NONLINEAR PARABOLIC EQUATIONS WITH MEASURE DATA

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Abstract

In this paper we give summability results for the gradients of solutions of nonlinear parabolic equations whose model is

$$u' - \text{div}\,(|\nabla u|^{p-2}\nabla u) = \mu \qquad \text{on } \Omega \times (0,T)\,, \tag{P}$$

with homogeneous Cauchy-Dirichlet boundary conditions, where $p > 1$ and μ is a bounded measure on $\Omega \times (0,T)$. We also study how the summability of the gradient improves if the measure μ is a function in $L^m(\Omega \times (0,T))$, with m "small".
Moreover we give a new proof of the existence of a solution for problem (P).

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1 Introduction and statement of results

Let Ω be a bounded domain in $\mathbf{R}^N$, $N \geq 2$. For $T > 0$, let us denote by Q the cylinder $\Omega \times (0,T)$, and by Γ the lateral surface $\partial\Omega \times (0,T)$. We will consider the following nonlinear parabolic Cauchy-Dirichlet problem:

$$\begin{cases} u' - \operatorname{div}(a(x,t,u,\nabla u)) = \mu & \text{in } Q, \\ u(x,0) = 0 & \text{in } \Omega, \\ u(x,t) = 0 & \text{on } \Gamma. \end{cases} \tag{1.1}$$

Here μ belongs to $\mathcal{M}(Q)$, the space of bounded Borel measures on Q, and the function $a(x,t,\sigma,\xi) : \Omega \times (0,T) \times \mathbf{R} \times \mathbf{R}^N \to \mathbf{R}^N$ is a Carathéodory function (i.e., it is continuous with respect to σ and ξ for almost every $(x,t) \in Q$, and measurable with respect to (x,t) for every $\sigma \in \mathbf{R}$ and $\xi \in \mathbf{R}^N$) which satisfies the following classical Leray-Lions assumptions:

$$a(x,t,\sigma,\xi) \cdot \xi \geq \alpha|\xi|^p \ , \tag{1.2}$$

$$|a(x,t,\sigma,\xi)| \leq \beta\,[\eta(x,t) + |\sigma|^{p-1} + |\xi|^{p-1}]\,, \tag{1.3}$$

$$[a(x,t,\sigma,\xi) - a(x,t,\sigma,\xi')] \cdot (\xi - \xi') > 0 \ , \tag{1.4}$$

for a.e. $(x,t) \in Q$, for every $\sigma \in \mathbf{R}$, ξ, $\xi' \in \mathbf{R}^N$, $\xi \neq \xi'$, where p is an exponent such that

$$p \geq 2 \tag{1.5}$$

(see Remarks 1.7 below for some comments about this bound on p), α and β are positive real numbers, and $\eta \in L^{p'}(Q)$, where $p' = \frac{p}{p-1}$ is the Hölder conjugate exponent of p.

Definition 1.1 We will say that a function u in $L^1(0,T;W_0^{1,1}(\Omega))$ is a weak solution of (1.1) if $a(x,t,u,\nabla u) \in (L^1(Q))^N$ and

$$-\int_Q u\frac{\partial \varphi}{\partial t}\,dx\,dt + \int_Q a(x,t,u,\nabla u)\cdot\nabla\varphi\,dx\,dt = \int_Q \varphi\,d\mu\,, \tag{1.6}$$

for every $\varphi \in C^\infty(\bar{Q})$ which is zero in a neighborhood of $\Gamma \cup (\Omega \times \{T\})$.

The first existence theorem for the elliptic problems corresponding to (1.1) is due to G. Stampacchia (see [25]) in the case of linear equations. Existence results have been proved in [2] for semilinear Dirichlet problems and in [7], [8], [15] (see also [4]) for nonlinear Dirichlet problems. In the parabolic case, existence theorems have been given by the authors in [7], [13]. The main aim of this paper (as in [5] for the case $p = 2$) is to precise the summability with respect to space and time of the gradients of solutions of (1.1) which are obtained, as in [7] or [13], by approximating μ with regular data. More precisely, we will show that every such solution u belongs to the space $L^r(0,T;W_0^{1,q}(\Omega))$, where r and q are two real numbers which are linked by a suitable relation, as stated in the Theorem 1.2 below. Moreover we study (see Theorems 1.8 and 1.9) how the summability of the gradient improves if the measure μ is replaced by a function f which belongs to $L^m(Q)$, with m "small".

Theorem 1.2 *Assume that hypotheses (1.2)–(1.5) hold, and that $\mu \in \mathcal{M}(Q)$. Then (1.1) has at least a solution u belonging to $L^r(0,T;W_0^{1,q}(\Omega))$ for every pair (q,r) such that*

$$1 \le q < \min\left\{\frac{N(p-1)}{N-1}, p\right\}, \quad 1 \le r \le p\,, \tag{1.7}$$

$$\frac{N(p-2)+p}{r} + \frac{N}{q} > N+1\,. \tag{1.8}$$

Remark 1.3 As far as the bound on q is concerned, let us remark that

$$\min\left\{\frac{N(p-1)}{N-1}, p\right\} = \begin{cases} \dfrac{N(p-1)}{N-1} & \text{if } p < N, \\ \\ p & \text{if } p \geq N. \end{cases}$$

Remark 1.4 If $p = 2$, the result of Theorem 1.2 has been obtained in the linear case by Baras and Pierre in [1], and by Casas in [10], using duality methods, and in the quasilinear case by the authors in [5]; in these cases, the relation (1.8) becomes

$$1 \leq q < \frac{N}{N-1}, \quad 1 \leq r < 2, \qquad \frac{2}{r} + \frac{N}{q} > N + 1 .$$

Remark 1.5 Let us notice that if we require $q = r$, then the bounds on q and r become

$$q < p - \frac{N}{N+1},$$

which is the same condition obtained in [7]. Observe that $p - \frac{N}{N+1}$ is smaller than both $\frac{N(p-1)}{N-1}$ and p.

Remark 1.6 The bound $q < \frac{N(p-1)}{N-1}$ is the same obtained in [7] for the $W_0^{1,q}(\Omega)$-regularity of solutions of the stationary problems associated to (1.1) in the case $p < N$. If q tends to $\frac{N(p-1)}{N-1}$, then r tends to $p - 1$.

Remark 1.7 If $2 - \frac{1}{N+1} < p < 2$, it is possible to prove a result similar to Theorem 1.2. However, in this case, the bounds on q become

$$1 \leq q < \frac{N}{3N + 1 - (N+1)p} .$$

The proof of such a result can be done exactly as in the proof of Theorem 1.2, keeping in mind that r must not be smaller than 1. In the case $1 < p \leq 2 - \frac{1}{N+1}$ one can still find a solution of problem (1.1) using the notion of *entropy solution* (see [22]), but in this case the gradient of the solution may not belong to $(L^1(Q))^N$.

In order to obtain the equality in (1.8) one has to impose, as in [9] and [13], a stronger assumption on the datum μ; more precisely, we will require that μ is a function on Q belonging to the space $L^1(0,T;L^1\log L^1(\Omega))$. The space $L^1\log L^1(\Omega)$ is defined as the set of all measurable functions v on Ω such that

$$\int_\Omega |v|\,\log(1+|v|)\,dx < +\infty\,.$$

This is a Banach space under the norm

$$\|v\|_{L^1\log L^1(\Omega)} = \inf\Big\{\lambda>0 \text{ such that } \int_\Omega \frac{|v|}{\lambda}\,\log(1+\frac{|v|}{\lambda})\,dx \le 1\Big\}\,.$$

The following result holds.

Theorem 1.8 *Assume that hypotheses (1.2)–(1.5) hold, and that the datum* $\mu = f$ *is a function in* $L^1(0,T;L^1\log L^1(\Omega))$. *Then (1.1) has at least a solution* u *which belongs to* $L^r(0,T;W_0^{1,q}(\Omega))$ *for every pair* (q,r) *such that*

$$1 \le q \le \frac{N(p-1)}{N-1} \quad \text{if } p<N, \qquad 1\le q<p \quad \text{if } p \ge N,$$

$$1 \le r \le p\,,$$

$$\frac{N(p-2)+p}{r} + \frac{N}{q} = N+1\,.$$

Finally, we study the regularity of solutions of problem (1.1) if the datum μ is a function f belonging to $L^m(Q)$, with $m>1$.

Theorem 1.9 *Assume that hypotheses (1.2)–(1.5) hold, and that the datum* $\mu = f$ *is a function in* $L^m(Q)$, *with*

$$1 < m < \frac{(N+2)p}{(N+2)p-N}\,. \qquad (1.9)$$

Then (1.1) has at least a solution u *which belongs to* $L^q(0,T;W_0^{1,q}(\Omega))$, *with* q *given by*

$$q = \frac{[N(p-1)+p]\,m}{N+2-m}\,. \qquad (1.10)$$

Moreover, u *belongs to* $L^\sigma(Q)$, *with* σ *given by*

$$\sigma = \frac{[N(p-1)+p]\,m}{N+p-mp}\,. \qquad (1.11)$$

Remark 1.10 For the sake of simplicity, we limit ourselves to the case of $(L^q(Q))^N$ summability for ∇u. However, the result of the previous theorem can be extended in order to find the summability of the gradient with respect to time and space as in theorems 1.2 and 1.8: see Remark 2.5 for the precise statement.

Remark 1.11 If m tends to 1, then q tends to $p - \frac{N}{N+1}$, which is the bound on q obtained in Theorem 1.2 if $r = q$ (see also Remark 1.5). If m tends to $\frac{(N+2)p}{(N+2)p-N}$, then q tends to p, and σ tends to $p\,\frac{N+2}{N}$. Observe that $p\,\frac{N+2}{N}$ is the embedding exponent for functions in $L^\infty(0,T;L^2(\Omega))\cap L^p(0,T;W_0^{1,p}(\Omega))$ (see [11], Proposition 3.1, and (2.3)), and that $\frac{(N+2)p}{(N+2)p-N}$ is its Hölder conjugate exponent. The result of Theorem 1.9 has been obtained in [6] in the case $p = 2$ by means of duality arguments. The result of Theorem 1.9 improves those obtained in [9] and [13]. Other related results, concerning the regularity of the solutions with respect to the regularity of initial datum (which for us is zero) have been obtained in [21] and in [23].

The proof of the results will be achieved in two steps. First of all, in Section 2, some *a priori* estimates for solutions with more regular data will be proved. In Section 3, we will approximate the datum μ with a sequence $\{f_n\}$ of regular functions, and consider the solutions u_n of problem (1.1) with data f_n. We will then prove that u_n converges to a solution u of (1.1). The main tool of the proof will be an almost everywhere convergence result for the gradients of the approximating solutions u_n; we will give a new proof of this result.

In the next sections, we will use the following functions of one real variable, defined for $k > 0$:

$$T_k(s) = \max\{-k, \min\{k, s\}\}, \quad \varphi_k(s) = T_1(s - T_k(s)). \tag{1.12}$$

2 A priori estimates

This section is devoted to the proof of some results which form the core of the regularity theorems stated in the previous section. We begin by recalling the well-known Gagliardo-Nirenberg embedding theorem.

Lemma 2.1 *Let v be a function in $W_0^{1,q}(\Omega) \cap L^\rho(\Omega)$, with $q \geq 1$, $\rho \geq 1$. Then there exists a positive constant C, depending on N, q and ρ, such that*

$$\|v\|_{L^\gamma(\Omega)} \leq C \, \|\nabla v\|^\theta_{(L^q(\Omega))^N} \, \|v\|^{1-\theta}_{L^\rho(\Omega)} \,, \tag{2.1}$$

for every θ and γ satisfying

$$0 \leq \theta \leq 1 \,, \quad 1 \leq \gamma < +\infty \,, \qquad \frac{1}{\gamma} = \theta \left(\frac{1}{q} - \frac{1}{N} \right) + \frac{1-\theta}{\rho} \,. \tag{2.2}$$

Proof. See [20], Lecture II. ∎

An immediate consequence of the previous lemma is the following embedding result:

$$\int_Q |v|^\sigma \leq C \, \|v\|^{\frac{\rho q}{N}}_{L^\infty(0,T;L^\rho(\Omega))} \int_Q |\nabla v|^q \,, \tag{2.3}$$

which holds for every function v in $L^q(0,T;W_0^{1,q}(\Omega)) \cap L^\infty(0,T;L^\rho(\Omega))$, with $q \geq 1$, $\rho \geq 1$ and $\sigma = q\frac{N+\rho}{N}$ (see [11], Propostion 3.1).

Lemma 2.2 *Assume that hypotheses (1.2)–(1.5) hold, and that $\mu = f$ belongs to $L^{p'}(Q)$. Then every solution u of problem (1.1) satisfies the following estimate*

$$\|u\|_{L^r(0,T;W_0^{1,q}(\Omega))} \leq c_1 \,, \tag{2.4}$$

for every pair (q,r) of exponents satisfying the hypotheses of Theorem 1.2, where c_1 is a constant (depending also on a, $\text{meas}(Q)$, N, p, q, r) which depends on f only through its norm in $L^1(Q)$.

Proof. We recall that if f belongs to $L^{p'}(Q)$, then a solution u of (1.1) belongs to $L^p(0,T;W_0^{1,p}(\Omega))$. Thus, it is possible to use $\varphi_k(u)$ (see (1.12) for the definition of φ_k) as test function in (1.1): like in [7], using the fact that $|\varphi_k(s)| \leq 1$, one obtains the estimates

$$\|u\|_{L^\infty(0,T;L^1(\Omega))} \leq c \,, \tag{2.5}$$

$$\int_{B_k} |\nabla u|^p \leq c \,, \quad \text{for every } k \in \mathbf{N}, \tag{2.6}$$

where $B_k = \{(x,t) \in Q : k \le |u(x,t)| < k+1\}$ (here we denote by c any constant depending on a, meas(Q), N, p, q, r and $\|f\|_{L^1(Q)}$, whose value may be different from line to line). Let λ be a real number such that $\lambda > 1$. From (2.6) one then obtains,

$$\begin{aligned} \int_Q \frac{|\nabla u|^p}{(1+|u|)^\lambda} &= \sum_{k=0}^{\infty} \int_{B_k} \frac{|\nabla u|^p}{(1+|u|)^\lambda} \\ &\le \sum_{k=0}^{\infty} \frac{1}{(1+k)^\lambda} \int_{B_k} |\nabla u|^p \le c \sum_{k=0}^{\infty} \frac{1}{(1+k)^\lambda} = c(\lambda)\,. \end{aligned} \tag{2.7}$$

If $1 \le q < p$, for almost every $t \in (0,T)$ we can write, using the Hölder inequality:

$$\begin{aligned} \int_\Omega |\nabla u(x,t)|^q \, dx &= \int_\Omega \frac{|\nabla u(x,t)|^q}{(1+|u(x,t)|)^{\frac{\lambda q}{p}}} (1+|u(x,t)|)^{\frac{\lambda q}{p}} \, dx \\ &\le \left[\int_\Omega \frac{|\nabla u(x,t)|^p}{(1+|u(x,t)|)^\lambda} \, dx \right]^{\frac{q}{p}} \left[\int_\Omega (1+|u(x,t)|)^{\frac{\lambda q}{p-q}} \, dx \right]^{\frac{p-q}{p}}. \end{aligned}$$

We raise to the power r/q and integrate over t. If $1 \le r < p$, using the Hölder inequality with respect to time and (2.7), we obtain

$$\begin{aligned} &\int_0^T \|\nabla u(t)\|^r_{(L^q(\Omega))^N} \, dt \\ &\quad \le \int_0^T \left[\int_\Omega \frac{|\nabla u|^p}{(1+|u|)^\lambda} \, dx \right]^{\frac{r}{p}} \left[\int_\Omega (1+|u|)^{\frac{\lambda q}{p-q}} \, dx \right]^{\frac{(p-q)r}{pq}} dt \\ &\quad \le \left[\int_Q \frac{|\nabla u|^p}{(1+|u|)^\lambda} \right]^{\frac{r}{p}} \left[\int_0^T \left[\int_\Omega (1+|u|)^{\frac{\lambda q}{p-q}} \, dx \right]^{\frac{(p-q)r}{(p-r)q}} dt \right]^{\frac{p-r}{p}} \\ &\quad \le c \left[1 + \left(\int_0^T \|u(t)\|^{\frac{\lambda r}{p-r}}_{L^{\frac{\lambda q}{p-q}}(\Omega)} \, dt \right)^{\frac{p-r}{p}} \right]. \end{aligned} \tag{2.8}$$

Applying Lemma 2.1 with $\rho = 1$ and $\gamma = \frac{\lambda q}{p-q}$, and recalling (2.5), we obtain, for almost every t in $[0,T]$,

$$\|u(t)\|_{L^{\frac{\lambda q}{p-q}}(\Omega)} \le C \, \|\nabla u(t)\|^\theta_{(L^q(\Omega))^N} \, \|u(t)\|^{1-\theta}_{L^1(\Omega)} \le c \, \|\nabla u(t)\|^\theta_{(L^q(\Omega))^N} \,,$$

where θ is such that

$$\frac{p-q}{\lambda q} = \theta\left(\frac{1}{q} - \frac{1}{N}\right) + \frac{1-\theta}{1}\,. \tag{2.9}$$

Raising to the power $\frac{r}{\theta}$ and integrating on $(0,T)$, we obtain

$$\int_0^T \|u(t)\|^{\frac{r}{\theta}}_{L^{\frac{\lambda q}{p-q}}(\Omega)}\,dt \le c\int_0^T \|\nabla u(t)\|^r_{(L^q(\Omega))^N}\,dt\,. \tag{2.10}$$

Now we assume that

$$\frac{r}{\theta} = \frac{\lambda r}{p-r}\,. \tag{2.11}$$

Thus, (2.8) and (2.10) imply

$$\int_0^T \|\nabla u(t)\|^r_{(L^q(\Omega))^N}\,dt \le c\left[1 + \left(\int_0^T \|\nabla u(t)\|^r_{(L^q(\Omega))^N}\,dt\right)^{\frac{p-r}{p}}\right].$$

Since $\frac{p-r}{p} < 1$, one easily obtains an *a priori* estimate on the norm of ∇u in $L^r(0,T;(L^q(\Omega))^N)$.

Putting together (2.9) and (2.11), we obtain

$$\lambda = \frac{Npq + pq - Nqr + Nr - qr - Nq}{Nq}\,, \qquad \theta = \frac{p-r}{\lambda}\,.$$

The conditions on the various parameters we have used above are the following:

$$1 \le q < p\,, \qquad 1 \le r < p\,, \tag{2.12}$$

$$\lambda > 1\,, \tag{2.13}$$

$$\frac{\lambda q}{p-q} \ge 1\,, \tag{2.14}$$

$$0 \le \theta \le 1\,. \tag{2.15}$$

Inequalities (2.13)–(2.15) are equivalent to

$$\begin{cases} \dfrac{N(p-2)+p}{r} + \dfrac{N}{q} > N+1\,, \\[2ex] 1 \le q < p\,, \qquad r \le p\,, \\[2ex] \dfrac{p-N}{r} + \dfrac{N}{q} \ge 1\,. \end{cases} \tag{2.16}$$

Since the two curves (in the varaibles q and r)

$$\frac{N(p-2)+p}{r}+\frac{N}{q}=N+1\,,\qquad \frac{p-N}{r}+\frac{N}{q}=1\,,$$

intersect for $q=\frac{N(p-1)}{N-1}$ and $r=p-1$, conditions (2.12) and (2.16), together with standard embedding theorems between Lebesgue spaces (we recall that Ω is bounded), imply the desired *a priori* estimate in $L^r(0,T;W_0^{1,q}(\Omega))$ for every q and r such that

$$1\le q<\min\left\{\frac{N(p-1)}{N-1},p\right\}\,,\quad 1\le r<p\,,\quad \frac{N(p-2)+p}{r}+\frac{N}{q}>N+1\,.$$

Thus we only have to deal with the case $r=p$, which, by (1.8), corresponds to $1\le q<\frac{p}{2}$. In order to obtain the desired estimates in this case, we observe that if $1\le q<\frac{p}{2}$, then it is possible to choose $\lambda=\frac{p-q}{q}>1$. With this choice of λ, choosing $r=p$ and using (2.5) and (2.7), one obtains reasoning as before:

$$\int_0^T\|\nabla u(t)\|^p_{(L^q(\Omega))^N}\,dt\le\int_0^T\left[\int_\Omega\frac{|\nabla u|^p}{(1+|u|)^\lambda}\,dx\right]\left[\int_\Omega(1+|u|)\,dx\right]^{\frac{p-q}{q}}dt\le c\,,$$

thus concluding the proof. ■

Lemma 2.3 *Assume that hypotheses (1.2)–(1.5) hold, and that $\mu=f$ belongs to $L^{p'}(Q)$. Then every solution u of problem (1.1) satisfies the following estimate*

$$\|u\|_{L^r(0,T;W_0^{1,q}(\Omega))}\le c_1\,,\tag{2.17}$$

for every pair (q,r) of exponents satisfying the hypotheses of Theorem 1.8, where c_1 is a constant (depending also on a, meas(Q)*, N, p, q, r) which depends on f only through its norm in the space $L^1(0,T;L^1\log L^1(\Omega))$.*

Proof. The key estimate for the proof is inequality (2.7) for $\lambda=1$, that is

$$\int_Q\frac{|\nabla u|^p}{1+|u|}\le c\,,\tag{2.18}$$

where c depends on f through its norm in the space $L^1(0,T;L^1\log L^1(\Omega))$. Estimate (2.18) was proved in [13], Lemma 2.2. From (2.18) and (2.5) (which still holds since $L^1\log L^1(\Omega)$ is continuosly embedded in $L^1(\Omega)$), using Lemma 2.1 as in the proof of the previous result, one obtains the desired estimate.

■

Lemma 2.4 *Let m be as in the statement of Theorem 1.9. Under the same hypotheses of Lemma 2.2, every solution u of problem (1.1) with datum $\mu = f$ belonging to $L^{p'}(Q)$ satisfies the following estimate*

$$\|u\|_{L^q(0,T;W_0^{1,q}(\Omega))} \le c_1\,, \tag{2.19}$$

for every q satisfying the hypotheses of Theorem 1.9, where c_1 is a constant (depending also on α, meas(Q)*, N, p, q) which depends on f only through its norm in the space $L^m(Q)$.*

Proof. Let λ be a real number, with $0<\lambda<1$, let t in $(0,T)$, and choose $v=\phi(u)=((1+|u|)^{1-\lambda}-1)\,\mathrm{sgn}\,(u)\,\chi_{(0,t)}$ as test function in (1.1). Using (1.2), we obtain

$$\int_\Omega \Phi(u(t))\,dx + (1-\lambda)\,\alpha\int_0^t\!\!\int_\Omega \frac{|\nabla u|^p}{(1+|u|)^\lambda}\,dx\,dt \le \int_0^t\!\!\int_\Omega |f|\,|\phi(u)|\,dx\,dt\,,$$

where we have defined

$$\Phi(s)=\int_0^s \phi(\sigma)\,d\sigma\,.$$

Observing that there exist two positive constant c_λ and d_λ such that $\Phi(s)\ge c_\lambda|s|^{2-\lambda}-d_\lambda$, we get, after taking the supremum for t in $(0,T)$,

$$\begin{aligned} c_\lambda\,\|u\|^{2-\lambda}_{L^\infty(0,T;L^{2-\lambda}(\Omega))} &+ \alpha\,(1-\lambda)\int_Q \frac{|\nabla u|^p}{(1+|u|)^\lambda} \\ &\le d_\lambda\,\mathrm{meas}\,(\Omega) + \|f\|_{L^m(Q)}\left(\int_Q (1+|u|)^{(1-\lambda)m'}\right)^{\frac{1}{m'}}\,. \\ &\le c + c\,\|f\|_{L^m(Q)}\left(\int_Q (1+|u|)^{(1-\lambda)m'}\right)^{\frac{1}{m'}}\,. \end{aligned} \tag{2.20}$$

Let $q < p$, and define $\sigma = \frac{(N+2-\lambda)q}{N}$. By inequality (2.3) with $\rho = 2 - \lambda$, we obtain

$$\int_Q |u|^\sigma \leq \|u\|_{L^\infty(0,T;L^{2-\lambda}(\Omega))}^{\frac{(2-\lambda)q}{N}} \int_Q |\nabla u|^q .$$

Moreover, by the Hölder inequality,

$$\begin{aligned} \int_Q |\nabla u|^q &= \int_Q \frac{|\nabla u|^q}{(1+|u|)^{\frac{\lambda q}{p}}} (1+|u|)^{\frac{\lambda q}{p}} \\ &\leq \left(\int_Q \frac{|\nabla u|^p}{(1+|u|)^\lambda}\right)^{\frac{q}{p}} \left(\int_Q (1+|u|)^{\frac{\lambda q}{p-q}}\right)^{1-\frac{q}{p}} . \end{aligned} \tag{2.21}$$

Putting the estimates together, and using (2.20), we get

$$\begin{aligned} \int_Q |u|^\sigma \leq & \left(\int_Q (1+|u|)^{\frac{\lambda q}{p-q}}\right)^{1-\frac{q}{p}} \\ & \times \left(c + c\|f\|_{L^m(Q)} \left(\int_Q (1+|u|)^{(1-\lambda)m'}\right)\right)^{\frac{q}{Nm'}+\frac{q}{pm'}} . \end{aligned}$$

Choose now λ and q such that

$$\frac{\lambda q}{p-q} = (1-\lambda)m' = \frac{(N+2-\lambda)q}{N} \quad (= \sigma) ,$$

that is,

$$\lambda = \frac{(N+2)(p-q)}{N+p-q} , \qquad q = \frac{[N(p-1)+p]\, m}{N+2-m}$$

(see (1.10)), which then yield

$$\sigma = \frac{[N(p-1)+p]\, m}{N+p-m\, p}$$

(see (1.11)). With this choice of λ and q, one has that λ belongs to $(0,1)$ if and only if $p > q$ and $q > p - \frac{N}{N+1}$ (which is true by the bound on m, see Remark 1.11). Thus, we obtain

$$\int_Q |u|^\sigma \leq c + c \left(\int_Q (1+|u|)^\sigma\right)^{1-\frac{q}{p}+\frac{q}{Nm'}+\frac{q}{pm'}} .$$

This inequality yields a bound on the norm of u in $L^\sigma(Q)$ if and only if $1-\frac{q}{p}+\frac{q}{Nm'}+\frac{q}{pm'}<1$. This is true if and only if $m<1+\frac{N}{p}$, which is satisfied, since $\frac{(N+2)p}{(N+2)p-N}<1+\frac{N}{p}$ for every $p>1$. Thus, from (2.20) we get a bound on the term

$$\int_Q \frac{|\nabla u|^p}{(1+|u|)^\lambda}\,,$$

which then yields, by (2.21), a bound on the norm of u in $L^q(0,T;W_0^{1,q}(\Omega))$.

∎

Remark 2.5 As stated in the Introduction, we quote here the result giving the $L^r(0,T;W_0^{1,q}(\Omega))$ *a priori* estimates for the solutions in the case of data in $L^m(Q)$, $1<m<\frac{(N+2)p}{(N+2)p-N}$. In this case the conditions on q and r are given by:

$$1\le q<\min\left\{N\frac{Nm(p-1)-p(p-2)(m-1)}{2p(m-1)+N(N-m)},p\right\}\,,$$

$$1\le r\le p\,,$$

$$\frac{N(p-2)+mp}{r}+\frac{N[1+(p-1)(m-1)]}{q}=N+2-m\,.$$

3 Almost everywhere convergence and proof of the results

In this section, we will prove the existence and regularity results stated in the introduction. In order to do this, assume that $\mu\in\mathcal{M}(Q)$, and let $\{f_n\}$ be a sequence of $L^{p'}(Q)$ functions such that

$$\|f_n\|_{L^1(Q)}\le c\,,\qquad f_n\to\mu\quad\text{in the weak* topology of measures.}$$

Let u_n be a solution of

$$\begin{cases} u_n'-\operatorname{div}\left(a(x,t,u_n,\nabla u_n)\right)=f_n & \text{in } Q,\\ u_n(x,0)=0 & \text{in } \Omega,\\ u_n(x,t)=0 & \text{on } \Gamma.\end{cases}\tag{3.1}$$

Such a solution exists by well-known results (see, for instance, [19]), and belongs to $L^p(0,T;W_0^{1,p}(\Omega))\cap C^0([0,T];L^2(\Omega))$. Since $\{f_n\}$ is bounded in $L^1(Q)$, then, by Lemma 2.2, $\{u_n\}$ is bounded in $L^r(0,T;W_0^{1,q}(\Omega))$ for every q and r as in the statement of Theorem 1.2. Therefore there exist a subsequence (still denoted by $\{u_n\}$) and a function u such that

$$u_n \rightharpoonup u \quad \text{weakly in } L^r(0,T;W_0^{1,q}(\Omega)).$$

Moreover, since from the equation one obtains that $\{u_n'\}$ is bounded in $L^1(0,T;W^{-1,1}(\Omega))$, using compactness arguments (see [24]) it is easy to see that

$$u_n \to u \quad \text{strongly in } L^1(Q). \tag{3.2}$$

On the other hand, choosing $T_k(u_n)$ as test function in (3.1), one easily obtains that there exists a positive constant c, independent on k, such that

$$\int_Q |\nabla T_k(u_n)|^p \le c\,k \qquad \forall k>0\,. \tag{3.3}$$

From (3.2), (3.3) and the continuity and boundedness of $T_k(s)$, it follows that the same subsequence $\{u_n\}$ satisfies

$$\begin{aligned} T_k(u_n) &\rightharpoonup T_k(u) \quad \text{weakly in } L^p(0,T;W_0^{1,p}(\Omega)),\\ T_k(u_n) &\to T_k(u) \quad \text{strongly in } L^p(Q). \end{aligned}$$

for every $k>0$.

Since the problem is nonlinear, the weak convergence of u_n in the space $L^r(0,T;W_0^{1,q}(\Omega))$ is not enough in order to prove that u is a solution of problem (1.1). To do this, we will prove the almost everywhere convergence of the gradients for a subsequence of the approximating solutions $\{u_n\}$, and this is the goal of Theorem 3.3 below. The result of almost everywhere convergence of the gradients is usually the main tool in the proof of existence of solutions for nonlinear equations with L^1 or measure data (see [7], [15], [4] for elliptic problems).

We begin by introducing a time-regularization of functions v belonging to $L^p(0,T;W_0^{1,p}(\Omega))\cap C^0([0,T];L^2(\Omega))$ (see [16]): given $\nu>0$, we define

$$v_\nu(x,t)=\nu\int_{-\infty}^t \tilde v(x,s)\,\mathrm{e}^{\nu\,(s-t)}\,ds\,, \tag{3.4}$$

where $\tilde{v}(x,s)$ is the zero extension of v for $s \notin [0,T]$. From now on, the letter ν will be only used with this meaning. We recall that v_ν converges to v strongly in $L^p(0,T;W_0^{1,p}(\Omega))$ as ν tends to infinity, and that $\|v_\nu\|_{L^p(\Omega)} \leq \|v\|_{L^p(\Omega)}$ for every $p \in [1,+\infty]$; moreover,

$$(v_\nu)' = \nu(v - v_\nu)\,, \tag{3.5}$$

in the sense of distributions (see [16] for the proof of these properties). Observe that, due to the regularity of $v - v_\nu$, we also have

$$\langle (v_\nu)', \varphi \rangle = \nu \int_Q (v - v_\nu)\,\varphi \qquad \forall \varphi \in L^p(0,T;W_0^{1,p}(\Omega)) \cap L^{p'}(Q)\,. \tag{3.6}$$

Here and in the following, $\langle \cdot, \cdot \rangle$ denotes the duality pairing between the spaces $L^{p'}(0,T;W^{-1,p'}(\Omega))$ and $L^p(0,T;W_0^{1,p}(\Omega))$.

If n, ν and k are positive integers and ε is a positive real number, we will denote by $\omega(n,\nu,k,\varepsilon)$ any quantity such that

$$\lim_{\varepsilon\to 0^+}\limsup_{k\to+\infty}\limsup_{\nu\to+\infty}\limsup_{n\to+\infty}|\omega(n,\nu,k,\varepsilon)| = 0\,.$$

Sometimes we will also use a subset of parameters: for instance, we will denote by $\omega^{\nu,k,\varepsilon}(n)$ a quantity such that, for any fixed ν, k, and ε,

$$\lim_{n\to+\infty}|\omega^{\nu,k,\varepsilon}(n)| = 0\,.$$

If the quantities that we are taking into account do not depend on some parameters, we will omit the dependence of ω from them. For example, $\omega(n,k)$ is a quantity that depends only on n and k, and such that

$$\lim_{k\to+\infty}\limsup_{n\to+\infty}|\omega(n,k)| = 0\,.$$

In the rest of this section the order in which we intend to pass to the limit in the various parameters will always be the same: first n, then respectively ν, k and ε.

We begin with the following result, which is a modified version of a result that has been proved in [14], Lemma 3.2 (see also [17], Proposition 3).

Lemma 3.1 *Let $\{v_n\}$ be a sequence in $L^p(0,T;W_0^{1,p}(\Omega))\cap C^0([0,T];L^2(\Omega))$ such that $v_n(\cdot,0)=0$, and $v_n'\in L^{p'}(0,T;W^{-1,p'}(\Omega))$; suppose that there exists v in $L^1(0,T;W_0^{1,1}(\Omega))$ such that, for every $k>0$, $T_k(v_n)$ converges strongly to $T_k(v)$ in $L^p(Q)$. Then, for every choice of ε, k and ν,*

$$\langle v_n', T_\varepsilon(v_n-(T_k(v))_\nu)\rangle \geq \omega^{\nu,k,\varepsilon}(n)\,. \tag{3.7}$$

Proof. We have

$$\begin{aligned}\langle v_n', T_\varepsilon(v_n-T_k(v)_\nu)\rangle &= \langle v_n'-(T_k(v)_\nu)', T_\varepsilon(v_n-T_k(v)_\nu)\rangle \\ &\quad +\langle (T_k(v)_\nu)', T_\varepsilon(v_n-T_k(v)_\nu)\rangle\,.\end{aligned} \tag{3.8}$$

For the first term, defining $\Phi_\varepsilon(s)=\int_0^s T_\varepsilon(\sigma)\,d\sigma$ and recalling that $v_n(x,0)=0$, we can write

$$\langle v_n'-(T_k(v)_\nu)', T_\varepsilon(v_n-T_k(v)_\nu)\rangle = \int_\Omega \Phi_\varepsilon(v_n-T_k(v)_\nu)(T)\,dx\geq 0\,, \tag{3.9}$$

since Φ_ε is positive. On the other hand, (3.5) and (3.6) imply

$$\begin{aligned}&\langle (T_k(v)_\nu)', T_\varepsilon(v_n-T_k(v)_\nu)\rangle \\ &= \nu\int_Q [T_k(v)-T_k(v)_\nu]\,T_\varepsilon(v_n-T_k(v)_\nu) \\ &= \nu\int_Q [T_k(v)-T_k(v)_\nu]\,T_\varepsilon(v-T_k(v)_\nu)+\omega^{\nu,k,\varepsilon}(n)\,,\end{aligned} \tag{3.10}$$

since, for $n\to+\infty$, $T_\varepsilon(v_n-T_k(v)_\nu)$ converges to $T_\varepsilon(v-T_k(v)_\nu)$ $*$-weakly in

$L^\infty(\Omega)$. We have

$$\int_Q [T_k(v) - T_k(v)_\nu]\, T_\varepsilon(v - T_k(v)_\nu)$$

$$= \int_{\{|v|\le k\}} [v - T_k(v)_\nu]\, T_\varepsilon(v - T_k(v)_\nu)$$

$$+ \int_{\{v>k\}} [k - T_k(v)_\nu]\, T_\varepsilon(v - T_k(v)_\nu)$$

$$+ \int_{\{v<-k\}} [-k - T_k(v)_\nu]\, T_\varepsilon(v - T_k(v)_\nu)\,,$$

and all three terms of the right hand side are positive since the integrand functions are positive, $s\,T_\varepsilon(s)$ being positive (for the last two integrals recall that $|T_k(v)_\nu| \le k$). Thus, from (3.10) one obtains

$$\langle (T_k(v)_\nu)', T_\varepsilon(v_n - T_k(v)_\nu)\rangle \ge \omega^{\nu,k,\varepsilon}(n)\,. \qquad (3.11)$$

Putting together (3.8), (3.9) and (3.11), one obtains (3.7). ∎

The following result is well known. We give the proof for completeness.

Lemma 3.2 *Let $\mathcal{O}$ be an open bounded subset of $\mathbf{R}^N$, $N \ge 1$, and let $\{v_n\}$ be a sequence of measurable functions on $\mathcal{O}$ such that v_n converges to some function v almost everywhere in $\mathcal{O}$. Then, for almost every h in $\mathbf{R}^+$,*

$$\chi_{\{|v_n|>h\}} \to \chi_{\{|v|>h\}} \qquad \text{strongly in } L^\rho(\mathcal{O}),\ \text{for every } 1 \le \rho < +\infty.$$

Here χ_E denotes the characteristic function of a set $E \subseteq \mathcal{O}$.

Proof. Since $\chi_{\{|v_n|>h\}} \le 1$, the only thing we have to prove in order to apply the Lebesgue dominated convergence theorem is that $\chi_{\{|v_n|>h\}}$ converges to $\chi_{\{|v|>h\}}$ almost everywhere in $\mathcal{O}$. Choose h such that

$$\text{meas}\,(\{|v| = h\}) = 0\,.$$

This is true for every $h \ge 0$ except a countable set. Let y in $\mathcal{O}$ be such that $v_n(y)$ converges to $v(y)$. If $|v(y)| > h$, then $|v_n(y)| > h$ for every n

large enough, and so both $\chi_{\{|v_n|>h\}}$ and $\chi_{\{|v|>h\}}$ are one. If $|v(y)| < h$, then $|v_n(y)| < h$ for every n large enough, and so both $\chi_{\{|v_n|>h\}}$ and $\chi_{\{|v|>h\}}$ are zero. The only set that may give problems is the set where $|v(y)| = h$, but with our choice of h it has measure zero. ∎

The result we are going to prove is the following.

Theorem 3.3 *Let $\{u_n\}$ be a sequence of solutions of (3.1) which converges to some u weakly in $L^q(0,T;W_0^{1,q}(\Omega))$ for some $q > 1$. Then, up to subsequences,*

$$\nabla u_n \to \nabla u \qquad \textit{almost everywhere in } Q.$$

Proof. We follow the method used in [4] for elliptic equations. By the monotonicity of $a(x,t,\sigma,\cdot)$, the result will be proved if, up to subsequences still denoted by u_n (for simplicity of notation, we will omit the dependence of a on x and t),

$$[a(u_n,\nabla u_n) - a(u_n,\nabla u)]\cdot\nabla(u_n - u) \to 0 \qquad \text{almost everywhere in } Q, \tag{3.12}$$

since in [18], Lemme 3.3, it is proved that, under our assumptions on the function $a(x,t,\sigma,\xi)$, the convergence (3.12) implies the result. Furthermore, (3.12) will be true for some subsequence if we show that

$$\lim_{n\to+\infty}\int_Q \{[a(u_n,\nabla u_n) - a(u_n,\nabla u)]\cdot\nabla(u_n - u)\}^\theta = 0\,, \tag{3.13}$$

for some $\theta > 0$. To do this, we will prove that

$$0 \le \int_Q \{[a(u_n,\nabla u_n) - a(u_n,\nabla u)]\cdot\nabla(u_n - u)\}^\theta \le \omega(n,\nu,k,\varepsilon)\,. \tag{3.14}$$

Since u belongs to $L^1(Q)$, the following estimate holds

$$\text{meas}\,(\{(x,t)\in Q : |u(x,t)| \ge k\}) = \omega(k)\,, \tag{3.15}$$

We can write

$$\begin{aligned}
&\int_Q \{[a(u_n,\nabla u_n)-a(u_n,\nabla u)]\cdot\nabla(u_n-u)\}^\theta \\
&\quad=\int_{\{|u|\geq k\}}\{[a(u_n,\nabla u_n)-a(u_n,\nabla u)]\cdot\nabla(u_n-u)\}^\theta \\
&\qquad+\int_{\{|u|<k\}}\{[a(u_n,\nabla u_n)-a(u_n,\nabla u)]\cdot\nabla(u_n-u)\}^\theta \\
&\quad= I_{n,k}+J_{n,k}\,.
\end{aligned}$$

Since $\{u_n\}$ is bounded in $L^q(0,T;W_0^{1,q}(\Omega))$ for some $q<p$, we can choose $\theta<\frac{q}{p}<1$, so that, using the Hölder inequality and (1.3), we obtain

$$\begin{aligned}
|I_{n,k}| &\leq c\left(\int_Q [\eta^{\frac{p'}{p}}+|\nabla u_n|+|\nabla u|+|u_n|]^q\right)^{\frac{\theta p}{q}}\ (\text{meas}\,(\{|u(x,t)|\geq k\}))^{1-\frac{\theta p}{q}} \\
&\leq c\,(\text{meas}\,(\{|u(x,t)|\geq k\}))^{1-\frac{\theta p}{q}}\,,
\end{aligned}$$

and so $I_{n,k}=\omega(n,k)$ by (3.15). On the other hand

$$\begin{aligned}
J_{n,k} &= \int_{\{|u|<k\}}\{[a(u_n,\nabla u_n)-a(u_n,\nabla T_k(u))]\cdot\nabla(u_n-T_k(u))\}^\theta \\
&\leq \int_Q \{[a(u_n,\nabla u_n)-a(u_n,\nabla T_k(u))]\cdot\nabla(u_n-T_k(u))\}^\theta\,,
\end{aligned}\tag{3.16}$$

since the integrand is positive. Now we assume that $\varepsilon>0$ satisfies

$$\begin{aligned}
\lim_{n\to+\infty}\chi_{\{|u_n-T_k(u)_\nu|>\varepsilon\}} &= \chi_{\{|u-T_k(u)_\nu|>\varepsilon\}} \quad \text{for every } \nu \text{ and } k \text{ in } \mathbf{N}, \\
\lim_{\nu\to+\infty}\chi_{\{|u-T_k(u)_\nu|>\varepsilon\}} &= \chi_{\{|u-T_k(u)|>\varepsilon\}} \quad \text{for every } k \text{ in } \mathbf{N},
\end{aligned}\tag{3.17}$$

and the limit is meant in $L^\rho(Q)$, for every $\rho\geq 1$. By Lemma 3.2, almost every ε satisfies (3.17). From now on, we will assume that ε tends to 0 satisfying (3.17). We split the last integral of (3.16) on the sets

$$\{(x,t)\in Q: |u_n-T_k(u)_\nu|\leq\varepsilon\}\,,\qquad \{(x,t)\in Q: |u_n-T_k(u)_\nu|>\varepsilon\}\,,$$

and we define

$$\Psi_{n,k} = [a(u_n, \nabla u_n) - a(u_n, \nabla T_k(u))] \cdot \nabla(u_n - T_k(u)) .$$

Then we have seen that

$$\begin{aligned} &\int_Q \{[a(u_n, \nabla u_n) - a(u_n, \nabla u)] \cdot \nabla(u_n - u)\}^\theta \\ &= \int_Q \Psi_{n,k}^\theta \, \chi_{\{|u_n - T_k(u)_\nu| \le \varepsilon\}} \\ &\quad + \int_Q \Psi_{n,k}^\theta \, \chi_{\{|u_n - T_k(u)_\nu| > \varepsilon\}} + \omega(n,k) . \end{aligned} \tag{3.18}$$

Since $\{\Psi_{n,k}^\theta\}$ is bounded in $L^{\frac{q}{\theta p}}(Q)$, and since $\chi_{\{|u - T_k(u)| > \varepsilon\}}$ converges to zero almost everywhere in Q as k tends to infinity, we have, by (3.17),

$$\int_Q \Psi_{n,k}^\theta \, \chi_{\{|u_n - T_k(u)_\nu| > \varepsilon\}} = \omega^\varepsilon(n, \nu, k) .$$

Thus, (3.18) becomes

$$\begin{aligned} &\int_Q \{[a(u_n, \nabla u_n) - a(u_n, \nabla u)] \cdot \nabla(u_n - u)\}^\theta \\ &= \int_Q \Psi_{n,k}^\theta \, \chi_{\{|u_n - T_k(u)_\nu| \le \varepsilon\}} + \omega^\varepsilon(n, \nu, k) . \end{aligned}$$

Using the Hölder inequality (with exponents $\frac{1}{\theta}$ and $\frac{1}{1-\theta}$), the last integral is smaller than

$$(\text{meas}\,(Q))^{1-\theta} \left(\int_Q \Psi_{n,k} \, \chi_{\{|u_n - T_k(u)_\nu| \le \varepsilon\}} \right)^\theta ,$$

so that (3.14) will be proved if we can show that

$$\int_Q \Psi_{n,k} \, \chi_{\{|u_n - T_k(u)_\nu| \le \varepsilon\}} = \omega(n, \nu, k, \varepsilon) . \tag{3.19}$$

Recalling the definition of $\Psi_{n,k}$, we can write

$$\begin{aligned}
&\int_Q \Psi_{n,k}\, \chi_{\{|u_n - T_k(u)_\nu| \le \varepsilon\}} \\
&\quad = \int_Q a(u_n, \nabla u_n) \cdot \nabla(u_n - T_k(u))\, \chi_{\{|u_n - T_k(u)_\nu| \le \varepsilon\}} \\
&\qquad - \int_Q a(u_n, \nabla T_k(u)) \cdot \nabla(u_n - T_k(u))\, \chi_{\{|u_n - T_k(u)_\nu| \le \varepsilon\}}\,.
\end{aligned} \tag{3.20}$$

By the properties of u_n, and since $|T_k(u)_\nu| \le k$, we can easily deal with the latter integral:

$$\begin{aligned}
&\int_Q a(u_n, \nabla T_k(u)) \cdot \nabla(u_n - T_k(u))\, \chi_{\{|u_n - T_k(u)_\nu| \le \varepsilon\}} \\
&= \int_Q a(T_{\varepsilon+k}(u), \nabla T_k(u)) \cdot \nabla(T_{\varepsilon+k}(u) - T_k(u))\, \chi_{\{|u - T_k(u)_\nu| \le \varepsilon\}} \\
&\quad + \omega^{\nu,k,\varepsilon}(n) \\
&= \int_Q a(u, \nabla T_k(u)) \cdot \nabla(u - T_k(u))\, \chi_{\{|u - T_k(u)_\nu| \le \varepsilon\}} + \omega^{\nu,k,\varepsilon}(n) \\
&= \omega^{\nu,k,\varepsilon}(n)\,,
\end{aligned} \tag{3.21}$$

since $a(u,\nabla T_k(u))\cdot\nabla(u-T_k(u))\equiv 0$. On the other hand,

$$
\begin{aligned}
&\int_Q a(u_n,\nabla u_n)\cdot\nabla(u_n-T_k(u))\,\chi_{\{|u_n-T_k(u)_\nu|\le\varepsilon\}}\\
&\quad=\int_Q a(u_n,\nabla u_n)\cdot\nabla(u_n-T_k(u)_\nu)\,\chi_{\{|u_n-T_k(u)_\nu|\le\varepsilon\}}\\
&\qquad+\int_Q a(u_n,\nabla u_n)\cdot\nabla(T_k(u)_\nu-T_k(u))\,\chi_{\{|u_n-T_k(u)_\nu|\le\varepsilon\}}\\
&\quad=\int_Q a(u_n,\nabla u_n)\cdot\nabla(u_n-T_k(u)_\nu)\,\chi_{\{|u_n-T_k(u)_\nu|\le\varepsilon\}}\\
&\qquad+\omega^{k,\varepsilon}(n,\nu)\,.
\end{aligned}
\tag{3.22}
$$

Indeed, by hypothesis (1.3) and by the Hölder inequality, we have

$$
\begin{aligned}
&\left|\int_Q a(u_n,\nabla u_n)\cdot\nabla(T_k(u)_\nu-T_k(u))\,\chi_{\{|u_n-T_k(u)_\nu|\le\varepsilon\}}\right|\\
&\le\|a(T_{\varepsilon+k}(u_n),\nabla T_{\varepsilon+k}(u_n))\|_{(L^{p'}(Q))^N}\,\|\nabla(T_k(u)_\nu-T_k(u))\|_{(L^p(Q))^N}\\
&=\omega^{k,\varepsilon}(n,\nu)\,,
\end{aligned}
$$

since $|a(T_{\varepsilon+k}(u_n),\nabla T_{\varepsilon+k}(u_n))|$ is bounded in $L^{p'}(Q)$ by (3.3), and $T_k(u)_\nu$ converges to $T_k(u)$ strongly in $L^p(0,T;W_0^{1,p}(\Omega))$. Thus (3.20), (3.21) and (3.22) imply

$$
\begin{aligned}
&\int_Q \Psi_{n,k}\,\chi_{\{|u_n-T_k(u)_\nu|\le\varepsilon\}}\\
&\quad=\int_Q a(u_n,\nabla u_n)\cdot\nabla(u_n-T_k(u)_\nu)\,\chi_{\{|u_n-T_k(u)_\nu|\le\varepsilon\}}+\omega^{k,\varepsilon}(n,\nu)\,.
\end{aligned}
\tag{3.23}
$$

Now we use the equation solved by u_n. Taking $T_\varepsilon(u_n-T_k(u)_\nu)$ as test function in (3.1), we obtain

$$\langle u_n', T_\varepsilon(u_n - T_k(u)_\nu)\rangle + \int_Q a(u_n, \nabla u_n)\cdot \nabla T_\varepsilon(u_n - T_k(u)_\nu)$$
$$= \int_Q f_n\, T_\varepsilon(u_n - T_k(u)_\nu)\,.$$

By Lemma 3.1, whose hypotheses are satisfied by u_n, we have

$$\langle u_n', T_\varepsilon(u_n - T_k(u)_\nu)\rangle \geq \omega^{\nu,k,\varepsilon}(n)\,,$$

while, by the properties of f_n,

$$\int_Q f_n\, T_\varepsilon(u_n - T_k(u)_\nu) \leq \varepsilon\, \|f_n\|_{L^1(Q)} \leq c\,\varepsilon\,.$$

Thus

$$\int_Q a(u_n, \nabla u_n)\cdot \nabla T_\varepsilon(u_n - T_k(u)_\nu) \leq \omega(n,\nu,k,\varepsilon)\,,$$

which, by (3.23), implies (3.19), and therefore (3.14). ∎

Proof of Theorem 1.2. We start from the weak form of the approximating problems (3.1), that is

$$-\int_Q u_n \frac{\partial \varphi}{\partial t} + \int_Q a(u_n, \nabla u_n)\cdot \nabla\varphi = \int_Q f_n\, \varphi\,, \tag{3.24}$$

for every φ as in Definition 1.1. Since u_n converges to u strongly in $L^1(Q)$, the first integral passes easily to the limit. The last integral tends to $\int_Q \varphi\, d\mu$ by the hypotheses on the sequence $\{f_n\}$. Now, by the *a priori* estimates, by (1.3), by the almost everywhere convergence result proved in Theorem 3.3, and by the continuity of $a(x,t,\cdot,\cdot)$, one has, using Vitali's theorem,

$$a(u_n, \nabla u_n) \to a(u, \nabla u) \qquad \text{strongly in } (L^1(Q))^N.$$

Thus, it is possible to pass to the limit in (3.24), obtaining (1.6). The proof of Theorem 1.2 is completed. ∎

Proof of Theorems 1.8 and 1.9. Let $\{f_n\}$ be a sequence of functions in $L^{p'}(Q)$, that converges to f strongly in $L^1(0,T; L^1 \log L^1(\Omega))$ (or in $L^m(Q)$, if we are in the case of Theorem 1.9); take, for instance, $f_n = T_n(f)$. The result is easily obtained combining the proof of Theorem 1.2 with the *a priori* estimate proved in Lemma 2.3 (or Lemma 2.4). ∎

Remark 3.4 In this paper we do not deal with uniqueness of solutions. If the operator is linear, then uniqueness is proved in [10] using duality methods. If the operator is strongly monotone, a positive answer to the problem of uniqueness has been given in [12], where it is proved that the solution obtained by approximation (as those obtained in the present paper) is unique if μ is a function in $L^1(Q)$ and the approximating sequence $\{f_n\}$ is weakly convergent in $L^1(Q)$. If the differential operator is monotone, then it is necessary to give a different definition of the solution in order to prove that it is unique: see [22] for uniqueness of *entropy solutions*, and [3] for uniqueness of *renormalized solutions*.

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References

[1] P. Baras, M. Pierre, Problèmes paraboliques semilinéaires avec données mesures, *Applicable Analysis*, **18** (1984), 111–149.

[2] P. Benilan, H. Brezis, M. Crandall, A semilinear equation in L^1, *Ann. Scuola Norm. Sup. Pisa*, **2** (1975), 523–555.

[3] D. Blanchard, F. Murat, Renormalized solutions of nonlinear parabolic problems with L^1 data: existence and uniqueness, preprint.

[4] L. Boccardo, Some nonlinear Dirichlet problems in L^1 involving lower order terms in divergence form, in *Proceedings of the Intenational Congress on Elliptic and Parabolic Partial Differential Equations*, Capri (1994), to appear.

[5] L. Boccardo, A. Dall'Aglio, T. Gallouët, L. Orsina, Quasi-linear parabolic equations with measure data, in *Proceedings of the International Conference on Nonlinear Differential Equations*, Kiev (1995), to appear.

[6] L. Boccardo, A. Dall'Aglio, T. Gallouët, L. Orsina, Regularity results for nonlinear parabolic equations, preprint.

[7] L. Boccardo, T. Gallouët, Nonlinear elliptic and parabolic equations involving measure data, *J. Funct. Anal.*, **87** (1989), 149–169.

[8] L. Boccardo, T. Gallouët, Nonlinear elliptic equations with right hand side measures, *Comm. Partial Differential Equations*, **17** n. 3&4 (1992), 641–655.

[9] L. Boccardo, T. Gallouët, J.L. Vazquez, Some regularity results for some nonlinear parabolic equations in L^1, *Rend. Sem. Mat. Univers. Politecn. Torino*, Fascicolo Speciale (1989), 69–74.

[10] E. Casas, Pontryagin's principle for state-constrained boundary control problems of semilinear parabolic equations, *IMA Preprint Series*, **1290** (1995).

[11] E. DiBenedetto, *Degenerate parabolic equations*, Springer-Verlag, New York (1993).

[12] A. Dall'Aglio, Approximated solutions of equations with L^1 data. Application to the H-convergence of quasi-linear parabolic equations, *Ann. Mat. Pura Appl.*, **170** (1996), 207–240. .

[13] A. Dall'Aglio, L. Orsina, Existence results for some nonlinear parabolic equations with nonregular data, *Differential Integral Equations*, **5** n. 6 (1992), 1335–1354.

[14] A. Dall'Aglio, L. Orsina, Nonlinear parabolic equations with natural growth conditions and L^1 data, *Nonlinear Anal.*, **27** n. 1 (1996), 59–73.

[15] T. Del Vecchio, Nonlinear elliptic equations with measure data, *Potential Analysis*, **4** (1995), 185–203.

[16] R. Landes, On the existence of weak solutions for quasilinear parabolic boundary value problems, *Proc. Royal Soc. Edinburgh* Sect. A, **89** (1981), 217–237.

[17] R. Landes, V. Mustonen, On parabolic initial-boundary value problems with critical growth for the gradient, *Ann. Inst. H. Poincaré Anal. non linéaire*, **2** (1994), 135–158.

[18] J. Leray, J.L. Lions, Quelques résultats de Višik sur les problèmes elliptiques non linéaires par les méthodes de Minty-Browder, *Bull. Soc. Math. France*, **93** (1965), 97–107.

[19] J.L. Lions, *Quelques méthodes de resolution des problèmes aux limites non linéaires*, Dunod, Gauthier-Villars, Paris (1969).

[20] L. Nirenberg, On elliptic partial differential equations, *Ann. Scuola Norm. Sup. Pisa*, **13** (1959), 116–162.

[21] A. Porretta, Regularity results for a class of parabolic equations with non regular initial datum, preprint

[22] A. Prignet, Existence and uniqueness of "entropy" solutions of parabolic problems with L^1 data, *Nonlinear Anal.*, to appear.

[23] S. Segura, Estimates for solutions of non-linear parabolic equations, *Boll. Un. Mat. Ital.*, to appear.

[24] J. Simon, Compact sets in the space $L^p(0,T;B)$, *Ann. Mat. Pura Appl.*, **146** (1987), 65–96.

[25] G. Stampacchia, Le problème de Dirichlet pour les équations elliptiques du second ordre à coefficients discontinus, *Ann. Inst. Fourier (Grenoble)*, **15** n. 1 (1965), 189–258.