\alpha),\mu(y,s,\beta))\;\tfrac{1}{2}\Big[v(x,t)+v(y,s)\Big]. grad\psi(\frac{x+y}{2},\frac{t+s}{2})\rho_{N,n}(x-y)\rho_{1,n}(t-s)dxdtd\alpha dydsd\beta, \tag{44}$$

$$X_{3n} = \int_{\mathcal{E}_3\times\mathcal{E}_3} F(\nu(x,t,\alpha),\mu(y,s,\beta))\;\Big[v(x,t)-v(y,s)\Big]. \psi(\frac{x+y}{2},\frac{t+s}{2})grad\rho_{N,n}(x-y)\rho_{1,n}(t-s)dxdtd\alpha dydsd\beta. \tag{45}$$

Perform in (43)-(45) the change of variables $x' = \frac{1}{2}(x+y)$, $y' = n(x-y)$, $t' = \frac{1}{2}(t+s)$ and $s' = n(t-s)$; denote the new variables x', t', y', s' by x, t, y, s and $\mathcal{E}_6 = I\!R^N \times]0,T[\times]0,1[\times B_{N,1} \times B_{1,1} \times]0,1[$ yields:

$$X_{1n} = \int_{\mathcal{E}_6} \begin{array}{l} |\nu(x+\frac{y}{2n}, t+\frac{s}{2n}, \alpha) - \mu(x-\frac{y}{2n}, t-\frac{s}{2n}, \beta)| \\ \psi_t(x,t)\rho_N(y)\rho_1(s)dxdtd\alpha dydsd\beta, \end{array} \tag{46}$$

$$X_{2n} = \int_{\mathcal{E}_6} \begin{array}{l} F(\nu(x+\frac{y}{2n}, t+\frac{s}{2n}, \alpha), \mu(x-\frac{y}{2n}, t-\frac{s}{2n}, \beta)) \\ \frac{1}{2}\Big[v(x+\frac{y}{2n}, t+\frac{s}{2n}) + v(x-\frac{y}{2n}, t-\frac{s}{2n})\Big]. \\ grad\psi(x,t)\rho_N(y)\rho_1(s)dxdtd\alpha dydsd\beta, \end{array} \tag{47}$$

$$X_{3n} = \int_{\mathcal{E}_6} \begin{array}{l} F(\nu(x+\frac{y}{2n}, t+\frac{s}{2n}, \alpha), \mu(x-\frac{y}{2n}, t-\frac{s}{2n}, \beta)) \\ \psi(x,t)\Big[v(x+\frac{y}{2n}, t+\frac{s}{2n}) - v(x-\frac{y}{2n}, t-\frac{s}{2n})\Big]. \\ ngrad\rho_N(y)\rho_1(s)dxdtd\alpha dydsd\beta. \end{array} \tag{48}$$

¿From (46),

$$\begin{array}{l} \left| \; X_{1n} - \displaystyle\int_{I\!R^N \times]0,T[\times]0,1[\times]0,1[} |\nu(x,t,\alpha) - \mu(x,t,\beta)| \; \psi_t(x,t)dxdtd\alpha d\beta \; \right| \leq \\ \displaystyle\int_{B_{N,1}\times B_{1,1}\times]0,1[\times]0,1[} \Big[\; \|\nu(.+\frac{y}{2n}, .+\frac{s}{2n}, \alpha) - \nu(.,.,\alpha)\|_{L^1(K_\psi)} + \\ \|\mu(.-\frac{y}{2n}, .-\frac{s}{2n}, \beta) - \mu(.,.,\beta)\|_{L^1(K_\psi)}\Big] \|\psi_t\|_\infty \rho_N(y)\rho_1(s)dydsd\alpha d\beta, \end{array} \tag{49}$$

where K_ψ is the (compact) support of ψ. Applying the theorem of continuity in mean to the measurable bounded (and therefore integrable on bounded sets) functions $\nu(.,.,\alpha)$ and $\mu(.,.,\beta)$ yields:

$$X_{1n} \to \int_{I\!R^N \times I\!R_+ \times]0,1[\times]0,1[} |\nu(x,t,\alpha) - \mu(x,t,\beta)| \; \psi_t(x,t)dxdtd\alpha d\beta \quad \text{as } n \to +\infty. \tag{50}$$

Let us now turn to X_{2n}, defined in (44); from (47), using the fact that $v \in C^1(I\!R^N \times [0,T], I\!R^N)$ and is therefore Lipschitz over the compact K_ψ:

$$
\begin{array}{rl}
|X_{2n} & -\int_{I\!R^N\times]0,T[\times]0,1[\times]0,1[} F(\nu(x,t,\alpha),\mu(x,t,\beta))\ v(x,t).grad\psi(x,t)dxdtd\alpha d\beta| \\
& \leq VM_f\int_{B_{N,1}\times B_{1,1}\times]0,1[\times]0,1[}\Big[\|\nu(.+\frac{y}{2n},.+\frac{s}{2n},\alpha)-\nu(.,.,\alpha)\|_{L^1(K_\psi)}+ \\
& \|\mu(.-\frac{y}{2n},.-\frac{s}{2n},\beta)-\mu(.,.,\beta)\|_{L^1(K_\psi)}\Big]\ \|grad\psi\|_\infty\rho_N(y)\rho_1(s)dydsd\alpha d\beta \\
& +\frac{1}{n}C(v,\psi,M_f,r_\nu,r_\mu),
\end{array}
\tag{51}
$$

where $C(v,\psi,M_f,r_\nu,r_\mu)$ depends only on v,ψ,M_f,r_ν,r_μ. Hence, by the theorem of continuity in mean:

$$
\begin{array}{l}
X_{2n}\rightarrow\int_{I\!R^N\times I\!R_+\times]0,1[\times]0,1[} F(\nu(x,t,\alpha),\mu(x,t,\beta))\ v(x,t).grad\psi(x,t)dxdtd\alpha d\beta \\
\text{as } n\rightarrow+\infty.
\end{array}
\tag{52}
$$

Let us now study X_{3n}, defined in (45). First note that, thanks to the properties of the support of ρ_N, and since $divv=0$, one has:

$$
\begin{array}{l}
\int_{B_{N,1}} v(x+\frac{y}{2n},t).grad\rho_N(y)dy=\int_{B_{N,1}} v(x-\frac{y}{2n},t).grad\rho_N(y)dy=0, \\
\text{for a.e. } (x,t)\in I\!R^N\times]0,T[,
\end{array}
\tag{53}
$$

Hence:

$$
\begin{array}{rl}
X_{4n} & =\int_{\mathcal{E}_6} F(\nu(x,t,\alpha),\mu(x,t,\beta))\psi(x,t)\Big[v(x+\frac{y}{2n},t+\frac{s}{2n})- \\
& v(x-\frac{y}{2n},t-\frac{s}{2n})\Big].ngrad\rho_N(y)\rho_1(s)dxdtd\alpha dydsd\beta=0,
\end{array}
\tag{54}
$$

Next, since v is Lipschitz over the support of the test functions, the expression $n\Big[v(x+\frac{y}{2n},t+\frac{s}{2n})-v(x-\frac{y}{2n},t-\frac{s}{2n})\Big]$ is bounded uniformly w.r.t. $y\in B_{N,1}$ and $s\in B_{1,1}$. Hence, by the theorem of continuity in mean, $X_{3n}-X_{4n}\rightarrow 0$, as $n\rightarrow+\infty$. Therefore $X_{3n}\rightarrow 0$, as $n\rightarrow+\infty$.

Passing to the limit in (42) yields (20), which concludes the proof of step 2.

Step 3: Proof of Relation (22).

The aim here is to prove that the function A defined in (21) is almost everywhere nondecreasing. Let $a\geq 0$ and recall that $\omega=VM_f$; let $0<t_1<t_2<\min(T,\frac{a}{\omega})$, $0<\varepsilon<\min(t_1,\min(T,\frac{a}{\omega})-t_2)$, and $\delta>0$. Let $\psi\in C_c^1(I\!R_+,[0,1])$ such that $\psi(r)=1\ \forall r\in[0,a]$, $\psi(r)=0\ \forall r\in[a+\delta,+\infty[$, and $\psi'\leq 0$. Define r_ε by:

$$r_\varepsilon(t) = \begin{cases} 0 & \text{if } 0 \le t \le t_1 - \varepsilon \\ \dfrac{t-(t_1-\varepsilon)}{\varepsilon} & \text{if } t_1 - \varepsilon \le t \le t_1 \\ 1 & \text{if } t_1 \le t \le t_2 \\ \dfrac{(t_2+\varepsilon)-t}{\varepsilon} & \text{if } t_2 \le t \le t_2+\varepsilon \\ 0 & \text{if } t_2+\varepsilon \le t < +\infty. \end{cases} \tag{55}$$

Let $|x|$ denote the Euclidean norm of x in $I\!R^N$; one can take in (20) the test function $\varphi(x,t) = \psi(|x|+\omega t)\, r_\varepsilon(t)$. Indeed, this is easily seen by considering regularizations of the functions r_ε. This yields:

$$\begin{array}{l} \dfrac{1}{\varepsilon}\displaystyle\int_{t_1-\varepsilon}^{t_1}\int_{I\!R^N}\int_0^1\int_0^1 |\nu(x,t,\alpha)-\mu(x,t,\beta)|\psi(|x|+\omega t)d\alpha d\beta dx dt - \\ \dfrac{1}{\varepsilon}\displaystyle\int_{t_2}^{t_2+\varepsilon}\int_{I\!R^N}\int_0^1\int_0^1 |\nu(x,t,\alpha)-\mu(x,t,\beta)|\psi(|x|+\omega t)d\alpha d\beta dx dt \quad \ge E, \end{array} \tag{56}$$

with:

$$\begin{cases} E = -\displaystyle\int_0^T\int_{I\!R^N}\int_0^1\int_0^1 \Big[& \omega|\nu(x,t,\alpha)-\mu(x,t,\beta)| + \\ & F(\nu(x,t,\alpha),\mu(x,t,\beta))\dfrac{v(x,t).x}{|x|}\Big] \\ & \psi'(|x|+\omega t) r_\varepsilon(t) d\alpha d\beta dx dt. \end{cases} \tag{57}$$

From:

$$|F(\nu(x,t,\alpha),\mu(x,t,\beta))\frac{v(x,t).x}{|x|}| \le M_f V \quad |\nu(x,t,\alpha)-\mu(x,t,\beta)|, \tag{58}$$

and since $M_f V = \omega$ and $\psi' \le 0$, we deduce that $E \ge 0$. Letting $\delta \to 0$ (and noting that the mapping $(x,t) \mapsto \displaystyle\int_0^1\int_0^1 |\nu(x,t,\alpha)-\mu(x,t,\beta)| d\alpha d\beta$ is in $L^\infty(I\!R^N \times]0,T[)$), (56) yields:

$$\begin{array}{l} \dfrac{1}{\varepsilon}\displaystyle\int_{t_1-\varepsilon}^{t_1}\int_{B_{N,a-\omega t_1}}\int_0^1\int_0^1 |\nu(x,t,\alpha)-\mu(x,t,\beta)| d\alpha d\beta dx dt - \\ \dfrac{1}{\varepsilon}\displaystyle\int_{t_2}^{t_2+\varepsilon}\int_{B_{N,a-\omega t_2}}\int_0^1\int_0^1 |\nu(x,t,\alpha)-\mu(x,t,\beta)| d\alpha d\beta dx dt \quad \ge 0, \end{array} \tag{59}$$

that is:

$$\frac{1}{\varepsilon}\int_{t_1-\varepsilon}^{t_1} A(t)dt - \frac{1}{\varepsilon}\int_{t_2}^{t_2+\varepsilon} A(t)dt \ge 0. \tag{60}$$

Note that $A \in L^1(]0,T[)$ $(0 \le A(t) \le (r_\nu + r_\mu)\lambda_N(B_{N,a-\omega t}))$; let t_1 and t_2 be Lebesgue points of the function A such that $0 < t_1 \le t_2 < \min(T, \dfrac{a}{\omega})$, one deduces from (60) letting ε tend to 0: $A(t_1) \ge A(t_2)$. This concludes the proof of Relation (22).

Step 4 : conclusion of the proof.
First remark that:

$$\int_{B_{N,a}} \int_0^1 \int_0^1 |\nu(x,t,\alpha) - \mu(x,t,\beta)| d\alpha d\beta dx \leq \int_{B_{N,a}} \int_0^1 |\nu(x,t,\alpha) - u_0(x)| d\alpha dx + \int_{B_{N,a}} \int_0^1 |\mu(x,t,\alpha) - u_0(x)| d\alpha dx. \tag{61}$$

Then, from (19) and (61):

$$\lim_{\tau \to 0} \frac{1}{\tau} \int_0^\tau \int_{B_{N,a}} \int_0^1 \int_0^1 |\nu(x,t,\alpha) - \mu(x,t,\beta)| d\alpha d\beta dx dt = 0, \tag{62}$$

and therefore : $\frac{1}{\tau} \int_0^\tau A(t)dt \to 0$ as $\tau \to 0$. Thus, since A is a.e. non-increasing on $]0, \tau[$, and $A(t) \geq 0$ for a.e. $t \in]0, \min(T, \frac{a}{\omega})[$, one has $A(t) = 0$ for a.e. $t \in]0, \min(T, \frac{a}{\omega})[$; since a is arbitrary,

$$\int_0^1 \int_0^1 |\nu(x,t,\alpha) - \mu(x,t,\beta)| d\alpha d\beta = 0 \text{ for a.e. } (x,t) \in I\!R^N \times]0, T[. \tag{63}$$

Therefore:

- there exists $u(x,t)$ such that $\mu(x,t,\alpha) = \nu(x,t,\beta) = u(x,t)$, for a.e. α and β,
- the function u is the weak entropy solution to problem (1).

This concludes the proof of the theorem.

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