Finite volumes and nonlinear diffusion equations.

R. Eymard[1], T. Gallouët[2], D. Hilhorst[3] and Y. Naït Slimane[4]

Abstract.

In this paper we prove the convergence of a finite volume scheme to the solution of a Stefan problem, namely the nonlinear diffusion equation $u_t - \Delta\varphi(u) = v$, together with a homogeneous Neumann boundary condition and an initial condition. This is done by means of a priori estimates in L^∞ and use of Kolmogorov's theorem on relative compactness of subsets of L^2.

1 Introduction.

In this paper we prove the convergence of explicit and implicit finite volume schemes for the numerical solution of the Stefan-type problem

$$u_t(x,t) - \Delta\varphi(u(x,t)) = v(x,t), \quad \text{for all } (x,t) \in \Omega \times \mathbb{R}_+^*, \tag{1}$$

together with the homogeneous Neumann boundary condition

$$\frac{\partial\varphi(u)}{\partial n}(x,t) = 0, \quad \text{for all } (x,t) \in \partial\Omega \times \mathbb{R}_+^*, \tag{2}$$

and the initial condition

$$u(x,0) = u_0(x), \quad \text{for all } x \in \Omega. \tag{3}$$

We suppose that the following hypotheses are satisfied:

$$\left.\begin{array}{ll} \text{(i)} & \Omega \text{ is a bounded open subset of } \mathbb{R}^N, \text{ with smooth boundary } \partial\Omega, \\ \text{(ii)} & \varphi \in C(\mathbb{R}) \text{ is a non decreasing locally Lipschitz continuous function,} \\ \text{(iii)} & u_0 \in L^\infty(\Omega), \\ \text{(iv)} & v \in L^\infty(\Omega \times (0,T)), \quad \text{for all } T > 0. \end{array}\right\} \tag{4}$$

Remark 1.1 *The cases of the Stefan problem and of the porous medium equations are both contained in the hypothesis (4.ii). For example, every function φ, which is constant in an interval and linearly increasing outside that interval, satisfies the hypothesis (4.ii).*

Equation (1) is a degenerate parabolic equation. Therefore it is useful to give a definition of a weak solution u to Problem (1, 2, 3).

Definition 1.1 *A measurable function u is a weak solution of (1, 2, 3) if*

[1] Laboratoire Central des Ponts et Chaussées, 58 bd Lefebvre, 75015 Paris and Université Paris Nord, France
[2] E.N.S. Lyon, 46 allée d'Italie, 69364 Lyon, France
[3] CNRS and Laboratoire d'Analyse Numérique, Université Paris-Sud (bât. 425), 91405 Orsay Cedex, France
[4] Université Paris Nord, Institut Galilée 93340 Villetaneuse, France

$$
\left.\begin{array}{l}
u \in L^{\infty}(\Omega \times (0,T)), \quad \textit{for all } T > 0, \\[2ex]
\displaystyle\int_0^T \int_\Omega \Big(u(x,t)\psi_t(x,t) + \varphi(u(x,t))\Delta\psi(x,t) + v(x,t)\psi(x,t) \Big)\ dx\ dt\ + \\
\displaystyle\int_\Omega u_0(x)\psi(x,0)dx = 0, \quad \textit{for all } T > 0, \quad \textit{for all } \psi \in \mathcal{A}_T,
\end{array}\right\} \tag{5}
$$

where $\mathcal{A}_T = \{\psi \in C^{2,1}(\overline{\Omega} \times [0,T]),\ \dfrac{\partial \psi}{\partial n} = 0$ *on* $\partial\Omega \times [0,T]$*, and* $\psi(.,T) = 0\}$.

The convergence of numerical schemes to the weak solution of Problem (1, 2, 3) has been proved by several authors:
(i) A finite difference scheme has been used by [9] to show the existence of a solution to the Stefan problem. Similar finite difference schemes were used by [2] and [12]. These authors show the convergence of the scheme.
(ii) Convergence proofs for finite element schemes have been proposed by [13], [15], [6] and [1].
(iii) The framework of semigroup theory has been used by [4] to prove the convergence of a time implicit scheme, and by [3] for the study of a "co-volume method", which is a special case of a finite volume method.

Finite volume schemes have first been developed by engineers in order to study complex coupled physical phenomena where the conservation of extensive quantities (such as masses, energy, impulsion...) must be carefully respected by the approximate solution. Another advantage of such schemes is that a large variety of meshes can be used. The basic idea is the following : one integrates the partial differential equations in each control volume and then approximates the fluxes across the volume boundaries. In this paper we prove the convergence of an explicit and an implicit finite volume scheme to the weak solution of Problem (1, 2, 3). Note that the function u satisfies the conservation law

$$
\int_\Omega u(x,t)dx = \int_\Omega u_0(x)dx + \int_0^t \int_\Omega v(x,t)dxdt, \tag{6}
$$

for all $t \in [0,T]$. The approximate solution computed by the finite volume method exactly satisfies a discrete analog of equality (6).
A proof of convergence for a stationary diffusive-convective problem is given by [8]. As far as we know, this article gives the first convergence proof in the case that a finite volume scheme on a general mesh is used for the space discretization of a degenerate parabolic equation. Our method is based on rather simple a priori estimates which are discrete versions of continuous estimates. It could certainly be extended to a large class of linear and semilinear parabolic equations.
We present the proofs in the case of the explicit scheme and show in several remarks how they can be extended to the case of the implicit scheme (which is easier to study). As in [6], a functional convergence property, which is proved here in a general setting, is being used. For the sake of completeness we recall in the appendix the proof of uniqueness of the weak solution of Problem (1, 2, 3) for the precise case that we consider.

2 Finite volume scheme for a nonlinear parabolic equation.

In this section, we construct approximate solutions to Problem (1, 2, 3). To this purpose, we introduce a time discretization and a finite volume space discretization. Let $\mathcal{T}$ be a mesh of Ω. The elements of $\mathcal{T}$ will be called control volumes in what follows. For any $(p,q) \in \mathcal{T}^2$ with $p \neq q$, we denote by $e_{pq} = \overline{p} \cap \overline{q}$ their common interface, which is supposed to be included in a hyperplane of $\mathbb{R}^N$, which does not intersect neither p nor q. Then $m(e_{pq})$ denotes the measure of e_{pq} for the Lebesgue measure of the hyperplane, and $\mathbf{n}_{pq}$ denotes the unit vector normal to e_{pq}, oriented from p to q. The set of pairs of adjacent control volumes is denoted by $\mathcal{E} = \{(p,q) \in \mathcal{T}^2, p \neq q, m(e_{pq}) \neq 0\}$, and for all $p \in \mathcal{T}$, $N(p) = \{q \in \mathcal{T}, (p,q) \in \mathcal{E}\}$ denotes the set of neighbors of p. We assume that there exist $h > 0$ and $x_p \in p$, for all $p \in \mathcal{T}$, such that:

$$\left.\begin{array}{ll}\text{(i)} & \delta(p) \leq h, \quad \text{for all } p \in \mathcal{T}, \\ & \\ \text{(ii)} & \dfrac{x_q - x_p}{|x_q - x_p|} = \mathbf{n}_{pq}, \text{ for all } (p,q) \in \mathcal{E},\end{array}\right\} \tag{7}$$

where $\delta(p)$ denotes the diameter of control volume p and $m(p)$ its measure in $\mathbb{R}^N$. We denote by $d_{pq} = |x_q - x_p|$ the euclidian distance between x_p and x_q, and we then set $T_{pq} = \dfrac{m(e_{pq})}{d_{pq}}$.

Remark 2.1 *For any domain Ω with smooth boundary $\partial\Omega$, it is possible to build meshes which satisfy the previous hypotheses. For example, let us consider, for any $h > 0$,*
$X_h = \{(\frac{k_1 h}{2\sqrt{N}}, \frac{k_2 h}{2\sqrt{N}}, \dots \frac{k_N h}{2\sqrt{N}}), k_1, k_2, \dots k_N \in \mathbb{Z}\} \cap \Omega$; *$X_h$ is a finite subset of Ω. For all $x \in X_h$, we define:*

$$p_x = \{y \in \Omega, |y - x| < \min_{z \in X_h, z \neq x} |y - z|\}. \tag{8}$$

We then note that, for h small enough, $\mathcal{T} = \{p_x, x \in X_h\}$ verifies the hypotheses (7).

Remark 2.2 *Another example of a mesh which satisfies the hypotheses (7) is the following. If $N = 2$ and if $\mathcal{T}$ is the dual mesh of a P^1 triangular finite element mesh, T_{pq} is an element of the rigidity matrix of an elliptic problem [3].*
However, in the general case, $\mathcal{T}$ cannot be seen as such a dual mesh.

The functions u_0, v and φ satisfying the hypotheses (4), the explicit finite volume scheme is then defined by the following equations, in which $k > 0$ denotes the time step.

(i) The initial condition for the scheme is

$$u_p^0 = \frac{1}{m(p)} \int_p u_0(x)dx, \text{ for all } p \in \mathcal{T}. \tag{9}$$

(ii) The source term is taken into account by defining values v_p^n such that

$$v_p^n = \frac{1}{km(p)} \int_{nk}^{(n+1)k} \int_p v(x,t)dxdt, \text{ for all } p \in \mathcal{T}, \text{ for all } n \in \mathbb{N}. \tag{10}$$

(iii) The explicit finite volume scheme is defined by

$$m(p)\frac{u_p^{n+1}-u_p^n}{k}-\sum_{q\in N(p)}T_{pq}\left(\varphi_q^n-\varphi_p^n\right)=m(p)v_p^n,\quad \text{for all } p\in\mathcal{T},\quad \text{for all } n\in\mathbb{N}, \tag{11}$$

where we set $\varphi_p^n=\varphi(u_p^n)$, for all $p\in\mathcal{T}$ and $n\in\mathbb{N}$. Equation (11) formally corresponds to integrating the equation (1) on the element $p\times(nk,(n+1)k)$ and defining a suitable approximation of the flux function across ∂p.
Scheme (11) allows to build an approximate solution, $u_{\mathcal{T},k}:\Omega\times\mathbb{R}^+\mapsto\mathbb{R}$ by

$$u_{\mathcal{T},k}(x,t)=u_p^n,\ \text{for all } x\in p,\ \text{for all } t\in[nk,(n+1)k). \tag{12}$$

We define in the same way the approximate $\varphi_{\mathcal{T},k}$ of $\varphi(u)$ by $\varphi_{\mathcal{T},k}(x,t)=\varphi(u_{\mathcal{T},k}(x,t))$, for all $(x,t)\in\Omega\times\mathbb{R}^+$.

Remark 2.3 *The implicit finite volume scheme is defined by*

$$m(p)\frac{u_p^{n+1}-u_p^n}{k}-\sum_{q\in N(p)}T_{pq}\left(\varphi_q^{n+1}-\varphi_p^{n+1}\right)=m(p)v_p^n,\quad \textit{for all } p\in\mathcal{T},\quad \textit{for all } n\in\mathbb{N}. \tag{13}$$

The proof of the existence of u_p^{n+1}, for any $n\in\mathbb{N}$, can be obtained using the following fixed point method:

$$u_p^{n+1,0}=u_p^n,\quad \textit{for all } p\in\mathcal{T}, \tag{14}$$

and

$$m(p)\frac{u_p^{n+1,m+1}-u_p^n}{k}-\sum_{q\in N(p)}T_{pq}\left(\varphi(u_q^{n+1,m})-\varphi(u_p^{n+1,m+1})\right)=m(p)v_p^n,\quad \textit{for all } p\in\mathcal{T},\quad \textit{for all } m\in\mathbb{N}. \tag{15}$$

Equation (15) gives a contraction property, which leads first to prove that for all $p\in\mathcal{T}$, $(\varphi(u_p^{n+1,m}))_{m\in\mathbb{N}}$ converges. Then we deduce that $(u_p^{n+1,m})_{m\in\mathbb{N}}$ converges as well.
We shall see, in remarks, that all results obtained for the explicit scheme are also true for the implicit scheme. The function $u_{\mathcal{T},k}$ is then defined by $u_{\mathcal{T},k}(x,t)=u_p^{n+1}$, for all $x\in p$, for all $t\in[nk,(n+1)k)$.

The mathematical problem is to study, under hypotheses (4) and (7), the convergence of $u_{\mathcal{T},k}$ to the weak solution of Problem (1, 2, 3), when $h\to 0$ and $k\to 0$.

3 A priori estimates

3.1 Maximum principle.

Lemma 3.1 *Under the hypotheses (4) and (7), let $T>0$, $U=||u_0||_{L^\infty(\Omega)}+T||v||_{L^\infty(\Omega\times(0,T))}$, $B=\sup\limits_{-U\le a<b\le U}\dfrac{\varphi(a)-\varphi(b)}{a-b}$. Assume that the condition*

$$k\le\frac{m(p)}{B\sum\limits_{q\in N(p)}T_{pq}},\quad \textit{for all } p\in\mathcal{T}, \tag{16}$$

is satisfied. Then the function $u_{\mathcal{T},k}$ defined by (9), (10), (11) and (12) verifies

$$\|u_{\mathcal{T},k}\|_{L^\infty(\Omega\times(0,T))} \le U. \tag{17}$$

PROOF.
Let $T>0$. Let $p\in\mathcal{T}$, $n\in\mathbb{N}$. The scheme (11) can be written as:

$$\begin{aligned} u_p^{n+1} = \quad & u_p^n\Big(1-\frac{k}{m(p)}\sum_{q\in N(p)} T_{pq}\frac{\varphi_q^n-\varphi_p^n}{u_q^n-u_p^n}\Big) + \\ & \frac{k}{m(p)}\sum_{q\in N(p)}\Big(T_{pq}\frac{\varphi_q^n-\varphi_p^n}{u_q^n-u_p^n}\Big)u_q^n + kv_p^n. \end{aligned} \tag{18}$$

Therefore, under condition (16), u_p^{n+1} is then an affine combination of u_q^n, $q\in\mathcal{T}$, with all coefficients positive, and their sum equal to 1. Hence the following inequality can be deduced:

$$|u_p^{n+1}| \le \sup_{q\in\mathcal{T}}|u_q^n| + k\|v\|_{L^\infty(\Omega\times(0,T))}. \tag{19}$$

Using (19), for $n=0,...[T/k]$, where we denote by $[x]=\max\limits_{n\in\mathbb{N}}\{n\le x\}$, and $p\in\mathcal{T}$, gives $|u_p^n|\le\|u_0\|_{L^\infty(\Omega)}+T\|v\|_{L^\infty(\Omega\times(0,T))}$, which leads to inequality (17).

Remark 3.1 *Under more regularity hypotheses on the mesh, there exists a value $C>0$ which does not depend on h such that the condition (16) is satisfied by any $k\le Ch^2$.*

Remark 3.2 *In view of (17) we deduce that there exists a function $u\in L^\infty(\Omega\times(0,T))$ and a subsequence of $(u_{\mathcal{T},k})$ which we denote again by $(u_{\mathcal{T},k})$ such that $(u_{\mathcal{T},k})$ converges to u for the weak star topology of $L^\infty(\Omega\times(0,T))$.*

Remark 3.3 *Estimate (17) is also true for the implicit scheme, because the fixed point method guarantees (19), without any condition on k.*

3.2 Space translates of approximate solutions.

We first define the following hypotheses and notations.

$$\left.\begin{array}{ll} \text{(i)} & T \text{ is a given real value with } T>0, \\ \text{(ii)} & U=\|u_0\|_{L^\infty(\Omega)}+T\|v\|_{L^\infty(\Omega\times(0,T))}, \\ \text{(iii)} & B=\sup\limits_{-U\le a<b\le U}\dfrac{\varphi(a)-\varphi(b)}{a-b}, \\ \text{(iv)} & \alpha \text{ is a given real value with } 0<\alpha<1, \\ \text{(v)} & k<T \text{ is a given real value with } k\le(1-\alpha)\dfrac{m(p)}{B\sum\limits_{q\in N(p)}T_{pq}}, \quad \text{for all } p\in\mathcal{T}, \\ \text{(vi)} & u_p^n \text{ is given by the definitions (9), (10) and (11) for all } p\in\mathcal{T} \text{ and } n\in\mathbb{N}. \end{array}\right\} \tag{20}$$

Lemma 3.2 *Under the hypotheses (4), (7) and (20), there exists a positive function F_1, which only depends on Ω, T, φ, u_0, v and α such that*

$$\sum_{n=0}^{[T/k]} k \sum_{(p,q)\in\mathcal{E}} T_{pq}(\varphi_p^n - \varphi_q^n)^2 \le F_1. \tag{21}$$

Proof of lemma 3.2.

We first remark that the condition (20.v) is stronger than (16). Therefore, the result of lemma 3.1 holds, i.e. $|u_p^n| \le U$, for all $p \in \mathcal{T}$, $n = 0, ...[T/k]$. Let us multiply the equation (11) by ku_p^n, and sum the result over $n = 0, ...[T/k]$ and $p \in \mathcal{T}$. We obtain

$$\begin{aligned}&\sum_{n=0}^{[T/k]}\sum_{p\in\mathcal{T}} m(p)(u_p^{n+1} - u_p^n)u_p^n - \\ &\sum_{n=0}^{[T/k]} k \sum_{p\in\mathcal{T}}\sum_{q\in N(p)} T_{pq}\Big(\varphi_q^n - \varphi_p^n\Big)u_p^n \quad = \sum_{n=0}^{[T/k]} k \sum_{p\in\mathcal{T}} m(p) v_p^n u_p^n.\end{aligned} \tag{22}$$

Next we consider the first term on the left-hand-side of (22). We have

$$(u_p^{n+1} - u_p^n)u_p^n = \frac{1}{2}(u_p^{n+1})^2 - \frac{1}{2}(u_p^n)^2 - \frac{1}{2}(u_p^{n+1} - u_p^n)^2. \tag{23}$$

In view of (11) we deduce from Cauchy-Schwarz inequality that

$$(u_p^{n+1} - u_p^n)^2 \le k^2(1+\alpha)\Big[\Big(\frac{1}{m(p)}\sum_{q\in N(p)} T_{pq}(\varphi_q^n - \varphi_p^n)\Big)^2 + \frac{(v_p^n)^2}{\alpha}\Big]. \tag{24}$$

Using again Cauchy-Schwarz inequality gives

$$(u_p^{n+1} - u_p^n)^2 \le \frac{k^2}{m(p)^2}(1+\alpha)\Big[\sum_{q\in N(p)} T_{pq}\Big]\Big[\sum_{q\in N(p)} T_{pq}\Big(\varphi_q^n - \varphi_p^n\Big)^2\Big] + \frac{1+\alpha}{\alpha}k^2(v_p^n)^2. \tag{25}$$

Using (20.v) we obtain

$$(u_p^{n+1} - u_p^n)^2 \le (1-\alpha^2)\frac{k}{Bm(p)}\Big[\sum_{q\in N(p)} T_{pq}\Big(\varphi_q^n - \varphi_p^n\Big)^2\Big] + \frac{1+\alpha}{\alpha}k^2(v_p^n)^2. \tag{26}$$

Relations (23) and (26) lead to

$$\begin{aligned}\sum_{n=0}^{[T/k]}\sum_{p\in\mathcal{T}} m(p)(u_p^{n+1} - u_p^n)u_p^n \ge \quad & \frac{1}{2}\sum_{p\in\mathcal{T}} m(p)\Big((u_p^{[T/k]+1})^2 - (u_p^0)^2\Big) \\ & -\frac{1-\alpha^2}{2B}\sum_{n=0}^{[T/k]} k \sum_{p\in\mathcal{T}}\Big[\sum_{q\in N(p)} T_{pq}\Big(\varphi_q^n - \varphi_p^n\Big)^2\Big] \\ & -\frac{k(1+\alpha)}{2\alpha}\sum_{n=0}^{[T/k]} k \sum_{p\in\mathcal{T}} m(p)(v_p^n)^2.\end{aligned} \tag{27}$$

We now handle the second term on the left-hand-side of (22). We first remark that

$$\int_c^d (\varphi(x) - \varphi(c))dx \ge \frac{1}{2B}(\varphi(d) - \varphi(c))^2, \quad \text{for all } c, d \in [-U, U]. \tag{28}$$

Indeed let us assume, for instance, that $c < d$ (the other case is similar); then, one has $\varphi(s) \geq h(s)$, for all $s \in [c,d]$, where $h(s) = \varphi(c)$ for $s \in [c, d-l]$ and $h(s) = \varphi(c) + (s-d+l)B$ for $s \in [d-l, d]$, where l is defined by $lB = \varphi(d) - \varphi(c)$, and therefore

$$\int_c^d (\varphi(s) - \varphi(c))ds \geq \int_c^d (h(s) - \varphi(c))ds = \frac{l}{2}(\varphi(d) - \varphi(c)) = \frac{1}{2B}(\varphi(d) - \varphi(c))^2, \tag{29}$$

which then yields (28).

Let $\phi \in C(\mathbb{R})$ be defined by $\phi(x) = x\varphi(x) - \int_{x_0}^x \varphi(y)dy$, where $x_0 \in \mathbb{R}$ is an arbitrary given real value. Then the following equality holds.

$$\phi(u_q^n) - \phi(u_p^n) = u_p^n(\varphi_q^n - \varphi_p^n) - \int_{u_p^n}^{u_q^n} (\varphi(x) - \varphi_q^n)dx. \tag{30}$$

We have therefore, using (28), (30) and the equality $\sum_{p \in \mathcal{T}} \sum_{q \in N(p)} T_{pq}(\phi(u_q^n) - \phi(u_p^n)) = 0$,

$$-\sum_{n=0}^{[T/k]} k \sum_{p \in \mathcal{T}} \sum_{q \in N(p)} T_{pq}\left(\varphi_q^n - \varphi_p^n\right)u_p^n \geq \frac{1}{2B}\sum_{n=0}^{[T/k]} k \sum_{p \in \mathcal{T}} \sum_{q \in N(p)} T_{pq}(\varphi_q^n - \varphi_p^n)^2. \tag{31}$$

Since $k < T$ we deduce from (17) that the right-hand-side of equation (22) satisfies

$$|\sum_{n=0}^{[T/k]} k \sum_{p \in \mathcal{T}} m(p) v_p^n u_p^n| \leq 2Tm(\Omega)U\|v\|_{L^\infty(\Omega\times(0,T))}. \tag{32}$$

Relations $k < T$, (22), (27), (31) and (32) lead to

$$\begin{aligned}\frac{\alpha^2}{2B}\sum_{n=0}^{[T/k]} k \sum_{p \in \mathcal{T}} \sum_{q \in N(p)} T_{pq}(\varphi_q^n - \varphi_p^n)^2 \leq \quad & 2Tm(\Omega)\|v\|_{L^\infty(\Omega\times(0,T))}\Big(U + \frac{1+\alpha}{2\alpha}\|v\|_{L^\infty(\Omega\times(0,T))}T\Big) \\ & + \frac{1}{2}m(\Omega)\Big(\|u_0\|_{L^\infty(\Omega)}\Big)^2,\end{aligned} \tag{33}$$

which concludes the proof of the lemma. Next we deduce the following result.

Lemma 3.3 *Under the hypotheses (4), (7) and (20), there exists a positive function F_1, which only depends on Ω, T, φ, u_0, v and α such that*

$$\int_{\Omega_\xi \times (0,T)} (\varphi_{\mathcal{T},k}(x+\xi, t) - \varphi_{\mathcal{T},k}(x,t))^2 dxdt \leq |\xi|(|\xi| + 2h)F_1, \tag{34}$$

for all $\xi \in \mathbb{R}^N$, where $\Omega_\xi = \{x \in \Omega, [x+\xi, x] \subset \Omega\}$.

Proof of lemma 3.3.
Let $\xi \in \mathbb{R}^N$. For all $x \in \Omega_\xi$ and for all $(p,q) \in \mathcal{E}$, we denote by $E(x,p,q)$ the function whose value is 1 if

1. the segment $[x+\xi, x]$ intersects p, q and e_{pq},
2. the value c_{pq} defined by $c_{pq} = \frac{\xi}{|\xi|} \cdot \mathbf{n}_{pq}$ verifies $c_{pq} > 0$,

else $E(x,p,q)=0$. For almost every $x\in\Omega$, we denote by $p(x)$ the element p of $\mathcal{T}$ such that $x\in p$. For almost every $x\in\Omega_\xi$, and $t\in(nk,(n+1)k)$, we have

$$\varphi_{\mathcal{T},k}(x+\xi,t)-\varphi_{\mathcal{T},k}(x,t)=\varphi^n_{p(x+\xi)}-\varphi^n_{p(x)}=\sum_{(p,q)\in\mathcal{E}}E(x,p,q)(\varphi^n_q-\varphi^n_p). \tag{35}$$

Using Cauchy-Schwarz inequality, we get

$$(\varphi_{\mathcal{T},k}(x+\xi,t)-\varphi_{\mathcal{T},k}(x,t))^2\leq\sum_{(p,q)\in\mathcal{E}}E(x,p,q)c_{pq}d_{pq}\sum_{(p,q)\in\mathcal{E}}E(x,p,q)\frac{(\varphi^n_q-\varphi^n_p)^2}{c_{pq}d_{pq}}. \tag{36}$$

For all $(p,q)\in\mathcal{E}$, the property $c_{pq}d_{pq}=\dfrac{\xi}{|\xi|}\cdot(x_q-x_p)$ holds. Therefore we have $\sum_{(p,q)\in\mathcal{E}}E(x,p,q)c_{pq}d_{pq}=\dfrac{\xi}{|\xi|}\cdot(x_{p(x+\xi)}-x_{p(x)})$. We then deduce

$$\sum_{(p,q)\in\mathcal{E}}E(x,p,q)c_{pq}d_{pq}\leq|\xi|+2h. \tag{37}$$

Using (36) and (37), we get

$$\int_{\Omega_\xi\times(0,T)}(\varphi_{\mathcal{T},k}(x+\xi,t)-\varphi_{\mathcal{T},k}(x,t))^2dxdt\leq\sum_{n=0}^{[T/k]}k(|\xi|+2h)\sum_{(p,q)\in\mathcal{E}}\frac{(\varphi^n_q-\varphi^n_p)^2}{c_{pq}d_{pq}}\int_{\Omega_\xi}E(x,p,q)dx \tag{38}$$

The value $\int_{\Omega_\xi}E(x,p,q)dx$ is the measure of a set of points of Ω which are located inside a cylinder, whose basis is e_{pq} and generator vector is $-\xi$. Thus $\int_{\Omega_\xi}E(x,p,q)dx\leq m(e_{pq})c_{pq}|\xi|$, because c_{pq} is the cosine of the angle between ξ and $\mathbf{n}_{pq}$. Then we finally get

$$\int_{\Omega_\xi\times(0,T)}(\varphi_{\mathcal{T},k}(x+\xi,t)-\varphi_{\mathcal{T},k}(x,t))^2dxdt\leq|\xi|(|\xi|+2h)\sum_{n=0}^{[T/k]}k\sum_{(p,q)\in\mathcal{E}}T_{pq}(\varphi^n_q-\varphi^n_p)^2, \tag{39}$$

which, using (21), gives (34).

Remark 3.4 *This lemma gives an estimate for the translates of $\varphi_{\mathcal{T},k}$ in space. The following paragraph gives an estimate for the translates in time.*

Remark 3.5 *Estimate (21) also holds for the implicit scheme, without any condition on k. One multiplies (13) by u_p^{n+1}: the last term on the right-hand-side of (23) appears with the opposite sign, which considerably simplifies the previous proof. Therefore estimate (34) can also be proved for the implicit scheme.*

3.3 Time translates.

We now study the translate in time of function $\varphi_{\mathcal{T},k}$.

Lemma 3.4 *Under the hypotheses (4), (7) and (20), there exists a positive function F_2, which only depends on Ω, T, φ, u_0, v and α such that*

$$\int_{\Omega\times(0,T-\tau)}(\varphi_{\mathcal{T},k}(x,t+\tau)-\varphi_{\mathcal{T},k}(x,t))^2dxdt \le \tau F_2, \tag{40}$$

for all $\tau \in (0,T)$.

Proof of lemma 3.4.

Let $\tau\in(0,T)$ and $t\in(0,T-\tau)$. Since φ is locally Lipschitz continuous with constant B, one has

$$\int_{\Omega\times(0,T-\tau)}(\varphi_{\mathcal{T},k}(x,t+\tau)-\varphi_{\mathcal{T},k}(x,t))^2dxdt \le B\int_0^{T-\tau}A(t)dt, \tag{41}$$

where, for almost every $t\in(0,T-\tau)$,

$$A(t)=\int_\Omega(\varphi_{\mathcal{T},k}(x,t+\tau)-\varphi_{\mathcal{T},k}(x,t))(u_{\mathcal{T},k}(x,t+\tau)-u_{\mathcal{T},k}(x,t))dx. \tag{42}$$

Using the definition (12), setting $n_0=[t/k]$ and $n_1=[(t+\tau)/k]$, we get

$$A(t)=\sum_{p\in\mathcal{T}}m(p)(\varphi_p^{n_1}-\varphi_p^{n_0})(u_p^{n_1}-u_p^{n_0}), \tag{43}$$

which also reads

$$A(t)=\sum_{p\in\mathcal{T}}(\varphi_p^{n_1}-\varphi_p^{n_0})\sum_{\substack{n\in\mathbb{N},\\ t<(n+1)k\le t+\tau}}m(p)(u_p^{n+1}-u_p^n). \tag{44}$$

We now use the scheme (11), and we get

$$A(t)=\sum_{\substack{n\in\mathbb{N},\\ t<(n+1)k\le t+\tau}}k\sum_{p\in\mathcal{T}}(\varphi_p^{n_1}-\varphi_p^{n_0})\Big(\sum_{q\in N(p)}T_{pq}(\varphi_q^n-\varphi_p^n)+m(p)v_p^n\Big). \tag{45}$$

We now gather by edges and we get

$$A(t)=\sum_{\substack{n\in\mathbb{N},\\ t<(n+1)k\le t+\tau}}k\Big(\sum_{(p,q)\in\mathcal{E}}T_{pq}(\varphi_p^{n_1}-\varphi_q^{n_1}-\varphi_p^{n_0}+\varphi_q^{n_0})(\varphi_q^n-\varphi_p^n)+\sum_{p\in\mathcal{T}}(\varphi_p^{n_1}-\varphi_p^{n_0})m(p)v_p^n\Big). \tag{46}$$

We can then use the inequality $2ab\le a^2+b^2$. We get

$$A(t)\le\frac{1}{2}A_0(t)+\frac{1}{2}A_1(t)+A_2(t)+A_3(t), \tag{47}$$

with

$$A_0(t)=\sum_{\substack{n\in\mathbb{N},\\ t<(n+1)k\le t+\tau}}k\sum_{(p,q)\in\mathcal{E}}T_{pq}(\varphi_q^{n_0}-\varphi_p^{n_0})^2, \tag{48}$$

$$A_1(t)=\sum_{\substack{n\in\mathbb{N},\\ t<(n+1)k\le t+\tau}}k\sum_{(p,q)\in\mathcal{E}}T_{pq}(\varphi_q^{n_1}-\varphi_p^{n_1})^2, \tag{49}$$

$$A_2(t)=\sum_{\substack{n\in\mathbb{N},\\ t<(n+1)k\le t+\tau}}k\sum_{(p,q)\in\mathcal{E}}T_{pq}(\varphi_q^{n}-\varphi_p^{n})^2, \tag{50}$$

and

$$A_3(t) = \sum_{\substack{n\in\mathbb{N},\\ t<(n+1)k\le t+\tau}} k\sum_{p\in\mathcal{T}}(\varphi_p^{n_1}-\varphi_p^{n_0})m(p)v_p^n. \tag{51}$$

We introduce the function χ such that $\chi(\text{true}) = 1$ and $\chi(\text{false}) = 0$. We have, for all $t\in\mathbb{R}^+$ and $n\in\mathbb{N}$, $\chi(t<(n+1)k\le t+\tau) = \chi((n+1)k-\tau\le t<(n+1)k)$. Therefore

$$\int_0^{T-\tau} A_0(t)dt \le \sum_{n_0=0}^{[T/k]} k \sum_{(p,q)\in\mathcal{E}} T_{pq}(\varphi_q^{n_0}-\varphi_p^{n_0})^2 \int_{n_0 k}^{(n_0+1)k} \sum_{n\in\mathbb{N}} \chi((n+1)k-\tau\le t\le(n+1)k)dt. \tag{52}$$

The property

$$\int_{n_0 k}^{(n_0+1)k} \sum_{n\in\mathbb{N}} \chi((n+1)k-\tau\le t<(n+1)k)dt = \sum_{n\in\mathbb{N}} \int_{(n_0-n-1)k+\tau}^{(n_0-n)k+\tau} \chi(0\le t<\tau)dt = \tau \tag{53}$$

gives, using (21) and (52),

$$\int_0^{T-\tau} A_0(t)dt \le \tau F_1. \tag{54}$$

We get exactly in the same way

$$\int_0^{T-\tau} A_1(t)dt \le \tau F_1. \tag{55}$$

We now turn to the study of $\int_0^{T-\tau} A_2(t)dt$. We have

$$\int_0^{T-\tau} A_2(t)dt \le \sum_{n=0}^{[T/k]} k \sum_{(p,q)\in\mathcal{E}} T_{pq}(\varphi_q^{n}-\varphi_p^{n})^2 \int_0^{T-\tau} \chi((n+1)k-\tau\le t<(n+1)k)dt. \tag{56}$$

Because $\int_0^{T-\tau}\chi((n+1)k-\tau\le t<(n+1)k)dt = \min(T-\tau,(n+1)k)-\max(0,(n+1)k-\tau)\le\tau$, we get

$$\int_0^{T-\tau} A_2(t)dt \le \tau F_1. \tag{57}$$

We have in the same way

$$\int_0^{T-\tau} A_3(t)dt \le \sum_{n=0}^{[T/k]} k \sum_{p\in\mathcal{T}} m(p)2BUV \int_0^{T-\tau} \chi((n+1)k-\tau\le t<(n+1)k)dt \le \tau 2Tm(\Omega)BUV. \tag{58}$$

Using Equations (54)-(58), we conclude (40).

Remark 3.6 *Estimate (40) is again true for the implicit scheme, without any condition on k.*

3.4 Relative compactness in $L^2(\Omega \times (0,T))$

In this section, we show how estimates (17), (34) and (40) can be used to derive a strong convergence property in $L^2(\Omega \times (0,T))$.

Lemma 3.5 *Let $(f_m)_{m\in\mathbb{N}}$ be a sequence of functions of $L^2(\Omega\times(0,T))$ which verifies*

1. *there exists $M_1 > 0$ such that for all $m \in \mathbb{N}$, $\|f_m\|_{L^\infty(\Omega\times(0,T))} \leq M_1$,*
2. *there exists $M_2 > 0$ such that for all $m \in \mathbb{N}$ and $\tau \in (0,T)$, $\int_{\Omega\times(0,T-\tau)}(f_m(x,t+\tau)-f_m(x,t))^2 dxdt \leq \tau M_2$,*
3. *there exist $M_3 > 0$ and a sequence of real positive values $(h_m)_{m\in\mathbb{N}}$ with $\lim_{m\to\infty} h_m = 0$ such that for all $m \in \mathbb{N}$, $\int_{\Omega_\xi\times(0,T)}(f_m(x+\xi,t)-f_m(x,t))^2 dxdt \leq |\xi|(|\xi|+h_m)M_3$, for all $\xi \in \mathbb{R}^N$, where $\Omega_\xi = \{x \in \Omega, [x+\xi, x] \subset \Omega\}$.*

Then there exists a subsequence of $(f_m)_{m\in\mathbb{N}}$ which converges for the strong topology of $L^2(\Omega\times(0,T))$ to an element of $L^2(0,T;H^1(\Omega))$.

Proof of lemma 3.5.
We first extend the definition of f_m, for $m \in \mathbb{N}$, by the value 0 outside of $\Omega \times (0,T)$. Using the measurability of the boundary $\partial\Omega$ of Ω, we get that, for all $\xi \in \mathbb{R}^N$, $m(\Omega \setminus \Omega_\xi) \leq |\xi| m(\partial\Omega)$. Therefore we get, for $m \in \mathbb{N}$, $\int_{\Omega\times(0,T)}(f_m(x+\xi,t)-f_m(x,t))^2 dxdt \leq |\xi|(((|\xi|+h_m)M_3 + Tm(\partial\Omega)M_1^2)$. We also get, for all $\tau \in (-T,T)$, $\int_{\Omega\times(0,T)}(f_m(x,t+\tau)-f_m(x,t))^2 dxdt \leq \tau(M_2 + m(\Omega)M_1^2)$. Therefore the sequence $(f_m)_{m\in\mathbb{N}}$ satisfies the hypotheses of Kolmogorov's theorem. Thus there exists a subsequence of $(f_m)_{m\in\mathbb{N}}$ which converges for the strong topology of $L^2(\Omega\times(0,T))$.

Let f be the limit of such a subsequence. It satisfies, for all $\xi \in \mathbb{R}^N$, $\int_{\Omega_\xi\times(0,T)}(f(x+\xi,t)-f(x,t))^2 dxdt \leq |\xi|^2 M_3$ because the sequence $(h_m)_{m\in\mathbb{N}}$ converges to zero as $m \to \infty$. Therefore, for all $\varepsilon > 0$, denoting $\Omega_\varepsilon = \{x \in \Omega,\ B(x,\varepsilon) \subset \Omega\}$, we get that $f \in L^2(0,T;H^1(\Omega_\varepsilon))$, with $\|f\|_{L^2(0,T;H^1(\Omega_\varepsilon))} \leq \sqrt{NM_3 + m(\Omega)TM_1^2}$. Therefore $f \in L^2(0,T;H^1(\Omega))$, with $\|f\|_{L^2(0,T;H^1(\Omega))} \leq \sqrt{NM_3 + m(\Omega)TM_1^2}$.

4 A functional convergence property.

We now show a property which is necessary in the next section.

Theorem 4.1 *Let $U > 0$ be a given constant, and $\varphi \in C([-U,U])$ a non decreasing function. Let $N \in \mathbb{N}^*$, and let E be a bounded open subset of $\mathbb{R}^N$. For any $n \in \mathbb{N}$, let $u_n \in L^\infty(E)$ such that*
(i) $-U \leq u_n \leq U$ a.e., for all $n \in \mathbb{N}$;
(ii) there exists $u \in L^\infty(E)$, such that $(u_n)_{n\in\mathbb{N}}$ converges to u for the weak star topology of $L^\infty(E)$;
(iii) there exists a function $\Phi \in L^1(E)$ such that $(\varphi(u_n))_{n\in\mathbb{N}}$ converges to Φ for the topology of $L^1(E)$.
Then $\Phi(x) = \varphi(u(x))$, for a.e. $x \in E$.

Proof of theorem 4.1.
First we extend the definition of φ by $\varphi(v) = \varphi(-U) + v + U$ for all $v < -U$ and $\varphi(v) = \varphi(U) + v - U$ for all $v > U$, and denote again by φ this extension of φ which now maps $\mathbb{R}$ into $\mathbb{R}$, is continuous and non decreasing as well.

Next we define $\alpha_\pm : \mathbb{R} \longmapsto \mathbb{R}$ by $\alpha_-(t) = \inf\{v \in \mathbb{R}, \varphi(v) = t\}$, and $\alpha_+(t) = \sup\{v \in \mathbb{R}, \varphi(v) = t\}$, for all $t \in \mathbb{R}$.
Note that the functions $\alpha_\pm$ are strictly increasing and that
(i) α_- is continuous from the left and therefore lower semi-continuous, that is

$$\alpha_-(t) \le \liminf_{x \to t} \alpha_-(x), \tag{59}$$

(ii) α_+ is continuous from the right and therefore upper semi-continuous, that is

$$\alpha_+(t) \ge \limsup_{x \to t} \alpha_+(x). \tag{60}$$

Thus, for a.e. $x \in E$

$$\alpha_-(\Phi(x)) \le \liminf_{n\to\infty} \alpha_-\Big(\varphi(u_n(x))\Big) \le \limsup_{n\to\infty} \alpha_+\Big(\varphi(u_n(x))\Big) \le \alpha_+(\Phi(x)). \tag{61}$$

We multiply the inequalities (61) by a non negative function $\psi \in L^1(E)$ and integrate over E. Because Fatou's lemma can be applied to the sequence of L^1 positive functions $\alpha_-\Big(\varphi(u_n(.))\Big)\psi(.) - \alpha_-(\varphi(-U))\psi(.)$, we get

$$\int_E \alpha_-(\Phi(x))\psi(x)dx \le \liminf_{n\to\infty} \int_E \alpha_-\Big(\varphi(u_n(x))\Big)\psi(x)dx. \tag{62}$$

and in the same way, we get

$$\limsup_{n\to\infty} \int_E \alpha_+\Big(\varphi(u_n(x))\Big)\psi(x)dx \le \int_E \alpha_+(\Phi(x))\psi(x)dx. \tag{63}$$

By the definition of the functions α_- and α_+, the following inequalities hold.

$$\alpha_-\Big(\varphi(u_n(x))\Big) \le u_n(x) \le \alpha_+\Big(\varphi(u_n(x))\Big), \tag{64}$$

which, combined with (62), (63) and the convergence of $(u_n)_{n\in\mathbb{N}}$ to u for the weak star topology of $L^\infty(E)$, implies that

$$\int_E \alpha_-(\Phi(x))\psi(x)dx \le \int_E u(x)\psi(x)dx \le \int_E \alpha_+(\Phi(x))\psi(x)dx. \tag{65}$$

Thus $\alpha_-(\Phi(x)) \le u(x) \le \alpha_+(\Phi(x))$ for a.e. $x \in E$, which implies that $\Phi(x) = \varphi(u(x))$ for a.e. $x \in E$. That completes the proof of Theorem 4.1.

5 Convergence

We now prove the following result.

Theorem 5.1 *Suppose that the hypotheses (4) are satisfied and let $T > 0$, $U = \|u_0\|_{L^\infty(\Omega)} + T\|v\|_{L^\infty(\Omega\times(0,T))}$ and $B = \sup_{-U\le x<y\le U} \dfrac{\varphi(x)-\varphi(y)}{x-y}$. Let $\alpha \in]0,1[$ be a given real value. Let $(\mathcal{T}_m, k_m)_{m\in\mathbb{N}}$ be a sequence of meshes and time steps such that there exists a sequence of positive real values $(h_m)_{m\in\mathbb{N}}$ with*
- for all $m \in \mathbb{N}$, hypotheses (7) are satisfied with $\mathcal{T} = \mathcal{T}_m$ and $h = h_m$;

- the sequence $(h_m)_{m\in\mathbb{N}}$ converges to zero;
- for all $m \in \mathbb{N}$, k_m satisfies the condition (20.v) for $\mathcal{T} = \mathcal{T}_m$ and $k = k_m$.
For all $m \in \mathbb{N}$, let $u_m = u_{\mathcal{T},k}$ be given by (9), (10), (11) and (12), for $\mathcal{T} = \mathcal{T}_m$ and $k = k_m$.

Then the sequence $(u_m)_{m\in\mathbb{N}}$ converges to the unique weak solution u of Problem (1, 2, 3) in the following sense.
(i) $(u_m)_{m\in\mathbb{N}}$ converges to u for the weak star topology of $L^\infty(\Omega \times (0,T))$,
(ii) $(\varphi(u_m))_{m\in\mathbb{N}}$ converges to $\varphi(u) \in L^2(0,T;H^1(\Omega))$ for the strong topology of $L^2(\Omega \times (0,T))$.

Proof of theorem 5.1.
We first remark that by (20.v) the sequence $(k_m)_{m\in\mathbb{N}}$ converges to zero. Because of the lemmas 3.1, 3.5 and theorem 4.1, we can extract from the sequence $(u_m)_{m\in\mathbb{N}}$ a subsequence $(u_{M(m)})_{m\in\mathbb{N}}$ such that there exists a function $u \in L^\infty(\Omega \times (0,T))$ with
(i) $(u_{M(m)})_{m\in\mathbb{N}}$ converges to u for the weak star topology of $L^\infty(\Omega \times (0,T))$,
(ii) $(\varphi(u_{M(m)}))_{m\in\mathbb{N}}$ converges to $\varphi(u)$ for the strong topology of $L^2(\Omega \times (0,T))$.

Next we show that u is a weak solution of Problem (1, 2, 3).

Let $m \in \mathbb{N}$. We use the notations $\mathcal{T} = \mathcal{T}_{M(m)}$, $h = h_{M(m)}$ and $k = k_{M(m)}$. Let $T > 0$ and $\psi \in \mathcal{A}_T$. We multiply (11) by $k\psi(x_p, nk)$, and sum the result on $n = 0, ...[T/k]$ and $p \in \mathcal{T}$. We obtain

$$T_{1m} + T_{2m} = T_{3m}, \tag{66}$$

with

$$T_{1m} = \sum_{n=0}^{[T/k]} \sum_{p\in\mathcal{T}} m(p)(u_p^{n+1} - u_p^n)\psi(x_p, nk), \tag{67}$$

$$T_{2m} = -\sum_{n=0}^{[T/k]} k \sum_{p\in\mathcal{T}} \sum_{q\in N(p)} T_{pq}\left(\varphi_q^n - \varphi_p^n\right)\psi(x_p, nk), \tag{68}$$

and

$$T_{3m} = \sum_{n=0}^{[T/k]} k \sum_{p\in\mathcal{T}} \psi(x_p, nk)m(p)v_p^n. \tag{69}$$

We first consider T_{1m}. We have that

$$\begin{aligned} T_{1m} = \quad & \sum_{n=1}^{[T/k]} \sum_{p\in\mathcal{T}} m(p)u_p^n\left(\psi(x_p,(n-1)k) - \psi(x_p, nk)\right) \quad + \\ & \sum_{p\in\mathcal{T}} m(p)\left(u_p^{[T/k]+1}\psi(x_p,[T/k]k) - u_p^0\psi(x_p, 0)\right). \end{aligned} \tag{70}$$

Let us suppose $k < T$ (it is necessarily true for m large enough). We remark that $u_p^{[T/k]+1} < U + T\|v\|_{L^\infty(\Omega\times(0,T))}$. Since $0 \le T - [T/k]k < k$, there exists a positive function $C_{1\psi}$, which only depends on ψ, T and Ω such that $|\psi(x_p,[T/k]k)| \le C_{1\psi}k$. This leads to the convergence of T_{1m} to

$-\int_0^T \int_\Omega u(x,t)\psi_t(x,t)dxdt - \int_\Omega u_0(x)\psi_t(x,0)dx$, as $m \to \infty$, in view of the convergence of $(u_{M(m)})_{m\in\mathbb{N}}$ for the weak star topology of $L^\infty(\Omega\times(0,T))$, and of the convergence of $\sum_{p\in\mathcal{T}} u_p^0\psi(x_p,0)\chi(.\in p)$ to $u_0(.)\psi(.,0)$ for the topology of $L^1(\Omega)$.

We now study T_{2m}. This term can be rewritten as

$$T_{2m} = -\frac{1}{2}\sum_{n=0}^{[T/k]} k \sum_{(p,q)\in\mathcal{E}} m(e_{pq})(\varphi_q^n - \varphi_p^n)\frac{\psi(x_p,nk)-\psi(x_q,nk)}{d_{pq}}. \tag{71}$$

It is useful to introduce the following expression.

$$\begin{aligned} T'_{2m} &= \sum_{n=0}^{[T/k]}\int_{nk}^{(n+1)k}\int_\Omega \varphi(u_{\mathcal{T},k}(x,t))\Delta(\psi(x,nk))dxdt \\ &= \sum_{n=0}^{[T/k]} k \sum_{p\in\mathcal{T}}\varphi_p^n\int_p \Delta(\psi(x,nk))dx \\ &= \frac{1}{2}\sum_{n=0}^{[T/k]} k \sum_{(p,q)\in\mathcal{E}}(\varphi_p^n-\varphi_q^n)\int_{e_{pq}}\nabla\psi(\gamma,nk).\mathbf{n}_{pq}d\gamma. \end{aligned} \tag{72}$$

Because of the convergence of $(\varphi(u_{M(m)}))_{m\in\mathbb{N}}$ for the topology of $L^2(\Omega\times(0,T))$ to $\varphi(u)$, the term T'_{2m} converges to $\int_0^T\int_\Omega \varphi(u(x,t))\Delta\psi(x,t))\ dx\ dt$ as $m\to\infty$. The term $T_{2m}+T'_{2m}$ can be written as

$$T_{2m}+T'_{2m} = \frac{1}{2}\sum_{n=0}^{[T/k]} k \sum_{(p,q)\in\mathcal{E}} m(e_{pq})(\varphi_p^n-\varphi_q^n)R_{pq}^n, \tag{73}$$

with

$$R_{pq}^n = \frac{1}{m(e_{pq})}\int_{e_{pq}}\nabla\psi(\gamma,nk).\mathbf{n}_{pq}d\gamma - \frac{\psi(x_q,nk)-\psi(x_p,nk)}{d_{pq}}. \tag{74}$$

In view of the regularity properties of ψ and of hypotheses (7.i) and (7.iv), there exists a positive function C_ψ, which only depends on ψ, such that $|R_{pq}^n| \le C_\psi h$. Then, using the estimate (34), we conclude that $T_{2m}+T'_{2m}\to 0$ as $m\to\infty$. The property $T_{3m}\to\int_0^T\int_\Omega\psi(x,t)v(x,t)dxdt$ as $m\to\infty$ results from convergences in $L^1(\Omega\times(0,T))$.

Therefore u is the unique weak solution of Problem (1, 2, 3) and the full sequences $(u_m)_{m\in\mathbb{N}}$ and $(\varphi(u_m))_{m\in\mathbb{N}}$ converge.

Remark 5.1 *In the linear case ($\varphi(.) = .$, i.e. in the case of the heat equation), the estimates on space and time translates of $\varphi(u_{\mathcal{T},k})$ are not necessary in order to only obtain a weak star convergence of $u_{\mathcal{T},k}$ to the unique solution of (5).*

Remark 5.2 *This convergence proof is quite similar in the case of the implicit scheme, with the additional condition that $(k_m)_{m\in\mathbb{N}}$ converges to zero, since condition (20.v) does not have to be satisfied.*

Appendix : Uniqueness of the solution.

The uniqueness of the weak solution to variants of Problem (1, 2, 3) has been proved by several authors. For precise references we refer to [11]. Also rather similar proofs have been given in [5] and [7]. The uniqueness of the weak solution to Problem (1, 2, 3) immediately results from the following property.

Theorem 5.2 *Suppose that Hypothesis (4) is satisfied. Let u_1 and u_2 be two solutions of Problem (1, 2, 3), with initial conditions u_{01} and u_{02} and source terms v_1 and v_2 respectively. Then for all $T > 0$,*

$$\int_0^T \int_\Omega |u_1(x,t) - u_2(x,t)| dx dt \le T \int_\Omega |u_{01}(x) - u_{02}(x)| dx + \int_0^T \int_\Omega (T-t)\ |v_1(x,t) - v_2(x,t)| dx dt. \quad (75)$$

Before proving Theorem 5.2, we first show the following auxiliary result.

Lemma 5.1 *Suppose that Hypothesis (4) is satisfied. Let $T > 0$, $w \in C_c^\infty(\Omega \times (0,T))$ such that $|w| \le 1$, and $g \in C^\infty(\Omega \times (0,T))$ such that there exists $r \in \mathbb{R}$ with $0 < r \le g(x,t)$, for all $(x,t) \in \Omega \times (0,T)$.*

Then there exists a unique function $\psi \in C^{2,1}(\overline{\Omega} \times [0,T])$ such that

$$\psi_t(x,t) + g(x,t)\Delta\psi(x,t) = w(x,t), \qquad \text{for all } (x,t) \in \Omega \times (0,T), \quad (76)$$

$$\frac{\partial \psi}{\partial n}(x,t)) = 0, \quad \text{for all } (x,t) \in \partial\Omega \times (0,T), \quad (77)$$

$$\psi(x,T) = 0, \quad \text{for all } x \in \Omega. \quad (78)$$

Moreover the function ψ satisfies

$$|\psi(x,t)| \le T - t, \quad \text{for all } (x,t) \in \Omega \times (0,T), \quad (79)$$

and

$$\int_0^T \int_\Omega g(x,t)\Big(\Delta\psi(x,t)\Big)^2 dx dt \le 4T \int_0^T \int_\Omega \Big(\nabla w(x,t)\Big)^2 dx dt. \quad (80)$$

Proof of lemma 5.1.

It will be useful in the following to point out that the right-hand-side of (80) does not depend on g. Since the function g is bounded away from zero, equations (76, 77, 78) define a boundary value problem for a usual heat equation in which the time variable is reversed. Since Ω, g and w are sufficiently smooth, this problem has a unique solution $\psi \in \mathcal{A}_T$ [10]. Since $|w| \le 1$, the functions $T - t$ and $-(T-t)$ are respectively upper and lower solutions of Problem (76, 78, 77). Hence we get (79).

In order to show (80), we multiply (76) by $\Delta\psi(x,t)$, integrate by parts on $\Omega \times (0,T)$. This gives

$$\frac{1}{2}\int_{\Omega}\Big(\nabla\psi(x,0)\Big)^2 dx \;-\; \frac{1}{2}\int_{\Omega}\Big(\nabla\psi(x,\tau)\Big)^2 dx \;+\; \int_0^\tau\int_{\Omega} g(x,t)\Big(\Delta\psi(x,t)\Big)^2 dxdt = \\ -\int_0^\tau\int_{\Omega}\nabla w(x,t)\nabla\psi(x,t)dxdt. \tag{81}$$

Since $\nabla\psi(.,T) = 0$, letting $\tau = T$ in (81) leads to

$$\frac{1}{2}\int_{\Omega}\Big(\nabla\psi(x,0)\Big)^2 dx + \int_0^T\int_{\Omega} g(x,t)\Big(\Delta\psi(x,t)\Big)^2 dxdt = \\ -\int_0^T\int_{\Omega}\nabla w(x,t)\nabla\psi(x,t)dxdt. \tag{82}$$

Integrate (81) with respect to $\tau\in(0,T)$ leads to

$$\frac{1}{2}\int_0^T\int_{\Omega}\Big(\nabla\psi(x,\tau)\Big)^2 dxd\tau \leq \frac{T}{2}\int_{\Omega}\Big(\nabla\psi(x,0)\Big)^2 dx \;+\; T\int_0^T\int_{\Omega} g(x,t)\Big(\Delta\psi(x,t)\Big)^2 dxdt \;+\; T\int_0^T\int_{\Omega}|\nabla w(x,t)\nabla\psi(x,t)|dxdt. \tag{83}$$

Using (82) and (83), we get

$$\frac{1}{2}\int_0^T\int_{\Omega}\Big(\nabla\psi(x,\tau)\Big)^2 dxd\tau \leq 2T\int_0^T\int_{\Omega}|\nabla w(x,t)\nabla\psi(x,t)|dxdt. \tag{84}$$

Using Cauchy-Schwarz inequality, we get

$$\Big[\int_0^T\int_{\Omega}|\nabla w(x,t)\nabla\psi(x,t)|dxdt\Big]^2 \leq \int_0^T\int_{\Omega}\Big(\nabla\psi(x,t)\Big)^2 dxdt\;.\; \int_0^T\int_{\Omega}\Big(\nabla w(x,t)\Big)^2 dxdt. \tag{85}$$

Using (84) and (85), we get

$$\Big[\int_0^T\int_{\Omega}|\nabla w(x,t)\nabla\psi(x,t)|dxdt\Big]^2 \leq 4T\int_0^T\int_{\Omega}|\nabla w(x,t)\nabla\psi(x,t)|dxdt\;.\; \int_0^T\int_{\Omega}\Big(\nabla w(x,t)\Big)^2 dxdt. \tag{86}$$

Therefore

$$\int_0^T\int_{\Omega}|\nabla w(x,t)\nabla\psi(x,t)|dxdt \leq 4T\int_0^T\int_{\Omega}\Big(\nabla w(x,t)\Big)^2 dxdt. \tag{87}$$

Using (82) and (87), we finally get (80).

Proof of theorem (5.2).

Let u_1 and u_2 be two solutions of Problem (5), with initial conditions u_{01} and u_{02} and source terms v_1 and v_2 respectively. We set $u_d = u_1 - u_2$, $v_d = v_1 - v_2$ and $u_{0d} = u_{01} - u_{02}$. We also define, for all $(x,t)\in\Omega\times\mathbb{R}_+^*$, $q(x,t) = \dfrac{\varphi(u_1(x,t)) - \varphi(u_2(x,t))}{u_1(x,t) - u_2(x,t)}$ if $u_1(x,t)\neq u_2(x,t)$, else $q(x,t) = 0$. For all $T\in\mathbb{R}_+^*$ and for all $\psi\in\mathcal{A}_T$, we deduce from (5) that

$$\begin{aligned}\int_0^T \int_\Omega \Big[u_d(x,t)\Big(\psi_t(x,t) + q(x,t)\Delta\psi(x,t)\Big) + v_d(x,t)\psi(x,t)\Big]\ dx\ dt \quad &+ \\ \int_\Omega u_{0d}(x)\psi(x,0)dx \quad &= 0.\end{aligned} \tag{88}$$

Let $w \in C_c^\infty(\Omega \times (0,T))$, such that $|w| \leq 1$. Since φ is locally Lipschitz continuous, we can define its Lipschitz constant, say B_M, on $[-M, M]$, where $M = \max(||u_1||_{L^\infty(\Omega\times(0,T))}, ||u_2||_{L^\infty(\Omega\times(0,T))})$ so that $0 \leq q \leq B_M$ a.e.

Let $n \in \mathbb{N}^*$. Using mollifiers, one can find a function $q_{1n} \in C_c^\infty(\Omega\times(0,T))$ such that $||q_{1n} - q||_{L^2(\Omega\times(0,T))} \leq \frac{1}{n}$ and $0 \leq q_{1n} \leq B_M$. Let $q_n = q_{1n} + \frac{1}{n}$. We get

$$\frac{1}{n} \leq q_n(x,t) \leq B_M + \frac{1}{n}, \text{ for all } (x,t) \in \Omega \times (0,T), \tag{89}$$

and

$$\begin{aligned}\int_{\Omega\times(0,T)} \frac{(q_n(x,t) - q(x,t))^2}{q_n(x,t)} dxdt \leq 2\Big(\quad &\int_{\Omega\times(0,T)} \frac{(q_n(x,t) - q_{1n}(x,t))^2}{q_n(x,t)} dxdt + \\ &\int_{\Omega\times(0,T)} \frac{(q_{1n}(x,t) - q(x,t))^2}{q_n(x,t)} dxdt\Big),\end{aligned} \tag{90}$$

which shows that

$$\int_{\Omega\times(0,T)} \frac{(q_n(x,t) - q(x,t))^2}{q_n(x,t)} dxdt \leq 2n\Big(\frac{Tm(\Omega)}{n^2} + \frac{1}{n^2}\Big). \tag{91}$$

It leads to

$$||\frac{q_n - q}{\sqrt{q_n}}||_{L^2(\Omega\times(0,T))} \to 0 \quad \text{as} \quad n \to \infty. \tag{92}$$

Let $\psi_n \in \mathcal{A}_T$ be given by lemma 5.1, with $g = q_n$. Substituting ψ by ψ_n in (88), using (76) and (79) give

$$\begin{aligned}|\int_0^T \int_\Omega u_d(x,t)\Big(w(x,t) + (q(x,t) - q_n(x,t))\Delta\psi_n(x,t)\Big)dxdt| \quad &\leq \\ \int_0^T \int_\Omega |v_d(x,t)|(T-t)dxdt + T\int_\Omega |u_{0d}(x)|dx.&\end{aligned} \tag{93}$$

Using Cauchy-Schwarz inequality, we get

$$\begin{aligned}&\Big[\int_0^T \int_\Omega |u_d(x,t)||(q(x,t) - q_n(x,t))\Delta\psi_n(x,t)|dxdt\Big]^2 \leq 4M^2. \\ &\int_0^T \int_\Omega \Big(\frac{q(x,t) - q_n(x,t)}{\sqrt{q_n(x,t)}}\Big)^2\ dx\ dt \int_0^T \int_\Omega q_n(x,t)\Big(\Delta\psi_n(x,t)\Big)^2 dxdt.\end{aligned} \tag{94}$$

We deduce from (80) and (92) that the right hand side of (94) tends to zero as $n \to \infty$. Hence the left hand side of (94) also tends to zero as $n \to \infty$. Therefore letting $n \to \infty$ in (93), we get

$$\begin{aligned}|\int_0^T \int_\Omega u_d(x,t)w(x,t)dxdt| \leq \quad &\int_0^T \int_\Omega |v_d(x,t)|(T-t)dxdt + \\ &T\int_\Omega |u_{0d}(x)|dx.\end{aligned} \tag{95}$$

Inequality (95) holds for any function $w \in C_c^\infty(\Omega \times (0,T))$, with $|w| \leq 1$. We take as functions w the elements of a sequence $(w_m)_{m\in\mathbb{N}}$, such that for all $m \in \mathbb{N}$ $w_m \in C_c^\infty(\Omega \times (0,T))$ and $|w_m| \leq 1$, and the sequence $(w_m)_{m\in\mathbb{N}}$ converges to $\text{sign}(u_d(.,.))$ for the topology of $L^1(\Omega \times (0,T))$. Letting $m \to \infty$ yields

$$\int_0^T \int_\Omega |u_d(x,t)| dx dt \leq \int_0^T \int_\Omega |v_d(x,t)|(T-t) \; dx \; dt + T \int_\Omega |u_{0d}(x)| dx, \tag{96}$$

which concludes the proof of Theorem 5.2.

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