EXISTENCE AND NONEXISTENCE OF SOLUTIONS FOR SOME NONLINEAR ELLIPTIC EQUATIONS

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1 Introduction and statement of results

In this paper we consider nonlinear boundary value problems whose simplest model is the following:

$$\begin{cases} -\Delta u + u\,|\nabla u|^2 = \mu & \text{in } \Omega, \\ \qquad u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded open set in $\boldsymbol{R}^N$, $N \geq 2$, and μ is a Radon measure on Ω.

We are interested in existence and nonexistence of solutions for (1.1). More precisely, we will prove the existence of a solution u in $H_0^1(\Omega)$ for problem (1.1) if and only if the measure μ does not charge the sets of capacity zero in Ω. The main tool of our proof will be a characterization result, proved by the authors in [6], which states that every measure which is zero on sets of zero capacity is an element of $L^1(\Omega) + H^{-1}(\Omega)$ (see also Theorem 2.3, below). We will also prove that if we consider a sequence $\{u_n\}$ of solutions of (1.1) with $L^\infty(\Omega)$ data μ_n converging to a nonzero measure which is singular with respect to the capacity (such as, for example, a Dirac mass), then u_n converges to zero as n tends to infinity. For semilinear problems, an analogous result can be found in [10].

Our result is closely related to the work of H. Brezis and L. Nirenberg (see [11]), where (as a particular case of more general results) it is proved that if μ is a bounded $L^\infty(\Omega)$ function, and u is a smooth solution of (1.1) in $\Omega\backslash K$, with K a closed set of zero capacity, then u is smooth in the whole of Ω; that is to say, u cannot be singular on sets of zero capacity.

The link between the two results is due to the fact that, as a consequence of a theorem of [17] (see Proposition 3.1, below), every bounded Radon measure μ on Ω can be decomposed in a unique way as the sum of a measure μ_0 which is zero on the sets of zero capacity, and of a measure λ which is concentrated on a set E of zero capacity. Hence, problem (1.1) implies that

$$-\Delta u + u|\nabla u|^2 = \mu_0 \qquad \text{in } \Omega\backslash E.$$

The result of [11] states that if μ_0 is an $L^\infty(\Omega)$ function, then u is a solution of the problem with datum μ_0 in the whole Ω: that is to say, the term λ does not play any role. Our result states that if we look for solutions in the whole of Ω, then we have to take into account only the term μ_0, without the $L^\infty(\Omega)$ restriction on μ_0.

Problem (1.1), and some variants of it, has been widely studied in the literature. For example, if the nonlinear lower order term is independent of ∇u, and behaves like $|u|^{q-1}u$, with $q > 1$, there are existence and nonexistence results depending on the measure μ: see e.g., [10], [12], [13], [1], [18], [20]. If the nonlinear term is as in (1.1), there are existence results if μ belongs to $H^{-1}(\Omega)$ (see [2], [7] and [16]), or to $L^1(\Omega)$ (see [4], [5], and [21]).

Since we are going to consider more general nonlinear boundary value problems, we state now the assumptions that will hold throughout the paper.

Let Ω be a bounded, open subset of $\boldsymbol{R}^N$, $N \geq 2$. Let p be a real number such that $1 < p \leq N$, and let p' be its Hölder conjugate exponent (i.e., $1/p + 1/p' = 1$).

Let $a : \Omega \times \boldsymbol{R}^N \to \boldsymbol{R}^N$ be a Carathéodory function (i.e., $a(x, \cdot)$ is continuous on $\boldsymbol{R}^N$ for almost every x in Ω, and $a(\cdot, \xi)$ is measurable on Ω for every ξ in $\boldsymbol{R}^N$), such that the following holds:

$$a(x,\xi) \cdot \xi \geq \alpha\, |\xi|^p\,, \tag{1.2}$$

for almost every $x \in \Omega$, for every $\xi \in \boldsymbol{R}^N$, where α is a positive constant;

$$|a(x,\xi)| \leq \ell(x) + \beta\, |\xi|^{p-1}\,, \tag{1.3}$$

for almost every $x \in \Omega$, for every $\xi \in \boldsymbol{R}^N$, where β is a positive constant, and ℓ belongs to $L^{p'}(\Omega)$;

$$[a(x,\xi) - a(x,\eta)] \cdot (\xi - \eta) > 0\,, \tag{1.4}$$

for almost every $x \in \Omega$, for every ξ and η in $\boldsymbol{R}^N$, with $\xi \neq \eta$.

Let us define the differential operator

$$A(u) = -\mathrm{div}\,(a(x, \nabla u))\,.$$

Thanks to (1.2), (1.3) and (1.4), A is a monotone and coercive differential operator acting between $W_0^{1,p}(\Omega)$ and it dual $W^{-1,p'}(\Omega)$; hence, it is surjective on $W^{-1,p'}(\Omega)$ (see [19]).

Let $g : \Omega \times \boldsymbol{R} \times \boldsymbol{R}^N \to \boldsymbol{R}$ be a Carathéodory function (i.e. $g(\cdot, s, \xi)$ is measurable in Ω for any (s, ξ) in $\boldsymbol{R} \times \boldsymbol{R}^N$, and $g(x, \cdot, \cdot)$ is continuous in $\boldsymbol{R} \times \boldsymbol{R}^N$ for almost every $x \in \Omega$) such that

$$|g(x,s,\xi)| \leq b(|s|)\,[|\xi|^p + d(x)]\,, \tag{1.5}$$

with b a real valued, positive, increasing, continuous function, and d a non-negative function in $L^1(\Omega)$;

$$g(x,s,\xi)\,\mathrm{sgn}(s) \geq \rho\, |\xi|^p\,, \tag{1.6}$$

for almost every x in Ω, for every ξ in $\boldsymbol{R}^N$, for every s in $\boldsymbol{R}$ such that $|s| \geq \sigma$, where σ and ρ are two positive real numbers.

For every compact subset K of Ω the p-capacity of K with respect to Ω is defined as

$$\mathrm{cap}_p(K,\Omega) = \inf \left\{ \int_\Omega |\nabla u|^p\, dx : u \in C_0^\infty(\Omega),\ u \geq \chi_K \right\}\,,$$

where χ_K is the characteristic function of K; we will use the convention that $\inf \emptyset = +\infty$. The p-capacity of any open subset U of Ω is then defined by

$$\mathrm{cap}_p(U,\Omega) = \sup\left\{\mathrm{cap}_p(K,\Omega),\ K \text{ compact},\ K \subseteq U\right\},$$

and the p-capacity of any subset $B \subseteq \Omega$ by

$$\mathrm{cap}_p(B,\Omega) = \inf\left\{\mathrm{cap}_p(U,\Omega),\ U \text{ open},\ B \subseteq U\right\}.$$

We will denote by $\mathcal{M}_{\mathrm{b}}(\Omega)$ the space of all signed measures on Ω, i.e., the space of all σ-additive set functions μ with values in $\boldsymbol{R}$ defined on the Borel σ-algebra. Note that, if μ belongs to $\mathcal{M}_{\mathrm{b}}(\Omega)$, then $|\mu|$ (the total variation of μ) is a bounded positive measure on Ω. We will denote by $\mathcal{M}_0^p(\Omega)$ the space of all measures μ in $\mathcal{M}_{\mathrm{b}}(\Omega)$ such that $\mu(E) = 0$ for every set such that $\mathrm{cap}_p(E,\Omega) = 0$. Examples of measures in $\mathcal{M}_0^p(\Omega)$ are the $L^1(\Omega)$ functions, or the measures in $W^{-1,p'}(\Omega)$.

We define, for s and k in $\boldsymbol{R}$, with $k \geq 0$, $T_k(s) = \max(-k, \min(k,s))$ and $G_k(s) = s - T_k(s)$.

Our result is the following.

Theorem 1.1 *Let μ be a measure in $\mathcal{M}_{\mathrm{b}}(\Omega)$. Then there exists a solution u of*

$$\begin{cases} A(u) + g(x,u,\nabla u) = \mu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.7}$$

in the sense that u belongs to $W_0^{1,p}(\Omega)$, $g(x,u,\nabla u)$ belongs to $L^1(\Omega)$, and

$$\int_\Omega a(x,\nabla u)\cdot\nabla v\,dx + \int_\Omega g(x,u,\nabla u)\,v\,dx = \int_\Omega v\,d\mu\,, \tag{1.8}$$

for every v in $C_0^\infty(\Omega)$, if and only if μ belongs to $\mathcal{M}_0^p(\Omega)$.

Remark 1.2 Note that the solution of (1.7) given by the previous theorem belongs to $W_0^{1,p}(\Omega)$ even if μ belongs to $L^1(\Omega)$: this is in sharp contrast with the results in the case $g \equiv 0$, since the solutions of

$$\begin{cases} A(u) = \mu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

are known to belong only to $W_0^{1,q}(\Omega)$ for every $q < N(p-1)/(N-1)$ (see [3]).

The better regularity of the solution of (1.7) is due to assumption (1.6). Indeed, even if the nonlinear lower order term has a growth of order p with respect to the gradient, but does not satisfy (1.6), then we may not have solutions in $W_0^{1,p}(\Omega)$. To show this fact, let us consider the following example. Let $p = 2$, $\Omega = \{x \in \boldsymbol{R}^N : |x| < 1\}$, and $a(x,\xi) = \xi$, so that $A(u) = -\Delta u$, the laplacian operator. Let $m = \frac{N}{2} - 1$, and let $u(\rho) = \rho^{-m} - 1$, where $\rho = |x|$. Then u is solution of

$$-\Delta u + \frac{|\nabla u|^2}{1+u} = f(\rho)\,,$$

where

$$f(\rho) = \frac{(N-2)^2}{2\rho^{m+2}}\,.$$

It is easily seen that u does not belong to $H_0^1(\Omega)$, while f belongs to $L^s(\Omega)$, for every s in $[1, \frac{2N}{N+2})$.

Remark 1.3 The result of Theorem 1.1 explains the restriction $p \le N$ we made. Indeed, if $p > N$, then there are no sets of zero p-capacity (except the empty set): in other words, every measure in $\mathcal{M}_{\rm b}(\Omega)$ is in $\mathcal{M}_0^p(\Omega)$; moreover, due to the Sobolev embeddings, every measure in $\mathcal{M}_{\rm b}(\Omega)$ is in $W^{-1,p'}(\Omega)$.

2 Proof of Theorem 1.1

In the following, we will denote by c any constant which depends on the various quantities of the problem but not on n. The value of c may vary from line to line.

We begin with an existence result for problem (1.7) in the case in which the datum μ is regular.

Theorem 2.1 *Let f be an $L^\infty(\Omega)$ function, and let F be an element of $(L^s(\Omega))^N$, with $s > \frac{N}{p-1}$. Then there exists a solution u in $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ of*

$$\begin{cases} A(u) + g(x,u,\nabla u) = f - \operatorname{div}(F) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

in the sense that

$$\int_\Omega a(x,\nabla u)\cdot\nabla v\,dx + \int_\Omega g(x,u,\nabla u)\,v\,dx = \int_\Omega f\,v\,dx + \int_\Omega F\cdot\nabla v\,dx\,,$$

for every v in $W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$.

Remark 2.2 We remark that the result of the preceding theorem does not exist in the literature. Indeed, existence results in $W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$ have been given, under the same assumptions on f and F as in Theorem 2.1, for the problem

$$A(u) + g(x,u,\nabla u) + \alpha_0\,|u|^{p-2}\,u = f - \mathrm{div}(F)\,,$$

with $\alpha_0 > 0$ (see, for example, [9] and the references contained therein), or under a sign assumption on g: namely,

$$g(x,s,\xi)\,s \geq 0\,,$$

for almost every x in Ω, for every s in $\boldsymbol{R}$, for every ξ in $\boldsymbol{R}^N$ (see [2]). In our case, (1.6) gives a sign condition on g only for large values of s.

Proof of Theorem 2.1. For the sake of simplicity, we will give a proof of this result in the case $f\equiv 0$: the case of f different from zero can be dealt with by means of minor technical modifications.

Let n in $\boldsymbol{N}$, and let

$$g_n(x,s,\xi) = \frac{g(x,s,\xi)}{1+\frac{1}{n}\,|g(x,s,\xi)|}\,.$$

Then $g_n(x,s,\xi)$ is bounded, satisfies (1.5) and, thanks to (1.6), is such that

$$g_n(x,s,\xi)\,\mathrm{sgn}(s) \geq 0\,, \tag{2.2}$$

for almost every x in Ω, for every ξ in $\boldsymbol{R}^N$, and for every s in $\boldsymbol{R}$ with $|s|\geq\sigma$.

Since g_n is bounded, by classical results (see for example [19]) there exists a solution u_n in $W_0^{1,p}(\Omega)$ of

$$\begin{cases} A(u_n) + g_n(x,u_n,\nabla u_n) = -\mathrm{div}(F) & \text{in } \Omega,\\ \qquad\qquad u_n = 0 & \text{on } \partial\Omega, \end{cases}$$

in the sense that

$$\int_\Omega a(x, \nabla u_n) \cdot \nabla v \, dx + \int_\Omega g_n(x, u_n, \nabla u_n) \, v \, dx = \int_\Omega F \cdot \nabla v \, dx \,, \tag{2.3}$$

for every v in $W_0^{1,p}(\Omega)$.

As proved in [8], [9], if the sequence $\{u_n\}$ is bounded in $L^\infty(\Omega)$, then it is possible to extract a subsequence which converges strongly in $W_0^{1,p}(\Omega)$ to a solution of (2.1). Thus, the proof of the theorem will be achieved if we prove an $L^\infty(\Omega)$ *a priori* estimate on the sequence $\{u_n\}$.

To do this, we choose $v = G_k(u_n)$ as test function in (2.3), with $k \geq \sigma$. We obtain

$$\int_\Omega a(x, \nabla u_n) \cdot \nabla G_k(u_n) \, dx + \int_\Omega g_n(x, u_n, \nabla u_n) G_k(u_n) \, dx = \int_\Omega F \cdot \nabla G_k(u_n) \, dx.$$

Since $G_k(s)$ has the same sign as s, and since $G_k(s)$ is different from zero only where $|s| \geq k \geq \sigma$, (2.2) then implies

$$\int_\Omega g_n(x, u_n, \nabla u_n) \, G_k(u_n) \, dx \geq 0 \,.$$

On the other hand, setting

$$A_{k,n} = \{|u_n| \geq k\} \,,$$

we have, by the Young inequality,

$$\int_\Omega F \cdot \nabla G_k(u_n) \, dx \leq c \int_{A_{k,n}} |F|^{p'} \, dx + \frac{\alpha}{2} \int_\Omega |\nabla G_k(u_n)|^p \, dx \,.$$

Thus, using (1.2) we have

$$\frac{\alpha}{2} \int_\Omega |\nabla G_k(u_n)|^p \, dx \leq c \int_{A_{k,n}} |F|^{p'} \, dx \,.$$

¿From now on we will follow the method introduced by G. Stampacchia in order to prove $L^\infty(\Omega)$ *a priori* estimates for solutions of elliptic equations (see [22]). Since $|F|$ belongs to $L^s(\Omega)$, and $s > p'$, we have, by the Hölder inequality,

$$\int_{A_{k,n}} |F|^{p'} \, dx \leq \|F\|_{(L^s(\Omega))^N} \, \mathrm{meas}\,(A_{k,n})^{1-\frac{p'}{s}} \,.$$

On the other hand, by the Sobolev embedding, we have

$$\int_\Omega |\nabla G_k(u_n)|^p \, dx \geq c \left(\int_\Omega |G_k(u_n)|^{p^*} \, dx \right)^{\frac{p}{p^*}} ,$$

where $p^* = \frac{Np}{N-p}$ if $p < N$, and is any real number greater than $\frac{Ns(N-1)}{Ns-N-s}$ if $p = N$. Thus, we have

$$\left(\int_\Omega |G_k(u_n)|^{p^*} \, dx \right)^{\frac{p}{p^*}} \leq c \operatorname{meas}(A_{k,n})^{1-\frac{p'}{s}} .$$

Choosing $h > k$, and using the fact that $|G_k(u_n)| \geq h - k$ on $A_{h,n}$, we have

$$(h-k)^p \operatorname{meas}(A_{h,n})^{\frac{p}{p^*}} \leq c \operatorname{meas}(A_{k,n})^{1-\frac{p'}{s}} , \qquad \forall h > k \geq \sigma ,$$

that is

$$\operatorname{meas}(A_{h,n}) \leq \frac{c}{(h-k)^{p^*}} \operatorname{meas}(A_{k,n})^{\frac{p^*}{p}\left(1-\frac{p'}{s}\right)} , \qquad \forall h > k \geq \sigma .$$

Since, by our choice of p^* and s, we have

$$\frac{p^*}{p}\left(1 - \frac{p'}{s}\right) > 1 ,$$

a well-known result by G. Stampacchia (see [22], Lemme 4.1) implies that there exists a constant M (independent on n), such that

$$\operatorname{meas}(A_{k,n}) = 0 , \qquad \forall k \geq \sigma + M ,$$

that is to say,

$$\|u_n\|_{L^\infty(\Omega)} \leq \sigma + M .$$

This fact concludes the proof of the theorem. ∎

The next result is a decomposition theorem for measures in $\mathcal{M}_{\mathrm{b}}(\Omega)$, proved in [6], Theorem 2.1.

Theorem 2.3 *Let p be a real number such that $1 < p < +\infty$. Let μ be an element of $\mathcal{M}_{\mathrm{b}}(\Omega)$. Then $\mu \in L^1(\Omega) + W^{-1,p'}(\Omega)$ if and only if $\mu \in \mathcal{M}_0^p(\Omega)$.*

Finally, we give a technical lemma, whose proof is straightforward.

Lemma 2.4 *Let a and b be two nonnegative real numbers, and let*

$$\varphi(s) = s\,\mathrm{e}^{\theta\,s^2}\,,$$

with $\theta = \frac{b^2}{4a^2}$. Then

$$a\,\varphi'(s) - b\,|\varphi(s)| \geq \frac{a}{2}\,, \qquad \forall s \in \boldsymbol{R}\,. \tag{2.4}$$

Using the previous results, we can prove Theorem 1.1.

Proof of Theorem 1.1. It is clear that if there exists a solution u of (1.7), with u in $W_0^{1,p}(\Omega)$ and $g(x,u,\nabla u)$ in $L^1(\Omega)$, then, since $a(x,\nabla u)$ belongs to $(L^{p'}(\Omega))^N$ thanks to assumption (1.3), we have that μ belongs to $L^1(\Omega) + W^{-1,p'}(\Omega)$ and so is in $\mathcal{M}_0^p(\Omega)$ by Theorem 2.3.

On the other hand, suppose that μ belongs to $\mathcal{M}_0^p(\Omega)$. Thus, by Theorem 2.3, μ can be decomposed as $f - \mathrm{div}(F)$, with $f \in L^1(\Omega)$ and F in $(L^{p'}(\Omega))^N$.

Let $\{f_n\}$ be a sequence of $L^\infty(\Omega)$ functions that converges to f strongly in $L^1(\Omega)$, and let $\{F_n\}$ be a sequence of $(L^\infty(\Omega))^N$ functions that converges to F strongly in $(L^{p'}(\Omega))^N$.

By the result of Theorem 2.1, there exists a solution u_n of

$$\begin{cases} A(u_n) + g(x,u_n,\nabla u_n) = f_n - \mathrm{div}(F_n) & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.5}$$

in the sense that u_n belongs to $W_0^{1,p}(\Omega)$, $g(x,u_n,\nabla u_n)$ belongs to $L^1(\Omega)$, and

$$\int_\Omega a(x,\nabla u_n)\cdot\nabla v\,dx + \int_\Omega g(x,u_n,\nabla u_n)\,v\,dx = \int_\Omega f_n\,v\,dx + \int_\Omega F_n\cdot\nabla v\,dx\,, \tag{2.6}$$

for every $v \in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$.

Let us choose

$$v = \varphi(T_\sigma(u_n))\,,$$

as test function in (2.6), where σ is given by (1.6), and $\varphi(s)$ is as in Lemma 2.4, with $a = \frac{\alpha}{2}$ and $b = b(\sigma)$ ($b(s)$ is given by (1.5)).

Using (1.2), the Young inequality, and setting $\varphi'_\sigma = \varphi'(T_\sigma(u_n))$ and $\varphi_\sigma = \varphi(T_\sigma(u_n))$ for the sake of simplicity, we obtain

$$\begin{aligned}\alpha \int_\Omega |\nabla T_\sigma(u_n)|^p \, \varphi'_\sigma \, dx &+ \int_\Omega \varphi_\sigma \, g(x, u_n, \nabla u_n) \, dx \\ &\le \varphi(\sigma) \int_\Omega |f_n| \, dx + \varphi'(\sigma) \int_\Omega |F_n|^{p'} \, dx + \frac{\alpha}{2} \int_\Omega |\nabla T_k(u_n)|^p \, \varphi'_\sigma \, dx \,,\end{aligned}$$

so that, since $\{f_n\}$ is bounded in $L^1(\Omega)$, and $\{|F_n|\}$ is bounded in $L^{p'}(\Omega)$,

$$\frac{\alpha}{2} \int_\Omega |\nabla T_k(u_n)|^p \, \varphi'_\sigma \, dx + \int_\Omega \varphi_\sigma \, g(x, u_n, \nabla u_n) \, dx \le c \, (\varphi(\sigma) + \varphi'(\sigma)) \,. \tag{2.7}$$

We now have

$$\begin{aligned}\int_\Omega \varphi_\sigma \, g(x, u_n, \nabla u_n) \, dx &= \int_{\{|u_n| < \sigma\}} \varphi_\sigma \, g(x, u_n, \nabla u_n) \, dx \\ &\quad + \int_{\{|u_n| \ge \sigma\}} \varphi_\sigma \, g(x, u_n, \nabla u_n) \, dx \,.\end{aligned}$$

Using (1.5), we have

$$\left| \int_{\{|u_n| < \sigma\}} \varphi_\sigma \, g(x, u_n, \nabla u_n) \, dx \right| \le b(\sigma) \left(\int_\Omega |\nabla T_\sigma(u_n)|^p \, \varphi_\sigma \, dx + \varphi(\sigma) \, \|d\|_{L^1(\Omega)} \right) \,,$$

while, using (1.6), we get

$$\int_{\{|u_n| \ge \sigma\}} \varphi_\sigma \, g(x, u_n, \nabla u_n) \, dx \ge \rho \, \varphi(\sigma) \int_{\{|u_n| \ge \sigma\}} |\nabla u_n|^p \, dx \,.$$

We thus obtain

$$\int_\Omega |\nabla T_\sigma(u_n)|^p \left[\frac{\alpha}{2} \, \varphi'_\sigma - b(\sigma) |\varphi_\sigma| \right] \, dx + \rho \, \varphi(\sigma) \int_{\{|u_n| \ge \sigma\}} |\nabla u_n|^p \, dx \le c \,,$$

so that, by (2.4),

$$\int_\Omega |\nabla T_\sigma(u_n)|^p \, dx + \int_{\{|u_n| \ge \sigma\}} |\nabla u_n|^p \, dx \le c \,.$$

This proves that $\{u_n\}$ is bounded in $W_0^{1,p}(\Omega)$; hence, there exist a function $u \in W_0^{1,p}(\Omega)$ and a subsequence, still denoted by $\{u_n\}$, such that u_n converges to u weakly in $W_0^{1,p}(\Omega)$ and almost everywhere in Ω.

The next step of the proof is the strong convergence of u_n to u in $W_0^{1,p}(\Omega)$. We begin proving that we have

$$\lim_{k\to+\infty} \sup_{n\in N} \int_{\{|u_n|\geq k\}} |\nabla u_n|^p \, dx = 0\,. \tag{2.8}$$

We choose $v = \psi_{k-1}(u_n)$ as test function in (2.6), where

$$\psi_{k-1}(s) = T_1(G_{k-1}(s))\,,$$

and $k \geq \sigma + 1$. We get, using again (1.2), and the Young inequality,

$$\begin{aligned} &\alpha \int_\Omega |\nabla \psi_{k-1}(u_n)|^p \, dx + \int_\Omega \psi_{k-1}(u_n)\, g(x, u_n, \nabla u_n)\, dx \\ &\quad \leq \int_{\{|u_n|\geq k-1\}} |f_n| \, dx + \int_{\{k-1\leq|u_n|\leq k\}} |F_n|^{p'} \, dx + \frac{\alpha}{2} \int_\Omega |\nabla \psi_{k-1}(u_n)|^p \, dx\,. \end{aligned}$$

Using (1.6), the fact that $\psi_{k-1}(s)$ has the same sign as s, and that it is zero if $|s| \leq \sigma$, we have

$$g(x, u_n, \nabla u_n)\, \psi_{k-1}(u_n) \geq |g(x, u_n, \nabla u_n)|\, \chi_{\{|u_n|\geq k\}}\,,$$

so that we obtain, dropping positive terms,

$$\int_{\{|u_n|\geq k\}} |g(x, u_n, \nabla u_n)| \, dx \leq \int_{\{|u_n|\geq k-1\}} |f_n| \, dx + \int_{\{k-1\leq|u_n|\leq k\}} |F_n|^{p'} \, dx\,.$$

Since $\{u_n\}$ is bounded in $L^1(\Omega)$, we have

$$\lim_{k\to+\infty} \sup_{n\in N} \operatorname{meas}(\{|u_n| \geq k-1\}) = 0\,.$$

Thus, since f_n is strongly compact in $L^1(\Omega)$, and $|F_n|$ is strongly compact in $L^{p'}(\Omega)$, we have

$$\lim_{k\to+\infty} \sup_{n\in N} \int_{\{|u_n|\geq k-1\}} |f_n| \, dx + \int_{\{k-1\leq|u_n|\leq k\}} |F_n|^{p'} \, dx = 0\,,$$

and so

$$\lim_{k\to+\infty} \sup_{n\in N} \int_{\{|u_n|\geq k\}} |g(x, u_n, \nabla u_n)| \, dx = 0\,. \tag{2.9}$$

Using the fact that $k \geq \sigma$ and (1.6), (2.9) implies (2.8).

The next step is the proof that for every $k \geq \sigma$, the sequence $T_k(u_n)$ converges strongly to $T_k(u)$ in $W_0^{1,p}(\Omega)$. In order to do this, we shall closely follow the outline of [5]. Let $k \geq \sigma$ be fixed, and choose as test function in (2.6)

$$v = \varphi(T_k(u_n) - T_k(u))\,,$$

where $\varphi(s)$ is as in Lemma 2.4, with $a = 1$ and $b = b(k)/\alpha$. Such a function is admissible since it belongs to $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$. We thus have

$$\int_\Omega a(x, \nabla u_n) \cdot \nabla(T_k(u_n) - T_k(u))\, \varphi'(T_k(u_n) - T_k(u))\, dx \tag{A}$$

$$+ \int_\Omega g(x, u_n \nabla u_n)\, \varphi(T_k(u_n) - T_k(u))\, dx \tag{B}$$

$$= \int_\Omega f_n\, \varphi(T_k(u_n) - T_k(u))\, dx \tag{C}$$

$$+ \int_\Omega F_n \cdot \nabla(T_k(u_n) - T_k(u))\, \varphi'(T_k(u_n) - T_k(u))\, dx\,. \tag{D}$$

Here and in the following, we will denote by ε_n any quantity which converges to zero as n tends to infinity.

In order to deal with (A)–(D), we will set, for the sake of simplicity,

$$\varphi_n' = \varphi'(T_k(u_n) - T_k(u))\,, \qquad \varphi_n = \varphi(T_k(u_n) - T_k(u))\,,$$

and we will often use the fact that, in the weak$*$ topology of $L^\infty(\Omega)$, and almost everywhere in Ω, we have

$$\lim_{n \to +\infty} \varphi_n = \varphi(0) = 0\,, \qquad \lim_{n \to +\infty} \varphi_n' = \varphi'(0) = 1\,. \tag{2.10}$$

Since f_n is strongly compact in $L^1(\Omega)$, (2.10) then implies

$$(C) = \varepsilon_n\,. \tag{2.11}$$

Moreover, since $T_k(u_n)$ converges to $T_k(u)$ weakly in $W_0^{1,p}(\Omega)$, and F_n is strongly compact in $(L^{p'}(\Omega))^N$, (2.10) implies

$$(D) = \varepsilon_n\,. \tag{2.12}$$

We can then decompose (A) as

$$\int_\Omega a(x, T_k(u_n)) \cdot \nabla(T_k(u_n) - T_k(u))\, \varphi_n'\, dx \tag{E}$$

$$+ \int_\Omega a(x, G_k(u_n)) \cdot \nabla(T_k(u_n) - T_k(u))\, \varphi_n'\, dx\,. \tag{F}$$

Since $\nabla T_k(u_n)$ is zero where $\nabla G_k(u_n)$ is different from zero, and conversely, we have

$$(F) = -\int_\Omega a(x, G_k(u_n)) \cdot \nabla T_k(u)\, \varphi_n'\, dx\,.$$

Since $\nabla T_k(u) \equiv 0$ on the set $\{|u| \geq k\}$, we have that

$$\nabla T_k(u)\, \chi_{\{|u_n|\geq k\}} \to 0\,, \quad \text{almost everywhere in } \Omega.$$

In view of the fact that $\nabla T_k(u)$ belongs to $(L^p(\Omega))^N$, the Lebesgue theorem implies

$$\nabla T_k(u)\, \chi_{\{|u_n|\geq k\}} \to 0\,, \quad \text{strongly in } (L^p(\Omega))^N,$$

and so, since $a(x, \nabla G_k(u_n))$ is bounded in $(L^{p'}(\Omega))^N$ by (1.3), we have

$$(F) = \varepsilon_n\,. \tag{2.13}$$

As for (E), we can decompose it as

$$\int_\Omega [a(x, \nabla T_k(u_n)) - a(x, \nabla T_k(u))] \cdot \nabla (T_k(u_n) - T_k(u))\, \varphi_n'\, dx \tag{G}$$
$$+ \int_\Omega a(x, \nabla T_k(u)) \cdot \nabla (T_k(u_n) - T_k(u))\, \varphi_n'\, dx\,. \tag{H}$$

Since $T_k(u_n)$ converges to $T_k(u)$ weakly in $W_0^{1,p}(\Omega)$, and since $a(x, \nabla T_k(u))$ belongs to $(L^{p'}(\Omega))^N$ by (1.3), (2.10) then implies

$$(H) = \varepsilon_n\,. \tag{2.14}$$

Thus, putting together (2.13) and (2.14), we have

$$\begin{aligned}(A) = &\int_\Omega [a(x, \nabla T_k(u_n)) - a(x, \nabla T_k(u))] \cdot \nabla (T_k(u_n) - T_k(u))\, \varphi_n'\, dx \\ &+\varepsilon_n\,.\end{aligned} \tag{2.15}$$

We deal now with (B), and we decompose it as

$$\int_{\{|u_n|\geq k\}} g(x, u_n, \nabla u_n)\, \varphi_n\, dx \tag{I}$$
$$+ \int_{\{|u_n|<k\}} g(x, u_n, \nabla u_n)\, \varphi_n\, dx\,. \tag{J}$$

Using the fact that $k \geq \sigma$, assumption (1.6), and observing that, on the set $\{u_n \geq k\}$ we have

$$\varphi_n = \varphi(k - T_k(u)) \geq 0\,,$$

while on the set $\{u_n \leq -k\}$ we have

$$\varphi_n = \varphi(-k - T_k(u)) \leq 0\,,$$

we obtain

$$(I) \geq 0\,,$$

so that we can drop it. Using (1.5), we have

$$|(J)| \leq b(k) \int_\Omega d(x)\,|\varphi_n|\,dx + b(k) \int_\Omega |\nabla T_k(u_n)|^p\,|\varphi_n|\,dx\,,$$

where b is the function given in (1.5). Since d belongs to $L^1(\Omega)$, using (2.10) we have

$$\int_\Omega d(x)\,|\varphi_n|\,dx = \varepsilon_n\,,$$

so that, using (1.2), we have

$$|(J)| \leq \frac{b(k)}{\alpha} \int_\Omega a(x, \nabla T_k(u_n)) \cdot \nabla T_k(u_n)\,|\varphi_n|\,dx + \varepsilon_n\,.$$

Now we add and subtract to the above inequality the term

$$\int_\Omega a(x, \nabla T_k(u)) \cdot \nabla(T_k(u_n) - \nabla T_k(u))\,|\varphi_n|\,dx\,,$$

which converges to zero as n tends to infinity since $T_k(u_n)$ converges to $T_k(u)$ weakly in $W_0^{1,p}(\Omega)$ and $a(x, \nabla T_k(u))$ belongs to $(L^{p'}(\Omega))^N$), and the term

$$\int_\Omega a(x, \nabla T_k(u_n)) \cdot \nabla T_k(u)\,|\varphi_n|\,dx\,,$$

which is an ε_n since $a(x, \nabla T_k(u_n))$ is bounded in $(L^{p'}(\Omega))^N$ and since φ_n converges to zero by (2.10). We thus get

$$|(J)| \leq \frac{b(k)}{\alpha} \int_\Omega [a(x, \nabla T_k(u_n)) - a(x, \nabla T_k(u))] \cdot \nabla(T_k(u_n) - T_k(u))\,|\varphi_n|\,dx$$
$$+\varepsilon_n\,.$$

Putting together this inequality and (2.15) we thus obtain that

$$\int_\Omega [a(x, \nabla T_k(u_n)) - a(x, \nabla T_k(u))] \cdot \nabla(T_k(u_n) - T_k(u))[\varphi_n' - \frac{b(k)}{\alpha}|\varphi_n|]\, dx$$

is an ε_n. Hence, by (2.4),

$$\int_\Omega [a(x, \nabla T_k(u_n)) - a(x, \nabla T_k(u))] \cdot \nabla(T_k(u_n) - T_k(u))\, dx = \varepsilon_n\,.$$

This fact and the assumptions on a imply, by a result in [8] (see also [14]), that $T_k(u_n)$ converges strongly to $T_k(u)$ in $W_0^{1,p}(\Omega)$.

Let now E be a measurable subset of Ω. Then

$$\int_E |\nabla u_n|^p\, dx = \int_{E\cap\{|u_n|\le k\}} |\nabla u_n|^p\, dx + \int_{E\cap\{|u_n|>k\}} |\nabla u_n|^p\, dx\,.$$

Let $\varepsilon > 0$ be fixed. Since

$$\int_{E\cap\{|u_n|>k\}} |\nabla u_n|^p\, dx \le \int_{\{|u_n|>k\}} |\nabla u_n|^p\, dx\,,$$

(2.8) implies that there exists $k \ge \sigma$ such that

$$\int_{E\cap\{|u_n|>k\}} |\nabla u_n|^p\, dx \le \frac{\varepsilon}{2}\,, \qquad \forall n \in \boldsymbol{N}.$$

Once k is fixed, and since we have

$$\int_{E\cap\{|u_n|\le k\}} |\nabla u_n|^p\, dx \le \int_E |\nabla T_k(u_n)|^p\, dx\,,$$

the strong compactness of $T_k(u_n)$ in $W_0^{1,p}(\Omega)$ implies that there exists $\delta > 0$ such that if $\operatorname{meas}(E) < \delta$, we have

$$\int_{E\cap\{|u_n|\le k\}} |\nabla u_n|^p\, dx \le \frac{\varepsilon}{2}\,, \qquad \forall n \in \boldsymbol{N}.$$

Thus, for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $\operatorname{meas}(E) < \delta$, then

$$\int_E |\nabla u_n|^p\, dx \le \varepsilon\,, \qquad \forall n \in \boldsymbol{N},$$

that is to say, the sequence $\{|\nabla u_n|^p\}$ is equi-integrable. Since, up to a subsequence still denoted by u_n, ∇u_n is almost everywhere convergent to ∇u (as a consequence of the strong convergence of truncates), then u_n converges strongly to u in $W_0^{1,p}(\Omega)$.

In order to pass to the limit in the approximate equation, we are going to prove that

$$g(x,u_n,\nabla u_n) \to g(x,u,\nabla u) \qquad \text{strongly in } L^1(\Omega).$$

Since $g(x,u_n,\nabla u_n)$ converges almost everywhere to $g(x,u,\nabla u)$, it remains to prove the equi-integrability of the sequence $\{|g(x,u_n,\nabla u_n)|\}$. Once again, if E is a measurable subset of Ω, we have

$$\begin{aligned}\int_E |g(x,u_n,\nabla u_n)|\,dx &= \int_{E\cap\{|u_n|\le k\}} |g(x,u_n,\nabla u_n)|\,dx \\ &\quad + \int_{E\cap\{|u_n|>k\}} |g(x,u_n,\nabla u_n)|\,dx\,.\end{aligned}$$

Let $\varepsilon > 0$ be fixed. Since

$$\int_{E\cap\{|u_n|>k\}} |g(x,u_n,\nabla u_n)|\,dx \le \int_{\{|u_n|>k\}} |g(x,u_n,\nabla u_n)|\,dx\,,$$

we can use (2.9) in order to choose $k \ge \sigma$ such that

$$\int_{E\cap\{|u_n|>k\}} |g(x,u_n,\nabla u_n)|\,dx \le \frac{\varepsilon}{2}\,, \qquad \forall n \in \boldsymbol{N}.$$

Moreover, using (1.3), we have

$$\int_{E\cap\{|u_n|\le k\}} |g(x,u_n,\nabla u_n)|\,dx \le b(k) \int_E [d(x) + |\nabla T_k(u_n)|^p]\,dx\,,$$

and, since d belongs to $L^1(\Omega)$, and $T_k(u_n)$ is strongly compact in $W_0^{1,p}(\Omega)$, there exists $\delta > 0$ such that if $\operatorname{meas}(E) < \delta$, then

$$\int_{E\cap\{|u_n|\le k\}} |g(x,u_n,\nabla u_n)|\,dx \le \frac{\varepsilon}{2}\,, \qquad \forall n \in \boldsymbol{N}.$$

Thus, as before, we have proved that $\{|g(x,u_n,\nabla u_n)|\}$ is equi-integrable, and this allow us to pass to the limit in (2.6), in order to obtain (1.8). ∎

Remark 2.5 Since there exists a solution of (1.7) in the sense (1.8) if and only if μ belongs to $\mathcal{M}_0^p(\Omega)$, that is to say if and only if μ belongs to $L^1(\Omega) + W^{-1,p'}(\Omega)$, the class of admissible test functions in (1.8) can be extended by means of an easy density argument in order to consider test functions v in $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$.

Remark 2.6 The result of Theorem 1.1 refines the result of Theorem 2.3 in the sense that it states that if μ is a measure in $\mathcal{M}_0^p(\Omega)$, then there exists a function u in $W_0^{1,p}(\Omega)$ such that

$$\mu = A(u) + g(x, u, \nabla u)\,,$$

with $g(x, u, \nabla u)$ in $L^1(\Omega)$.

If, for example, $p = 2$, $a(x, \xi) = \xi$, and $g(x, s, \xi) = \operatorname{sgn}(s)\,|\xi|^2$, then for every nonnegative measure μ in $\mathcal{M}_0^2(\Omega)$ there exists a function u in $H_0^1(\Omega)$ such that

$$\mu = -\operatorname{div}(\nabla u) + |\nabla u|^2\,.$$

It is also easy to see, taking for instance $T_k(u^-)$ as test function in (1.8), that such a function u is nonnegative.

Remark 2.7 The same result of Theorem 1.1 holds true if the function a depends also on u, and satisfies the following assumptions:

$$a(x, s, \xi) \cdot \xi \geq \alpha\,|\xi|^p\,,$$
$$|a(x, s, \xi)| \leq \ell(x) + \beta\,[|s|^{p-1} + |\xi|^{p-1}]\,,$$
$$[a(x, s, \xi) - a(x, s, \eta)] \cdot (\xi - \eta) > 0\,,$$

for almost every $x \in \Omega$, for every s in $\boldsymbol{R}$, for every ξ, η in $\boldsymbol{R}^N$ ($\xi \neq \eta$), where α and β are positive constants, and ℓ belongs to $L^{p'}(\Omega)$.

3 A nonexistence result

We recall that if μ is a Radon measure, and E is a borelian subset of Ω, the *restriction* of μ to E is the measure $\lambda = \mu \llcorner E$ defined by

$$\lambda(B) = \mu(E \cap B)\,,$$

for every borelian subset B of Ω. We say that a measure λ is *concentrated* on a borelian set E if $\lambda = \lambda \llcorner E$.

We recall the following result, proved in [17], Lemma 2.1.

Proposition 3.1 *Let μ be a measure in $\mathcal{M}_{\mathrm{b}}(\Omega)$, and let $1 < p \leq N$. Then μ can be decomposed in a unique way as $\mu_0 + \lambda$, where*

1) μ_0 belongs to $\mathcal{M}_0^p(\Omega)$;

2) $\lambda = \mu \llcorner E$, and $\mathrm{cap}_p(E, \Omega) = 0$.

Given a measure μ in $\mathcal{M}_{\mathrm{b}}(\Omega)$, we can decompose it, by the previous Proposition, as $\mu_0 + \lambda$. The result of Theorem 1.1 then states that problem (1.7) has a solution with datum μ if and only if $\lambda = 0$. Suppose now that $\mu_0 = 0$, so that $\mu = \lambda$ is singular with respect to the p-capacity. What does it happen if we try (as in the proof of Theorem 1.1) to approximate the measure λ with a sequence $\{f_n\}$ of $L^\infty(\Omega)$ functions, and solve the corresponding problems with data f_n? If we take the sequence $\{f_n\}$ bounded in $L^1(\Omega)$, then the same technique used in the proof of Theorem 1.1 yields that the corresponding sequence of solutions $\{u_n\}$ is bounded in $W_0^{1,p}(\Omega)$; hence, it converges weakly in $W_0^{1,p}(\Omega)$ towards some function u. Clearly, u cannot be a solution of the problem with datum λ, since such a solution does not exists. What can we say about u? The answer is given by the following theorem, under a sign condition on the lower order nonlinearity g.

Theorem 3.2 *Let λ be a positive measure in $\mathcal{M}_{\mathrm{b}}(\Omega)$, concentrated on a set E such that $\mathrm{cap}_p(E, \Omega) = 0$, and let $\{f_n\}$ be a sequence of nonnegative $L^\infty(\Omega)$ functions such that*

$$\lim_{n \to +\infty} \int_\Omega f_n \, \varphi \, dx = \int_\Omega \varphi \, d\lambda \qquad \forall \varphi \in C^0(\bar{\Omega}).$$

Suppose that, in addition to assumptions (1.5) and (1.6), g also satisfies

$$g(x, s, \xi) \, s \geq 0, \tag{3.1}$$

for almost every x in Ω, for every s in $\boldsymbol{R}$, and for every ξ in $\boldsymbol{R}^N$. Let u_n be a solution of the equation

$$\begin{cases} A(u_n) + g(x, u_n, \nabla u_n) = f_n & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.2}$$

in the sense (2.6) (written with $F \equiv 0$). Then there exists $k > 0$ (depending on g and α), such that

$$T_k(u_n) \to 0 \qquad \text{strongly in } W_0^{1,p}(\Omega).$$

Moreover, u_n converges weakly to zero in $W_0^{1,p}(\Omega)$, and

$$\lim_{n\to+\infty} \int_\Omega g(x, u_n, \nabla u_n)\, \varphi\, dx = \int_\Omega \varphi\, d\lambda \qquad \forall \varphi \in C_0^1(\Omega)\,.$$

In the following we will denote by ε_δ and $\varepsilon_{n,\delta}$ respectively any sequence of real numbers such that

$$\lim_{\delta\to 0^+} \varepsilon_\delta = 0\,, \qquad \lim_{\delta\to 0^+} \lim_{n\to+\infty} \varepsilon_{n,\delta} = 0\,.$$

Before giving the proof of the theorem, we need to construct as in [15] a suitable sequence of cut-off functions.

Lemma 3.3 *Let λ be a nonnegative measure in $\mathcal{M}_{\mathrm{b}}(\Omega)$ which is concentrated on a set E of zero p-capacity. Then there exists a sequence $\{\psi_\delta\}$ of $C_0^\infty(\Omega)$ functions such that*

$$\int_\Omega |\nabla \psi_\delta|^p\, dx = \varepsilon_\delta\,, \qquad 0 \le \psi_\delta \le 1\,, \qquad \int_\Omega (1-\psi_\delta)\, d\lambda = \varepsilon_\delta\,. \tag{3.3}$$

Proof. Since λ belongs to $\mathcal{M}_{\mathrm{b}}(\Omega)$, for every $\delta > 0$ there exists a compact set $K_\delta \subseteq E$ such that $\lambda(E\backslash K_\delta) \le \delta$. Since K_δ is compact, and its p-capacity is zero, there exists a $C_0^\infty(\Omega)$ function ψ_δ which is equal to 1 in a neighbourhood of K_δ and that satisfies the first and the second of (3.3). As for the third, we have

$$0 \le \int_\Omega (1-\psi_\delta)\, d\lambda = \int_{E\backslash K_\delta} (1-\psi_\delta)\, d\lambda \le \lambda(E\backslash K_\delta) \le \delta\,,$$

and this completes the proof. ∎

Remark 3.4 If E is compact, we can choose $K_\delta = E$ for every $\delta > 0$. As a consequence of (3.3), we have that ψ_δ converges to zero both strongly in $W_0^{1,p}(\Omega)$, almost everywhere in Ω, and in the weak$*$ topology of $L^\infty(\Omega)$.

Proof of Theorem 3.2. Since f_n is nonnegative, the sign assumption (3.1) on g implies that also u_n is nonnegative. As in the proof of Theorem 1.1 it is easy to see, using the assumptions on a and g, that the sequence $\{u_n\}$ is bounded in $W_0^{1,p}(\Omega)$. Thus, up to subsequences, still denoted by u_n, there exists a function u in $W_0^{1,p}(\Omega)$ and an element G in $(L^{p'}(\Omega))^N$ such that

$$u_n \to u \qquad \text{weakly in } W_0^{1,p}(\Omega) \text{ and almost everywhere in } \Omega,$$

$$a(x, \nabla u_n) \to G \qquad \text{weakly in } (L^{p'}(\Omega))^N;$$

the latter convergence is due to the fact that, by (1.3), $a(x, \nabla u_n)$ is bounded in $(L^{p'}(\Omega))^N$.

Let $k > 0$ be such that

$$b(k)\, k \leq \frac{\alpha}{2}\,, \tag{3.4}$$

where b is the function that appears in (1.5). Such a k exists since b is continuous.

We choose now as test function in (2.6) the function

$$v = (k - T_k(u_n))\, \psi_\delta\,,$$

which is admissible since it belongs to $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$. We obtain:

$$- \int_\Omega a(x, \nabla T_k(u_n)) \cdot \nabla T_k(u_n)\, \psi_\delta\, dx \tag{A}$$

$$+ \int_\Omega [a(x, \nabla u_n) \cdot \nabla \psi_\delta]\, (k - T_k(u_n))\, dx \tag{B}$$

$$+ \int_\Omega g(x, u_n, \nabla u_n)\, (k - T_k(u_n))\, \psi_\delta\, dx \tag{C}$$

$$= \int_\Omega f_n\, (k - T_k(u_n))\, \psi_\delta\, dx\,. \tag{D}$$

Since $k - T_k(u_n)$ converges to $k - T_k(u)$ both in the weak$*$ topology of $L^\infty(\Omega)$ and almost everywhere in Ω, we have that $\nabla \psi_\delta\, (k - T_k(u_n))$ converges to $\nabla \psi_\delta (k - T_k(u))$ strongly in $(L^p(\Omega))^N$, and so

$$(B) = \int_\Omega G \cdot \nabla \psi_\delta\, (k - T_k(u))\, dx + \varepsilon_n = \varepsilon_{n,\delta}\,, \tag{3.5}$$

where the last passage is due to the first of (3.3). We then have, by (1.5), and since $k - T_k(u_n) = 0$ for $u_n > k$,

$$|(C)| \leq \int_{\{0 \leq u_n \leq k\}} b(u_n)\, (k - T_k(u_n))\, \psi_\delta\, [d(x) + |\nabla u_n|^p]\, dx\,.$$

Thus,

$$\int_{\{0\le u_n\le k\}} b(u_n)\,(k-T_k(u_n))\,\psi_\delta\, d(x)\,dx \le \frac{\alpha}{2}\int_\Omega d(x)\,\psi_\delta\,dx = \varepsilon_\delta\,,$$

by the choice of k, and since ψ_δ converges to zero in the weak$*$ topology of $L^\infty(\Omega)$ as a consequence of (3.3). Moreover, again by the choice of k,

$$\int_{\{0\le u_n\le k\}} b(u_n)\,(k-T_k(u_n))\,\psi_\delta\,|\nabla u_n|^p\,dx \le \frac{\alpha}{2}\int_\Omega |\nabla T_k(u_n)|^p\,\psi_\delta\,dx\,.$$

We thus have

$$|(B)| \le \frac{\alpha}{2}\int_\Omega |\nabla T_k(u_n)|^p\,\psi_\delta\,dx + \varepsilon_\delta\,.$$

Using (1.2) we also have

$$(A) \ge -\alpha\int_\Omega |\nabla T_k(u_n)|^p\,\psi_\delta\,dx\,.$$

Thus, since (D) is nonnegative, we have

$$\alpha\int_\Omega |\nabla T_k(u_n)|^p\,\psi_\delta\,dx \le \frac{\alpha}{2}\int_\Omega |\nabla T_k(u_n)|^p\,\psi_\delta\,dx + \varepsilon_{n,\delta}\,,$$

that is to say

$$\int_\Omega |\nabla T_k(u_n)|^p\,\psi_\delta\,dx = \varepsilon_{n,\delta}\,. \tag{3.6}$$

The next step consists in choosing as test function in (2.6) the function

$$v = T_k(u_n)(1-\psi_\delta)\,,$$

which is again admissible. We get

$$\int_\Omega a(x,\nabla T_k(u_n))\cdot\nabla T_k(u_n)\,(1-\psi_\delta)\,dx \tag{E}$$

$$-\int_\Omega [a(x,\nabla u_n)\cdot\nabla\psi_\delta]\,T_k(u_n)\,dx \tag{F}$$

$$+\int_\Omega g(x,u_n,\nabla u_n)\,T_k(u_n)\,(1-\psi_\delta)\,dx \tag{G}$$

$$= \int_\Omega f_n\,T_k(u_n)\,(1-\psi_\delta)\,dx\,. \tag{H}$$

The term (G) is nonnegative by (3.1), and so we drop it. Reasoning as before, we have

$$(F) = \varepsilon_{n,\delta}\,,$$

while we have

$$(H) \leq k \int_\Omega f_n\,(1-\psi_\delta)\,dx = k \int_\Omega (1-\psi_\delta)\,d\lambda + \varepsilon_n = \varepsilon_{n,\delta}\,,$$

by the third of (3.3). Thus, using (1.2) in (E), we get

$$\alpha \int_\Omega |\nabla T_k(u_n)|^p\,(1-\psi_\delta)\,dx = \varepsilon_{n,\delta}\,. \tag{3.7}$$

Putting together (3.6) and (3.7) we easily obtain

$$\int_\Omega |\nabla T_k(u_n)|^p\,dx = \varepsilon_n\,,$$

that is to say, $T_k(u_n)$ converges strongly to zero in $W_0^{1,p}(\Omega)$. Since the limit is independent of the choice of the subsequence, the whole sequence $\{u_n\}$ is such that $T_k(u_n)$ converges to zero strongly in $W_0^{1,p}(\Omega)$. Thus, $u = 0$, and so u_n converges weakly to zero in $W_0^{1,p}(\Omega)$.

In order to prove the second part of the theorem, we observe that from the strong convergence to zero of $T_k(u_n)$ follows the almost everywhere convergence to zero of ∇u_n, and this implies that $G \equiv 0$. Now we choose a test function φ in $C_0^1(\Omega)$ in (2.6); we obtain

$$\int_\Omega a(x,\nabla u_n)\cdot\nabla\varphi\,dx + \int_\Omega g(x,u_n,\nabla u_n)\,\varphi\,dx = \int_\Omega f_n\,\varphi\,dx\,. \tag{3.8}$$

Since $G = 0$, we have

$$\int_\Omega a(x,\nabla u_n)\cdot\nabla\varphi\,dx = \varepsilon_n\,,$$

while

$$\int_\Omega f_n\,\varphi\,dx = \int_\Omega \varphi\,d\lambda + \varepsilon_n\,.$$

Thus, from (3.8) we obtain by difference,

$$\lim_{n\to+\infty} \int_\Omega g(x,u_n,\nabla u_n)\,\varphi\,dx = \int_\Omega \varphi\,d\lambda\,,$$

for every φ in $C_0^1(\Omega)$, and this concludes the proof of the theorem. ∎

Remark 3.5 The result of the previous theorem also holds, with minor technical modifications in the proof, if λ is a signed measure.

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