# On the convergence of the parabolic approximation of a conservation law in several space dimensions

**Thierry GALLOUËT et Florence HUBERT**

Université de Provence, C.M.I., 39 rue F. Joliot Curie, 13453 Marseille Cedex 13, France.

**Abstract.** *We give a proof of the convergence of the solution of the parabolic approximation $u_t^\varepsilon + div\ f(x,t,u^\varepsilon) = \varepsilon\Delta u^\varepsilon$ towards the entropic solution of the scalar conservation law $u_t + div\ f(x,t,u) = 0$ in several space dimensions. For any initial condition $u_0 \in L^\infty(\mathbb{R}^N)$ and for a large class of flux $f$, we prove the strong convergence in any $L^p_{loc}$ space, using the notion of entropy process solution, which is a generalization of the measure-valued solutions of DiPerna.*

## 1 Introduction

We give a proof of the strong convergence in $L^p_{loc}, p < \infty$ of the solution of the parabolic approximation:

$$u_t^\varepsilon + \text{div } f(x,t,u^\varepsilon) = \varepsilon\Delta u^\varepsilon,\ x \in \mathbb{R}^N,\ t > 0 \tag{1}$$

$$u^\varepsilon(x,0) = u_0^\varepsilon(x),\ x \in \mathbb{R}^N \tag{2}$$

towards the entropic solution to the scalar conservation law

$$u_t + \text{div } f(x,t,u) = 0,\ x \in \mathbb{R}^N,\ t > 0 \tag{3}$$

$$u(x,0) = u_0(x),\ x \in \mathbb{R}^N \tag{4}$$

where $u_0 \in L^\infty(\mathbb{R}^N)$, $u_0^\varepsilon$ denotes some approximation of $u_0$ such that

$$\begin{cases} \|u_0^\varepsilon\|_{L^\infty} \leq M \\ u_0^\varepsilon \to u_0 \text{ strongly in } L^1_{loc}(\mathbb{R}^N); \end{cases} \tag{5}$$

and the flux $f$ satisfies:

$$\begin{cases} f \in \mathcal{C}^1(\mathbb{R}^N \times \mathbb{R}^+ \times \mathbb{R}) \\ \text{div}_x f(x,t,s) = 0, \forall (x,t,s) \in \mathbb{R}^N \times \mathbb{R}^+ \times \mathbb{R} \\ \dfrac{\partial f}{\partial s} \in W^{1,\infty}_{loc}(\mathbb{R}^N \times \mathbb{R}^+ \times \mathbb{R}) \end{cases} \tag{6}$$

The convergence of the approximate solutions $(u^\varepsilon)$ of (1)(2) towards the entropic solution $u$ of the hyperbolic equation with initial condition (3)(4) has been studied in several way. In 1970, Kruzkov gave in [6] a proof of these convergence using the $L^1_{loc}$-compactness of the approximate solutions (see also [9]). Our proof follows ideas of Tartar ([11]) and DiPerna ([2]).

For any uniformly bounded initial data $u_0^\varepsilon \in L^\infty(\mathbb{R}^n)$ ($\|u_0^\varepsilon\|_{L^\infty(\mathbb{R}^N)} \le M$), it is classical to see that the parabolic approximation (1) (2) admits a smooth solution, at least $\mathcal{C}^2(\mathbb{R}^N \times (0,+\infty))$, which is uniformly bounded : $\|u^\varepsilon\|_{L^\infty(\mathbb{R}^N \times \mathbb{R}^+)} \le M$ (See [8][10]).
Thanks to this $L^\infty$ estimate, as in Tartar's work, one can construct a Young measure as the limit of a sequence of approximate solutions $(u^\varepsilon)$ : for all $g \in \mathcal{C}^0(\mathbb{R}^N \times \mathbb{R}^+ \times \mathbb{R})$ and for all $\varphi : \mathbb{R}^N \times [0,\infty) \to \mathbb{R}$ compactly supported, $\varphi \in L^1(\mathbb{R}^N \times [0,\infty))$ :

$$(7)\ \int_0^\infty \int_{\mathbb{R}^N} g(x,t,u^\varepsilon(x,t))\,\varphi(x,t)\,dx\,dt \to \int_0^\infty \int_{\mathbb{R}^N} < g(x,t,\lambda), \nu_{x,t}(\lambda) > \ \varphi(x,t)\,dx\,dt.$$

This result is well known for continuous functions on $\mathbb{R}$ ($g$ does not depends on $(x,t) \in \mathbb{R}^N \times \mathbb{R}^+$) and has been generalized to functions in $\mathcal{C}^0(\mathbb{R}^N \times \mathbb{R}^+ \times \mathbb{R})$ in the work of [1] using "process functions", which were introduced by Eymard-Gallouët-Herbin in [3] :

**Theorem:**
*Let $(u_n)_{n\in\mathbb{N}}$ be bounded in $L^\infty(\mathbb{R}^N \times \mathbb{R}^+)$, there exists a function $\mu \in L^\infty(\mathbb{R}^N \times \mathbb{R}^+ \times (0,1))$ and a subsequence, still denoted by $(u_n)$, such that $u_n$ converges towards $\mu$ in a "nonlinear weak-$\star$ sense" as $n \to \infty$, that is for any $g \in \mathcal{C}^0(\mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^N)$, and for any $\varphi \in L^1(\mathbb{R}^N \times \mathbb{R}^+)$ compactly supported :*

$$(8)\ \int_{\mathbb{R}^+} \int_{\mathbb{R}^N} g(x,t,u_n(x,t))\,\varphi(x,t)\,dx\,dt \to \int_{\mathbb{R}^+}\int_{\mathbb{R}^N}\int_0^1 g(x,t,\mu(x,t,\alpha))\,\varphi(x,t)\,d\alpha\,dx\,dt.$$

*We call "process function", the function $\mu$.*

**Remarks:**

- $\star$ To each "process function" $\mu$ can be associated a Young measure family $\nu_{x,t}$ which satisfies (7) and such that

$$(9)\qquad \int_0^1 g(x,t,\mu(x,t,\alpha))\,d\alpha =< \nu_{x,t}(\lambda), g(x,t,\lambda) > \quad a.e.\ (x,t).$$

- $\star$ Conversely, to each Young measure family $\nu_{x,t}$, associated to a bounded sequence $(u_n)$, we can associate a "process function" which verifies (9).

- $\star$ Let $u$ be in $L^\infty(\mathbb{R}^N \times \mathbb{R}^+)$, we suppose that "$u_n \to \mu$ " and so that "$u_n \to \nu_{x,t}$", then

$$u_n \text{ in } L^p_{loc} \Leftrightarrow \mu(x,t,\alpha) = u(x,t)\ a.e.\ (x,t,\alpha) \Leftrightarrow \nu_{x,t} = \delta_{u(x,t)} a.e.\ (x,t)$$

Tartar [11] shows, using compensated compactness, that in one space dimension, the Young measure $\nu_{x,t}$ is supported in an interval on which $f$ is affine, if $u_0^\varepsilon$ converges weakly to $u_0$ in $L^1_{loc}$.
In the next section, we prove that if $u_0^\varepsilon$ converge strongly towards $u_0$ , then for all flux $f$ satisfying (6), $\nu_{x,t}$ is a Dirac measure at the point $u(x,t)$ for $a.e.$ $(x,t)$ and by the way the

familly $u^\varepsilon$ strongly converges in $L^p_{loc}$ for all $p < \infty$ towards $u$, the unique entropic solution of the hyperbolic problem (3)(4).
For that, we prove that a "process function" $\mu$ associated to the familly $(u^\varepsilon)$ is an "entropy process solution" of the hyperbolic problem (3)(4) (see [1][3]):

**Definition:**
*A function $\mu \in L^\infty(\mathbb{R}^N \times \mathbb{R}^+ \times (0,1), \mathbb{R})$ is called "entropy process solution" of the hyperbolic problem (3)(4) if, for all $\varphi \in \mathcal{C}^1_c(\mathbb{R}^N \times \mathbb{R}^+, \mathbb{R}^+)$, and for all $\kappa \in \mathbb{R}$ :*

$$\int_{\mathbb{R}^+}\int_{\mathbb{R}^N}\int_0^1 [\eta_\kappa(\mu(x,t,\alpha))\varphi_t(x,t) + q_\kappa(x,t,\mu(x,t,\alpha))\nabla\varphi(x,t)]\, d\alpha\, dx\, dt \\ + \int_{\mathbb{R}^N} \eta(u_0(x))\varphi(x,0)\, dx \geq 0.$$

*where*

(10) $$\eta_\kappa(\mu) = \mid \mu - \kappa \mid \quad et\ q_\kappa(x,t,\mu) = (f(x,t,\mu\top\kappa) - f(x,t,\mu\bot\kappa)),$$

*with $a\top b = \max(a,b)$ and $a\bot b = \min(a,b)$.*
We can check that the Young measure associated to an "entropy process solution" is a "measure valued solution" of the hyperbolic equation (3) in the sense of DiPerna [2] which satisfies the initial condition in the sense:

(11) $$\lim_{\tau\to 0}\int_0^\tau\int_K < \nu_{x,t}(\lambda), |\lambda - u_0(x)| > dx\, dt = 0, \text{ for all compact } K \subset \mathbb{R}^N.$$

To conclude, we use a generalization of the theorem of uniqueness of measure valued solutions of DiPerna :

**Theorem:**
*If $\mu$ is an "entropy process solution" of the hyperbolic equation with initial condition (3)(4), then*

$$\mu(x,t,\alpha) = u(x,t) \text{ for almost every } (x,t,\alpha) \in \mathbb{R}^N \times \mathbb{R}^+ \times (0,1)$$

*where $u$ denotes the unique entropic solution of (3)(4).*
This result, first proved in [3] and then extended in [1][4][5], enables us to deduce that the parabolic approximation (1)(2) converges towards the entropic solution of (3)(4), this convergence is, moreover, strong in each $L^p_{loc}$-space for $p < \infty$.

# 2 Convergence of the parabolic approximation towards the entropic solution of the conservation law

## 2.1 Convergence towards the "entropy process solution"

**Theorem 1** *Under assumptions (5)(6), the solution $u^\varepsilon$ of the parabolic approximation (1)(2) converges in the "nonlinear weak-$\star$ sense" towards an "entropy process solution" of the problem (3)(4).*

**Sketch of proof :**
Under assumptions (5)(6), the problem (1)(2) has a smooth bounded solution $u^\varepsilon \in L^\infty(\mathbb{R}^N \times (0,+\infty)) \cap \mathcal{C}^2(\mathbb{R}^N \times (0,+\infty))$ such that $\|u^\varepsilon\|_\infty \leq M$. Moreover, we can derive an energy estimate : for any compact set $K$ of $\mathbb{R}^N \times [0,+\infty)$ ,

$$\varepsilon \int_K |\nabla u^\varepsilon|^2 dx\, dt \leq C, \tag{12}$$

where $C$ only depends on $K$ and on $M$.
We can then easily check that for any function $\varphi \in \mathcal{C}_c^1(\mathbb{R}^N \times [0,+\infty), \mathbb{R}^+)$ and for any constant $\kappa \in \mathbb{R}$,

$$\int_0^\infty \int_{\mathbb{R}^N} \eta_\kappa(u^\varepsilon)\, \varphi_t + q_\kappa(x,t,u^\varepsilon)\, .\nabla\varphi\, dx\, dt + \int_{\mathbb{R}^N} \eta_\kappa(u_0^\varepsilon(x))\, \varphi(x,0)\, dx \geq \xi(\varepsilon), \tag{13}$$

where $\eta_\kappa$ et $q_\kappa$ are defined by (10) and

$$\xi(\varepsilon) = \varepsilon \int_0^\infty \int_{\mathbb{R}^N} \text{sgn }(u^\varepsilon - \kappa)\nabla u^\varepsilon . \nabla\varphi dx\, dt \to 0 \text{ as } \varepsilon \to 0.$$

Therefore,

$$|\xi(\varepsilon)| \leq \sqrt{\varepsilon} \quad \|\nabla\varphi\|_\infty \left[\varepsilon \int_K |\nabla u^\varepsilon|^2 dx\, dt\right]^{\frac{1}{2}} mes(K)^{\frac{1}{2}}$$

where $K$ denotes the support of $\varphi$.
If $(u_\varepsilon)$ converges (as $\varepsilon \to 0$) in the "nonlinear weak-$\star$ sense" towards $\mu$, we then deduce, passing to the limit in Equation (13) and using the fact that $u_0^\varepsilon$ converges strongly in $L^1_{loc}(\mathbb{R}^N)$ towards $u_0$, that for any function $\varphi \in L^1(\mathbb{R}^N \times \mathbb{R}^+)$ and any $\kappa \in \mathbb{R}$

$$\begin{aligned} &\int_{\mathbb{R}^+} \int_{\mathbb{R}^N} \int_0^1 [\eta_\kappa(\mu(x,t,\alpha))\, \varphi_t(x,t) + q_\kappa(x,t,\mu(x,t,\alpha))\, \nabla\varphi(x,t)]\, d\alpha\, dx\, dt \\ &+ \int_{\mathbb{R}^N} \eta_\kappa(u^0(x))\, \varphi(x,0)\, dx \geq 0. \end{aligned} \tag{14}$$

Hence, $u^\varepsilon$ converges in the "nonlinear weak-$\star$ sense" towards an "entropy process solution" of the hyperbolic problem (3)(4).

■

## 2.2 Convergence to the entropic solution :

**Theorem 2** *Let $u_0 \in L^\infty(\mathbb{R}^N)$, and assume (5)(6). The solution $u^\varepsilon$ of the parabolic approximation (1)(2) converges strongly in $L^p_{loc}(\mathbb{R}^N \times \mathbb{R}^+)$ for all $p < \infty$ towards the entropic solution of the problem (3)(4).*

**Proof of theorem 2 :**
The uniqueness of the "entropy process solution" proved in [1][3] ensures the convergence in the "nonlinear weak-$\star$ sense" of $u^\varepsilon$ towards an "entropy process solution" $\mu$ of the hyperbolic problem (3)(4) and this entropy process solution is given by the entropic solution

$u$ ($\mu(x,t,\alpha) = u(x,t)$ *a.e.*) of the hyperbolic problem (3)(4). The strong convergence remains to be proved. So, if $K$ denotes any compact set of $\mathbb{R}^N \times \mathbb{R}^+$, using the nonlinear weak-$\star$ convergence of $u^\varepsilon$ towards $\mu$ for $g(s) = s^2$, we check that

$$u^\varepsilon \to u \text{ weakly in } L^2(K) \text{ and } \|u^\varepsilon\|_{L^2(K)} \to \|u\|_{L^2(K)}.$$

We immediately conclude that $u^\varepsilon$ strongly converges towards $u$ in $L^2(K)$ for any compact set $K$ of $\mathbb{R}^N \times \mathbb{R}^+$ . As the family $(u^\varepsilon)$ is bounded in $L^\infty(K)$ (for any $K$), we deduce that the convergence is strong in all $L^p_{loc}$, $p < \infty$.

■

**Remarks:**

- $\star$ For an initial condition $u_0 \in BV(\mathbb{R}^N)$, and $u_0^\varepsilon = u_0$ *a.e.*, Kuznetsov [7] get some estimate of the error $u^\varepsilon - u$ in the space $L^\infty(0,T;L^1(\mathbb{R}^N))$ :

$$\|u^\varepsilon - u(.,t)\|_{L^1(\mathbb{R}^N)} \leq C(\|u_0\|_{BV(\mathbb{R}^N)})\sqrt{t\varepsilon}.$$

- $\star$ For an initial data $u_0 \in L^\infty(\mathbb{R}^N)$, we can also obtain an estimate in the space $L^\infty(0,T;L^1_{loc}(\mathbb{R}^N))$ following the proof of [4][5].

## References


[1] C. Chainais-Hillairet. Preprint 205, UMPA, ENS Lyon, to appear in M2AN, 1998.

[2] R. DiPerna. Measure-valued solutions to conservation laws, *Arch. Rat. Mech. Anal.*, vol. 88, pp. 223-270, 1985.

[3] R. Eymard, T. Gallouët, R. Herbin. Existence and uniqueness of the entropy solution to a nonlinear hyperbolic equation, *Chin. Ann. of Math.*, vol. 16B, 1, pp.1-14, 1995.

[4] R. Eymard, T. Gallouët, M. Ghilani, R. Herbin. Error estimates for the approximate solutions of a nonlinear hyperbolic equation given by finite volume schemes, preprint 97-7 LATP (Université de Marseille, France, accepted for publication in IMAJNA.

[5] R. Eymard, T. Gallouët, R. Herbin. The Finite Volume Method, to appear in Handbook of Numerical Analysis, North-Holland, P.G. Ciarlet et J.L. Lions Eds, 1998.

[6] S. N. Kruskov. First Order Quasilinear Equations in Several Indepedant Variables, *Math. USSR Sbornik*, vol. 10, 2, pp. 1489-1502, 1970.

[7] N.N. Kuznetsov. Accuracy of some approximate methods for computing the weak solutions of a first-order quasi-linear equation, *Zh. $\nu\bar{\nu}$ chisl. Mat. mat. Fiz.*, vol. 16, 6, pp. 1489-1502, 1976.

[8] O.A. Ladyzenskaya, V.A. Solonnikov, N.N. Ural'ceva. *Linear and Quasilinear Equations of Parabolic Type*, vol. 23 of *Transaction of Mathematical Monograph.*

[9] J. Malek, J. Necas, M. Rokyta, M. Ruzicka. *Weak and measure-valued solutions to evolutionary PDEs* Chapman & Hall, London, 1996.

[10] M.H. Protter and H.F. Weinberger. *Maximum principles in Differential Equations*, Prentice-Hall, INC., 1967.

[11] L. Tartar. *Compensated compactness*, Herriot Watt Symposium vol. 4, Pitman,1978.