Modelling wells in porous media flows

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Abstract In this paper, we prove the existence of weak solutions for mathematical models of miscible and immiscible flow through porous medium. An important difficulty comes from the modelization of the wells, which does not allow us to use classical variational formulations of the equations.

Keywords: Nonlinear parabolic equation, flow through porous media, existence of solutions, measure data
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1 Introduction

This paper is devoted to the mathematical analysis of some models of flows in porous media. These models are those used for reservoir simulation's in petroleum engineering see [3], [4], [6], [8]. In these problems, wells have to be conveniently modeled. In particular, since the diameter of a well (about 10 cm) is very small compared to the length of the reservoir or to the mesh size of a "reasonable" discretization (10 to 100m...), one has to consider the action of a well as some spatial "measures" (instead of classical functions) whose supports shrink to a point (for two dimensionnal models) or to a segment (for three dimensionnal models).

In this paper, existence of weak solutions for miscible and immiscible flows are proven. Actually, using the Schauder fixed point theorem, one proves existence of weak solution for "regularized wells". Then through some estimates on these solutions, one proves the general case, passing to the limit on the solutions of regularized equations.

2 Problems and main results

2.1 Model problem

A first basic model is described in this section. Roughly speaking, it consists in a parabolic equation (in the unknown u) having a "measure" (instead of a classical function) as a source term and with a nonlinear convection term involving a velocity field given by the solution of an elliptic equation (on the unknown p with a given u) with also a measure as source term. To complete the problem, some initial and boundary conditions are given. Such models appear in the modelization of fluid flows through porous media. More complete models are described hereafter (Section 2.2) but the main difficulties are present in the basic model (in particular, the occurence of measures as source terms).

The unknowns of this model problem are p (a pressure in realistic models), u (a concentration or a saturation) and $\mathbf{v}$ (a filtration velocity). The equations satisfied by these unknowns are the following: for $(x,t) \in \Omega \times (0,T)$,

$$u_t(x,t) + \operatorname{div}(\mathbf{v}\, f(u))(x,t) - \operatorname{div}(\mathbf{D}(\mathbf{v})\nabla u)(x,t) + u\, b(x,t)\mu(x) - c\, a(x,t)\mu(x) = 0, \tag{2.1}$$

$$\begin{aligned} &\mathbf{v}(x,t) + A(x,t,u(x,t))\,(\nabla p(x,t) - \mathbf{g}(x,t,u(x,t))) = 0, \\ &\operatorname{div} \mathbf{v}(x,t) - a(x,t)\mu(x) + b(x,t)\mu(x) = 0, \end{aligned} \tag{2.2}$$

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where u_t denotes the time derivative of u and where div and ∇ denote the usual spatial differential operators. The functions f, $\mathbf{D}$, b, c, a, A and $\mathbf{g}$ are given and μ is a given measure on Ω (see below).
To complete the problem, the following initial and boundary conditions are given:

$$u(x,0) = u_0(x), \; x \in \Omega, \tag{2.3}$$

$$\mathbf{D}(\mathbf{v}(x,t))\nabla u(x,t) \cdot \mathbf{n}(x) = 0, (x,t) \in \partial\Omega \times (0,T), \tag{2.4}$$

$$\mathbf{v}(x,t) \cdot \mathbf{n}(x) = 0, (x,t) \in \partial\Omega \times (0,T). \tag{2.5}$$

where $\mathbf{n}(x)$ is the outward normal unit vector at the boundary $\partial\Omega$ of Ω at the point x.
Let us give the assumptions on the data.

$$T \in \mathbb{R}_+^\star, \; \Omega \text{ is a bounded open set of } \mathbb{R}^d, \; d = 2 \text{ or } 3, \text{ with a Lipschitz continuous boundary,} \tag{2.6}$$

$$\begin{array}{l} c \in L^\infty((0,T); C(\overline{\Omega})), \; 0 \le c \le 1, \\ u_0 \in L^\infty(\Omega), \; 0 \le u_0 \le 1, \end{array} \tag{2.7}$$

$$\begin{array}{l} f \in C(\mathbb{R}, \mathbb{R}) \text{ is Lipschitz continuous and} \\ f(y) = 0 \text{ for } y \le 0, \\ f(y) = 1 \text{ for } y \ge 1, \end{array} \tag{2.8}$$

$$\begin{array}{l} A \text{ is a Caratheodory function from } \Omega \times (0,T) \times \mathbb{R} \text{ into } M_d(\mathbb{R}), \text{ see [20] p.p. 133, satisfying:} \\ \exists \alpha > 0, \; A(x,t,s)\xi \cdot \xi \ge \alpha|\xi|^2, \text{ for a.e. } (x,t) \in \Omega \times (0,T), \; \forall s \in \mathbb{R}, \; \forall \xi \in \mathbb{R}^d, \\ \exists \beta \in \mathbb{R}, \; |A(x,t,s)| \le \beta, \text{ for a.e. } (x,t) \in \Omega \times (0,T), \; \forall s \in \mathbb{R}, \end{array} \tag{2.9}$$

$$\begin{array}{l} \mathbf{g} \text{ is a Caratheodory function from } \Omega \times (0,T) \times \mathbb{R} \text{ into } \mathbb{R}^d \text{ satisfying:} \\ \exists \bar{g} \in \mathbb{R}, \; |\mathbf{g}(x,t,s)| \le \bar{g}, \text{ for a.e. } (x,t) \in \Omega \times (0,T), \; \forall s \in \mathbb{R}, \end{array} \tag{2.10}$$

$$\begin{array}{l} \mathbf{D} \text{ is a continuous and bounded function from } \mathbb{R}^d \text{ to } M_d(\mathbb{R}) \text{ and satisfies} \\ \exists \gamma > 0, \; \mathbf{D}(\mathbf{s})\xi \cdot \xi \ge \gamma|\xi|^2, \; \forall \mathbf{s} \in \mathbb{R}^d, \; \forall \xi \in \mathbb{R}^d. \end{array} \tag{2.11}$$

Note that $x \cdot y$ denotes the usual scalar product of x and y in $\mathbb{R}^d$, $M_d(\mathbb{R})$ is the set of $d \times d$ matrices with real coefficients and $|\cdot|$ denotes the euclidian norm in $\mathbb{R}^d$ and the induced norm in $M_d(\mathbb{R})$.

It remains to make more precise the assumptions on $a\mu$ and $b\mu$. Let $\mathcal{M}(\Omega)$ denote the set of finite measures on Ω. Recall that a finite measure μ on Ω is a σ-additive application from the Borel σ-algebra of Ω in $\mathbb{R}$. If $\mu \in \mathcal{M}(\Omega)$ takes its values in $\mathbb{R}_+$, then $\mu \in \mathcal{M}_+(\Omega)$. Given μ in $\mathcal{M}(\Omega)$, it is identified as usually with the linear continuous application induced on $C(\overline{\Omega})$ ($C(\overline{\Omega})$ is endowed with the $L^\infty(\Omega)$-norm). Then, $\langle \mu, \varphi \rangle_{(C(\overline{\Omega}))', C(\overline{\Omega})} = \int_\Omega \varphi(x) d\mu(x)$, for any $\varphi \in C(\overline{\Omega})$. Note that a linear continuous application from $C(\overline{\Omega})$ to $\mathbb{R}$ is not necessarily given by some $\mu \in \mathcal{M}(\Omega)$ (but it is always given by some $\mu \in \mathcal{M}(\overline{\Omega})$). The topological dual space of a Banach space E is denoted by E'.
The assumptions on a, b and μ are the following

$$\mu \in \mathcal{M}_+(\Omega) \tag{2.12}$$

and

$$\begin{array}{l} a, b \in L^\infty((0,T); C(\overline{\Omega})) \\ a(x,t) \ge 0, \; b(x,t) \ge 0, \end{array} \tag{2.13}$$

According to the boundary condition (2.5), one has to assume the following compatibility condition

$$\int_\Omega a(x,t)\, d\mu(x) = \int_\Omega b(x,t)\, d\mu(x). \tag{2.14}$$

In the three-dimensional case an additional assumption is required. It reads

$$\mu \in (W^{1,q}(\Omega))', \ \forall q > 2. \tag{2.15}$$

In the two-dimensional case, this assumption is not an additional assumption since $W^{1,q}(\Omega) \subset C(\overline{\Omega})$ for $q > 2$ and then $\mathcal{M}(\Omega) \subset (W^{1,q}(\Omega))'$. When $d = 2$ and for the modelization of porous media flows, the measure μ has often its support reduced to some points, this support consisting in the localization of the wells. In the three-dimensional case, the assumption (2.15) is realistic for the applications, at least for flows through porous media. Indeed, a convenient modelization of the wells leads to a measure μ which consists in a finite number of terms such as $h\nu$ where h is a bounded measurable function and ν is the one dimensional Lebesgue measure on a one dimensionnal curve (indeed, it will be sufficient for us to assume that $h \in L^2(\Omega, \nu)$). Such a measure belongs to $(W^{1,q}(\Omega))'$ for $q > 2$ since there exists a continuous "trace operator" from $W^{1,q}(\Omega)$ to $L^q(S)$, when S is a segment in $\overline{\Omega}$.

Remark 2.1 *Note that the composition of the injected fluid is given, it appears in the term $ca\mu$ in Equation (2.1), but the composition of the produced fluid is an unknown, it appears in the term $ub\mu$ in Equation (2.1). This structure of Equation (2.1), together with the second part of Equation (2.2) (that is* $\operatorname{div}\mathbf{v} = a\mu - b\mu$*) and Condition (2.8) on f (and $0 \le u_0 \le 1$), is crucial to obtain the existence of a solution such that $0 \le u \le 1$.*

For a measure ν on the Borel $\sigma-$algebra on Ω, one denotes $L^p(\Omega, \nu)$ ($p \in [0, \infty]$) the usual L^p space related to ν. If ν is the usual Lebesgue measure (denoted by λ in the sequel), the space $L^p(\Omega, \nu)$ is denoted by $L^p(\Omega)$.

Definition 2.1 *let (2.6)-(2.15) hold. Then (u, p) is a weak solution of (2.1)-(2.5) if*

$$\begin{array}{l} u \in L^2((0,T); H^1(\Omega)), \ 0 \le u \le 1, \\ u \in L^\infty((0,T); L^1(\Omega,\mu)), \ 0 \le u(x,t) \le 1, \ \ \textit{for } \mu\textit{-a.e. } x \in \Omega, \ \ \textit{for a.e. } t \in (0,T), \end{array} \tag{2.16}$$

$$u_t \in L^2((0,T); (W^{1,s}(\Omega))'), \ \forall s > d, \tag{2.17}$$

$$u \in C([0,T]; (W^{1,s}(\Omega))'), \ \forall s > d, \ u(.,0) = u_0 \ (\textit{in } W^{1,s}(\Omega))'), \tag{2.18}$$

$$p \in L^\infty((0,T); W^{1,q}(\Omega)), \ \forall q \in [1,2), \tag{2.19}$$

$$\begin{array}{l} \langle u_t(.,t), \varphi \rangle_{(W^{1,r})', W^{1,r}} + \displaystyle\int_\Omega \mathbf{D}(\mathbf{v}(x,t)) \nabla u(x,t) \cdot \nabla \varphi(x) dx \\ - \displaystyle\int_\Omega \mathbf{v}(x,t) \cdot \nabla \varphi(x) f(u(x,t)) dx + \int_\Omega u(x,t) \varphi(x) b(x,t) d\mu(x) - \int_\Omega c(x,t) \varphi(x) a(x,t) d\mu(x) = 0, \\ \qquad \forall \varphi \in W^{1,r}(\Omega), \ \forall r > d, \ \textit{for a.e. } t \in (0,T), \end{array} \tag{2.20}$$

$$\begin{array}{l} \mathbf{v}(x,t) = -A(x,t,u(x,t)) \left(\nabla p(x,t) - \mathbf{g}(x,t,u(x,t)) \right), \ \ \textit{for a.e. } (x,t) \in \Omega \times (0,T), \\ - \displaystyle\int_\Omega \mathbf{v}(x) \cdot \nabla \psi(x) dx = \int_\Omega \psi(x) a(x,t) d\mu(x) - \int_\Omega \psi(x) b(x,t) d\mu(x), \\ \qquad \forall \psi \in W^{1,r}(\Omega), \ \forall r > d, \ \textit{for a.e. } t \in (0,T). \end{array} \tag{2.21}$$

Note that $\mathbf{v} \in L^\infty((0,T); (L^q(\Omega))^d)$ for all $q \in [1,2)$.
The main result of this paper is the following one:

Theorem 2.1 *Let (2.6)-(2.15) hold. There exists at least one weak solution to (2.1)-(2.5), in the sense of Definition 2.1.*

Remark 2.2 *Condition (2.15) allows us to obtain a stronger existence result (if $d = 3$) for (2.1)-(2.5). To be more precise, the proof of Theorem 2.1 gives, under the hypotheses (2.6)-(2.15), an existence result with $u_t \in L^2((0,T);(W^{1,s}(\Omega))')$ and $u \in C([0,T];(W^{1,s}(\Omega))')$ for all $s > 2$, instead of for all $s > d$. See the proof of Theorem 2.1 in Section 5.*

In order to prove Theorem 2.1 (in Section 5), one first proves the existence of a solution to (2.1)-(2.5) when the data are more regular (in Section 4), namely when $\mu \in L^2(\Omega)$. This existence result, for regular data, reads as follows:

Proposition 2.1 *Let (2.6)-(2.14) hold and assume $\mu \in L^2(\Omega)$. There exists a solution (u,p) of (2.1)-(2.5) satisfying :*

$$\begin{array}{l} u \in L^2((0,T);H^1(\Omega)) \cap C([0,T];L^2(\Omega)),\ 0 \le u \le 1, \\ u_t \in L^2((0,T);(H^1(\Omega))'), \end{array} \tag{2.22}$$

$$u(.,0) = u_0 \quad in\ L^2(\Omega), \tag{2.23}$$

$$p \in L^\infty((0,T);H^1(\Omega)),\ \mathbf{v} \in L^\infty((0,T);(L^2(\Omega))^d), \tag{2.24}$$

$$\begin{array}{l} \langle u_t(.,t),\varphi\rangle_{(H^1)',H^1} + \displaystyle\int_\Omega \mathbf{D}(\mathbf{v}(x,t))\nabla u(x,t)\cdot\nabla\varphi(x)dx \\ -\displaystyle\int_\Omega \mathbf{v}(x,t)\cdot\nabla\varphi(x)f(u(x,t))dx + \int_\Omega u(x,t)\varphi(x)b(x,t)\mu(x)dx - \int_\Omega c(x,t)\varphi(x)a(x,t)\mu(x)dx = 0, \\ \hfill \forall\varphi \in H^1(\Omega),\ for\ a.e.\ t \in (0,T), \end{array} \tag{2.25}$$

$$\mathbf{v}(x,t) = -A(x,t,u(x,t))\left\{\nabla p(x,t) - \mathbf{g}(x,t,u(x,t))\right\}, \tag{2.26}$$

$$\begin{array}{l} -\displaystyle\int_\Omega \mathbf{v}(x,t)\cdot\nabla\psi(x)dx = \int_\Omega \psi(x)a(x,t)\mu(x)dx - \int_\Omega \psi(x)b(x,t)\mu(x)dx, \\ \hfill \forall\psi \in H^1(\Omega),\ for\ a.e.\ t \in (0,T). \end{array} \tag{2.27}$$

The proof of Proposition 2.1 is carried out with the Schauder's fixed point theorem. Moreover we establish some estimates on the solutions given by Proposition 2.1 yielding the proof of Theorem 2.1 upon passing to the limit on approximate solutions.

2.2 Applications

In this section we give two applications arising from petroleum engineering [3], [4], [6], [8]. The first one deals with miscible flows through porous media, the second one deals with immiscible flows. In both applications the measures terms modelize the wells.

2.2.1 Miscible flow

According to [7], mass transfer of a substrate with concentration c in a saturated porous medium satisfies the following equations (mass conservation and Darcy law).

$$\begin{array}{c} c_t(x,t) + \mathrm{div}(\mathbf{v}c)(x,t)) - \mathrm{div}\big((\lambda(c) + \mathbf{D}(\mathbf{v}))\nabla c\big)(x,t)) \\ +c(x,t)\,a(x,t)\mu(x) - \bar{c}(x,t)\,b(x,t)\mu(x) = 0,\ (x,t) \in \Omega\times(0,T), \end{array} \tag{2.28}$$

$$\begin{array}{l} \mathbf{v}(x,t) + \dfrac{\mathbf{K}(x)}{\nu(c(x,t))}(\nabla p(x,t) - \rho(c(x,t))\mathbf{g}) = 0,\ (x,t) \in \Omega\times(0,T), \\ \mathrm{div}\mathbf{v}(x,t) - a(x,t)\mu(x) + b(x,t)\mu(x) = 0,\ (x,t) \in \Omega\times(0,T), \end{array} \tag{2.29}$$

with the following initial and boundary conditions

$$c(x,0) = c_0(x),\ x \in \Omega, \tag{2.30}$$

$$(\lambda(c(x,t)) + \mathbf{D}(\mathbf{v}(x,t)))\,\nabla c(x,t)\cdot \mathbf{n} = 0,\ (x,t) \in \partial\Omega \times (0,T), \tag{2.31}$$

$$\mathbf{v}(x,t)\cdot \mathbf{n} = 0,\ (x,t) \in \partial\Omega \times (0,T). \tag{2.32}$$

In this model, the concentration of the substrat is denoted by c and the given concentration at the injection well is denoted by $\bar{c}$. The filtration velocity is denoted by $\mathbf{v}$ and the pressure by p. We note $\lambda(c)$ the molecular diffusion, $\mathbf{D}(\mathbf{v})$ the dispersion tensor, $\mathbf{K}$ the permeability tensor, $\nu(c)$ is the viscosity, $\rho(c)$ denotes the density and $\mathbf{g}$ is the gravity vector. Next $a\mu$ and $b\mu$ represent the flow rates at the injection and production wells, $a\mu$ and $b\mu$ satisfy (2.12)-(2.13) and the compatibility condition (2.14) is fulfilled.

The following hypothesis are placed on the data

$$\begin{gathered}\lambda \in C(\mathbb{R},\mathbb{R}),\ \exists \lambda_m > 0;\ \lambda_m \le \lambda(\sigma),\ \forall \sigma \in \mathbb{R},\\ \mathbf{D} = (d_{ij})_{i,j=1,\dots,d},\ d_{ij} \in C_b(\mathbb{R}^d,\mathbb{R});\ \mathbf{D}(\mathbf{v})\xi\cdot\xi \ge 0,\ \forall \mathbf{v} \in \mathbb{R}^d,\ \forall \xi \in \mathbb{R}^d,\end{gathered} \tag{2.33}$$

where $C_b(E,F)$ denotes the set of continuous and bounded functions from E to F.

$$\begin{gathered}\mathbf{K} = (k_{ij})_{i,j=1,\dots,d},\ k_{ij} \in L^\infty(\Omega);\ \exists \alpha > 0;\ \mathbf{K}(x)\xi\cdot\xi \ge \alpha,\ \forall \xi \in \mathbb{R}^d,\ \text{ a.e. in } x \in \Omega,\\ \nu \in C(\mathbb{R},\mathbb{R}),\ \exists \beta > 0;\ \beta \le \nu(\sigma),\ \forall \sigma \in \mathbb{R},\\ \rho \in C(\mathbb{R},\mathbb{R}),\end{gathered} \tag{2.34}$$

$$\begin{gathered}\bar{c} \in L^\infty((0,T);C(\overline{\Omega})),\ 0 \le \bar{c} \le 1,\\ c_0 \in L^\infty(\Omega),\ 0 \le c_0 \le 1.\end{gathered} \tag{2.35}$$

Definition 2.2 *Let (2.6), (2.33)-(2.35) and (2.12)-(2.15) hold. Then (c,p) is a weak solution of (2.28)-(2.32) if*

$$c \in L^2((0,T);H^1(\Omega)),\ 0 \le c \le 1,$$

$$c \in L^\infty((0,T);L^1(\Omega,\mu)),\ 0 \le c(x,t) \le 1,\ \textit{for } \mu\textit{-a.e. } x \in \Omega,\ \textit{for a.e. } t \in (0,T),$$

$$c_t \in L^2((0,T);(W^{1,s}(\Omega))',\ \forall s > d,$$

$$c \in C([0,T];(W^{1,s}(\Omega))'),\ \forall s > d,\ c(\cdot,0) = c_0,$$

$$p \in L^\infty((0,T);W^{1,q}(\Omega)),\ \forall q \in [1,2),$$

and

$$\begin{aligned}&\langle c_t(.,t),\varphi\rangle_{(W^{1,q})',W^{1,q}} + \int_\Omega (\lambda(c(x,t)) + \mathbf{D}(\mathbf{v}(x,t)))\,\nabla c(x,t)\cdot\nabla\varphi(x)dx\\ &-\int_\Omega \mathbf{v}(x,t)\cdot\nabla\varphi(x)c(x,t)dx + \int_\Omega c(x,t)\varphi(x)b(x,t)d\mu(x)\\ &-\int_\Omega \bar{c}(x,t)\varphi(x)a(x,t)d\mu(x) = 0,\\ &\qquad\qquad \forall \varphi \in W^{1,q}(\Omega),\ \forall q > d,\ \textit{for a.e. } t \in (0,T),\end{aligned} \tag{2.36}$$

$$\begin{aligned}&\mathbf{v}(x,t) = -\frac{\mathbf{K}(x)}{\nu(c(x,t))}\left(\nabla p(x,t) - \rho(c(x,t))\mathbf{g}\right),\ \textit{for a.e. } (x,t) \in \Omega\times(0,T),\\ &-\int_\Omega \mathbf{v}(x,t)\cdot\nabla\psi(x)dx = \int_\Omega \psi(x)a(x,t)\mu(x)dx - \int_\Omega \psi(x)b(x,t)\mu(x)dx,\\ &\qquad\qquad \forall \psi \in W^{1,q}(\Omega),\ \forall q > d,\ \textit{for a.e. } t \in (0,T).\end{aligned} \tag{2.37}$$

This system is very similar to that of the model problem. The only difference lies in the diffusion coefficient in Equation (2.28) which corresponds to Equation (2.1) in the model problem, note that the unknown denoted here by c corresponds to u in the model problem. The diffusion coefficient is $\lambda(c)+\mathbf{D}(\mathbf{v})$ instead of $\mathbf{D}(\mathbf{v})$ for the model problem. This dependence on c of the diffusion coefficient does not lead to additionnal difficulties and we obtain the following theorem which is proved in Section 5:

Theorem 2.2 *Assuming (2.6), (2.33)-(2.35) and (2.12)-(2.15) to hold, there exists at least one solution of (2.28)-(2.32) in the sense of Definition 2.2.*

2.2.2 Immiscible flow

The model for incompressible immiscible flow in a homogeneous porous medium reads

$$\begin{array}{l} s_t(x,t) + \mathrm{div}(\mathbf{v}_T f(s))(x,t) - \mathrm{div}(h(s)\nabla s)(x,t) \\ -\mathrm{div}(\mathbf{k}(s))(x,t) + s(x,t)\, b(x,t)\mu(x) - a(x,t)\mu(x) = 0,\ (x,t) \in \Omega \times (0,T), \end{array} \tag{2.38}$$

$$\begin{array}{rcl} \mathbf{v}_T(x,t) + M(s(x,t))(\nabla\pi(x,t) - \overline{\rho}(s(x,t))\mathbf{g}) & = & 0,\ (x,t) \in \Omega\times(0,T), \\ \mathrm{div}(\mathbf{v}_T)(x,t) - a(x,t)\mu(x) + b(x,t)\mu(x) & = & 0,\ (x,t)\in\Omega\times(0,T), \end{array} \tag{2.39}$$

with initial and boundary conditions

$$s(x,0) = s_0(x),\ x\in\Omega, \tag{2.40}$$

$$(h(s(x,t))\nabla s(x,t) + \mathbf{k}(s(x,t)))\cdot\mathbf{n} = 0,\ (x,t)\in\partial\Omega\times(0,T), \tag{2.41}$$

$$\mathbf{v}_T(x,t)\cdot\mathbf{n} = 0,\ (x,t)\in\partial\Omega\times(0,T). \tag{2.42}$$

In this model, $\mathbf{v}_T$ is the total velocity, π is the total pressure, s is the saturation of water, $f(s)$ is the fractional flow of water, $h(s)$ takes into account the capillary effects, $\mathbf{k}(s)$ is a gravity term, $\overline{\rho}$ is a means density, and $\mathbf{g}$ is the gravity [6], [8].

As for the miscible flows, $T\in\mathbb{R}_+^\star$, Ω is a bounded open set of $\mathbb{R}^d$ ($d=2$ or 3) with a Lipschitz continuous boundary, and $a\mu$ and $b\mu$ represent the flow rates at the injection and production wells, they have the form (2.12)-(2.15). The compatibility condition (2.14) is due to the boundary condition (2.42).

Remark 2.3 *The additional difficulty with respect to the model problem lies in the fact that the parabolic equation is degenerate since (in the "real model") the function $h(s)$ vanishes for $s=1$ and $s=0$. For somme existence and local regularity results, we can see [1], [2] and [8].*

Now let us give the assumptions on the data:

$$s_0\in L^\infty(\Omega),\ 0\le s_0\le 1, \tag{2.43}$$

$$f\in C(\mathbb{R},\mathbb{R}) \text{ is locally Lipschitz continuous, } f(\sigma)=0 \text{ for } \sigma\le 0,\ f(\sigma)=1 \text{ for } \sigma\ge 1, \tag{2.44}$$

$$\begin{array}{l} h\in C(\mathbb{R},\mathbb{R}),\ h(\sigma)>0,\ \forall\sigma\in(0,1), \\ H(\sigma) = \int_0^\sigma h(\tau)d\tau,\ \forall\sigma\in[0,1], \end{array} \tag{2.45}$$

$$M\in C(\mathbb{R},\mathbb{R}) \text{ is such that } \exists\alpha>0,\ M(\sigma)\ge\alpha,\ \forall\sigma\in\mathbb{R}, \tag{2.46}$$

$$\mathbf{k}\in C(\mathbb{R},\mathbb{R}),\ \bar{\rho}\in C(\mathbb{R},\mathbb{R}). \tag{2.47}$$

Definition 2.3 *Let (2.6), (2.12)-(2.15) and (2.43)-(2.47) hold. One says that (s,π) is a weak solution of (2.38)-(2.42) if*

$$\begin{array}{l} s \in L^2((0,T);H^1(\Omega)),\ 0 \le s \le 1, \\ s \in L^\infty((0,T);L^1(\Omega,\mu)),\ 0 \le s(x,t) \le 1,\ \ \textit{for } \mu\textit{-a.e. } x \in \Omega,\ \ \textit{for a.e. } t \in (0,T), \end{array} \tag{2.48}$$

$$s_t \in L^2((0,T);(W^{1,q}(\Omega))'),\ \forall q > d, \tag{2.49}$$

$$s \in C([0,T];(W^{1,q}(\Omega))')\ \forall q > d,\ s(.,0) = s_0\ (\textit{in } W^{1,q}(\Omega))'), \tag{2.50}$$

$$\pi \in L^\infty((0,T);W^{1,q}(\Omega)),\ \forall q \in [1,2), \tag{2.51}$$

$$\begin{array}{l} \langle s_t(.,t),\varphi\rangle_{(W^{1,r})',W^{1,r}} + \displaystyle\int_\Omega \nabla H(s)(x,t)\cdot\nabla\varphi(x)dx + \int_\Omega \mathbf{k}(s(x,t))\cdot\nabla\varphi(x)dx \\ -\displaystyle\int_\Omega \mathbf{v}(x,t)\cdot\nabla\varphi(x)f(s(x,t))dx + \int_\Omega s(x,t)\varphi(x)b(x,t)d\mu(x) - \int_\Omega \varphi(x)a(x,t)d\mu(x) = 0, \\ \qquad\qquad \forall\varphi \in W^{1,r}(\Omega),\ \forall r > d,\ \textit{for a.e. } t \in (0,T), \end{array} \tag{2.52}$$

$$\begin{array}{l} \mathbf{v}_T(x,t) = -M(s(x,t))\left(\nabla\pi(x,t) - \bar\rho(s(x,t))\mathbf{g}\right),\ \ \textit{for a.e. } (x,t) \in \Omega\times(0,T), \\ -\displaystyle\int_\Omega \mathbf{v}_T(x,t)\cdot\nabla\psi(x)dx = \int_\Omega \psi(x)a(x,t)d\mu(x) - \int_\Omega \psi(x)b(x,t)d\mu(x), \\ \qquad\qquad \forall\psi \in W^{1,r}(\Omega),\ \forall r > d,\ \textit{for a.e. } t \in (0,T). \end{array} \tag{2.53}$$

Under these hypotheses, we shall prove, see section 5, the following result:

Theorem 2.3 *Assuming (2.6), (2.43)-(2.47) and (2.12)-(2.15), there exists at least one solution of (2.38)-(2.42) in the sense of definition 2.3.*

This system is very similar to the model problem, except for the degenerate diffusion term $\mathrm{div}(h(s)\nabla s)$. But this lack of coerciveness is classical and does not lead to additional difficulties compared to the proof for the model problem (see Section 5).

3 Analysis of an auxilary elliptic equation

In this section we prove existence, uniqueness and stability results for the elliptic problem involved in (2.1)-(2.5). Indeed, the problem to be considered is

$$\begin{array}{l} -\mathrm{div}\big(A(\nabla p - F)\big)(x) = \mu(x),\ \text{in } \Omega, \\ A(x)(\nabla p(x) - F(x))\cdot\mathbf{n} = 0,\ \text{in } \partial\Omega, \end{array} \tag{3.1}$$

under the following hypotheses on the data (recall that $M_d(\mathbb{R})$ denotes the set of $d\times d$ matrices with coefficients in $\mathbb{R}$):

$$\begin{array}{l} A\text{is Caratheodory function from } \Omega to M_d(\mathbb{R}) \text{ satisfying} \\ \exists\alpha > 0 \text{ such that } A(x)\xi\cdot\xi \ge \alpha\xi\cdot\xi,\ \forall\xi\in\mathbb{R}^d,\ \text{for a.e. } x\in\Omega, \\ \exists\beta\in\mathbb{R} \text{ such that } |A(x)| \le \beta,\ \text{for a.e. } x\in\Omega, \end{array} \tag{3.2}$$

$$\begin{array}{l} F = (F_1,\cdots,F_d)^t \in (L^2(\Omega))^d, \\ \mu \in \mathcal{M}(\Omega),\ \mu(\Omega) = 0. \end{array} \tag{3.3}$$

Recall that $\mathbf{n}$ denotes the outward unit normal vector to Ω on $\partial\Omega$.

3.1 Existence and uniqueness

One first proves a general existence result as in [5] and [15].

Proposition 3.1 (Existence for the elliptic equation)
Let (3.2)-(3.3) hold. There exists a solution p to (3.1) in the following sense:

$$p \in W^{1,q}(\Omega),\ \forall q \in [1, \frac{d}{d-1}),\ \int_\Omega p(x)dx = 0,$$
$$\int_\Omega A(x)\nabla p(x)\cdot\nabla\varphi(x)dx = \int_\Omega A(x)F(x)\cdot\nabla\varphi(x)dx + \int_\Omega \varphi(x)d\mu(x),\ \forall\varphi \in \bigcup_{r>d} W^{1,r}(\Omega). \tag{3.4}$$

Proof of Proposition 3.1

The proof proceeds in two steps and is very similar to that of [5]. In Step 1, some bounds on the unique solution of an approximate problem are given. In Step 2, passing to the limit in the approximate problem supplies the existence result.

Step 1. (Approximate problem)

Let f in $L^2(\Omega)$ be such that $\int_\Omega f(x)dx = 0$. A straightforward application of the Lax-Milgram Lemma leads to the existence and uniqueness of a solution p of

$$p \in H^1(\Omega),\ \int_\Omega p(x)dx = 0,$$
$$\int_\Omega A(x)\nabla p(x)\cdot\nabla\varphi(x)dx = \int_\Omega A(x)F(x)\cdot\nabla\varphi(x)dx + \int_\Omega \varphi(x)f(x)dx,\ \forall\varphi \in H^1(\Omega). \tag{3.5}$$

Note that without the hypothesis $\int_\Omega f(x)dx = 0$ the Lax-Milgram Lemma gives the existence and the uniqueness of a solution p of (3.5) but with $\varphi \in H^1(\Omega)$ such that $\int_\Omega \varphi(x)dx = 0$ instead of $\varphi \in H^1(\Omega)$.
A first estimate on the, solution p of (3.5) is obtained by taking $\varphi = \varphi_k(p)$, with $k \in \mathbb{R}_+$ where

$$\begin{aligned}
&\varphi_k(s) = -1,\ s \le -k-1,\\
&\varphi_k(s) = s+k,\ -k-1 < s < -k,\\
&\varphi_k(s) = 0,\ -k \le s \le k,\\
&\varphi_k(s) = s-k,\ k < s < k+1,\\
&\varphi_k(s) = 1,\ k+1 \le s.
\end{aligned}$$

From a classical result of Stampacchia, $\varphi_k(p) \in H^1(\Omega)$ and $\nabla\varphi_k(p) = \varphi_k'(p)\nabla p$, a.e. on Ω. With this choice of φ in (3.5), one gets

$$\int_{B_k} |\nabla p(x)|^2 \le C_1 + \frac{2}{\alpha}||f||_{L^1(\Omega)}, \tag{3.6}$$

where C_1 depends only on α, $||F||_2$, $||A||_\infty$ ($||F||_2$ denotes the $L^2(\Omega)$-norm of $|F|$ and $||A||_\infty$ the $L^\infty(\Omega)$-norm of $|A|$) and where

$$B_k = \{x \in \Omega,\ k \le |p(x)| \le k+1\}.$$

Given $q < d$, from the Sobolev inequality, there exists C_2 depending only on Ω and q such that

$$||w - \bar{w}||_{L^{q^\star}(\Omega)} \le C_2||\nabla w||_{(L^q(\Omega))^d},\ \forall w \in W^{1,q}(\Omega), \tag{3.7}$$

where $\bar{w}$ is the mean value of w over Ω, $q^\star = qd/(d-q)$ and

$$||\nabla w||_{L^q(\Omega)^d} = (\int_\Omega |\nabla w(x)|^q dx)^{\frac{1}{q}}.$$

Using (3.6) and (3.7) (with $w = p$) we shall now prove that, for all $1 \le q < d/(d-1)$, there exists C_3, depending only on α, $||F||_2$, $||A||_\infty$, Ω, $||f||_{L^1(\Omega)}$ and q, such that

$$\|p\|_{W^{1,q}(\Omega)} \le C_3. \tag{3.8}$$

Furthermore C_3 may be chosen to be nondecreasing with respect to $\|f\|_{L^1(\Omega)}$, $\|A\|_\infty$ and $\|F\|_2$.

In order to prove (3.8), we assume $1 \le q < d/(d-1)$. Since $q < d$ and since the mean value of p over Ω is 0, (3.7) gives

$$\|p\|^q_{L^{q^\star}(\Omega)} \le (C_2)^q \int_\Omega |\nabla p(x)|^q dx = (C_2)^q \sum_{k=0}^{\infty} \int_{B_k} |\nabla p(x)|^q dx. \tag{3.9}$$

Since $q < 2$, using the Hölder inequality in (3.9) leads to

$$\|p\|^q_{L^{q^\star}(\Omega)} \le (C_2)^q \sum_{k=0}^{\infty} (C_1 \|f\|_{L^1(\Omega)})^{\frac{q}{2}} (\text{meas}(B_k))^{1-\frac{q}{2}}.$$

Let $C_4 = (C_2)^q (C_1\|f\|_{L^1(\Omega)})^{\frac{q}{2}} (\text{meas}(\Omega))^{1-\frac{q}{2}}$ and $C_5 = (C_2)^q (C_1 \|f\|_{L^1(\Omega)})^{\frac{q}{2}}$. Noting that

$$k^{q^\star} \text{meas}(B_k) \le \int_{B_k} |p(x)|^{q^\star} dx,$$

the previous inequality yields, for any $n \ge 1$,

$$\|p\|^q_{L^{q^\star}(\Omega)} \le nC_4 + C_5 \sum_{k=n}^{\infty} \left(\frac{1}{k}\right)^{\frac{dq(2-q)}{2(d-q)}} \left(\int_{B_k} |p(x)|^{q^\star} dx\right)^{1-\frac{q}{2}}$$

and, using once more the Hölder inequality,

$$\left(\int_\Omega |p(x)|^{q^\star} dx\right)^{\frac{q}{q^\star}} = \|p\|^q_{L^{q^\star}(\Omega)} \le nC_4 + C_5 \left(\int_\Omega |p(x)|^{q^\star} dx\right)^{\frac{2-q}{2}} \sum_{k=n}^{\infty} \left(\frac{1}{k}\right)^{\frac{d(2-q)}{d-q}}. \tag{3.10}$$

Since $q < d/(d-1)$, then $d(2-q)/(d-q) > 1$ and the serie in (3.10) is convergent. Furthermore, $q/q^\star = (d-q)/d < (2-q)/2$ if $d = 3$ and $q/q^\star = (d-q)/d = (2-q)/2$ if $d = 2$. If $d = 3$ $n = 1$, if $d = 2$ one chooses n such that $C_5 \sum_{k=n}^{\infty} 1/k^2 \le 1/2$. Then, the inequality (3.10) gives the existence of C_6, depending only on α, $\|A\|_\infty$, $\|F\|_2$, Ω, $\|f\|_{L^1(\Omega)}$ and q, such that

$$\|p\|_{L^{q^\star}(\Omega)} \le C_6.$$

Therefore, going back to (3.9) there exists C_3, depending only on α, $\|A\|_\infty$, $\|F\|_2$, Ω, $\|f\|_{L^1(\Omega)}$ and q, such that (3.8) holds. This completes Step 1.

Step 2. (Passing to the limit)

There exists a sequence $(f_n)_{n\in\mathbb{N}} \subset L^\infty(\Omega)$ such that

1. $f_n \to \mu$ for the weak-$\star$ topology of $(C(\overline{\Omega}))'$ as $n \to \infty$ (i.e. $\int_\Omega f_n(x)\varphi(x)dx \to \int_\Omega \varphi(x)d\mu(x)$, as $n \to \infty$, for all $\varphi \in C(\overline{\Omega})$),
2. $\|f_n\|_{L^1(\Omega)} \le |\mu|(\Omega)$ for all $n \in \mathbb{N}$,
3. $\int_\Omega f_n(x)dx = 0$ for all $n \in \mathbb{N}$.

Recall that $|\mu| = \mu^+ + \mu^-$, where μ^+ and μ^- are the classical "positive" and "negative" parts of the measure μ (so that $\mu = \mu^+ - \mu^-$), and that $|\mu|(\Omega)$ is also the norm of μ in the topological dual space of $C(\overline{\Omega})$ endowed with the $L^\infty(\Omega)$-norm.

For $n \in \mathbb{N}$, let p_n be the solution of (3.5) with f_n instead of f. Step 1 shows that the sequence $(p_n)_{n\in\mathbb{N}}$ is bounded in $W^{1,q}(\Omega)$ for all $1 \le q < d/(d-1)$. Then, one may assume, up to a subsequence, that $p_n \to p$ in $W^{1,q}(\Omega)$ for the weak topology for all $1 \le q < d/(d-1)$. Therefore, passing to the limit as $n \to \infty$ in (3.5) (with p_n and f_n instead of p and f) one gets that p is a solution of (3.4). In order to pass to the limit in the right hand side of (3.5), note that $W^{1,r}(\Omega) \subset C(\overline{\Omega})$, if $r > d$. This concludes Step 2 and the proof of Proposition 3.1. ■

Remark 3.1 *Proposition 3.1 is quite general since it gives an existence result for any measure μ such that $\mu(\Omega) = 0$. Note also that this proof may be adapted to give existence for nonlinear operators such as Leray-Lions operators as in [5]. The solution given in Proposition 3.1, that is the solution of (3.4) is unique if $d = 2$ - this will be proven in the next proposition -, unfortunately this uniqueness result is no longer true if $d = 3$. A counterexample to uniqueness can be constructed as in [14] which follows essentially the work of Serrin [16]. In order to also have an existence and uniqueness result for $d = 3$, we shall use an additional assumption, i.e.(2.15). This is done in Proposition 3.2 below. Another way to obtain existence and uniqueness for any measure μ would be to use a "duality method" as in Stampacchia's work [19], which would lead to solutions of (3.1) in a "stronger sense" than that of (3.4).*

Proposition 3.2 Existence and uniqueness for the elliptic equation *Let (3.2)-(3.3) and (2.15) hold (note that (2.15) is always true in the case $d = 2$). There exists a unique solution p to (3.1) in the sense that :*

$$\begin{aligned} & p \in W^{1,q}(\Omega),\ \forall q \in [1,2),\ \int_\Omega p(x)dx = 0, \\ & \int_\Omega A(x)\nabla p(x) \cdot \nabla\varphi(x)dx = \int_\Omega A(x)F(x)\cdot\nabla\varphi(x)dx + \int_\Omega \varphi(x)d\mu(x), \\ & \qquad \forall \varphi \in \bigcup_{r>d} W^{1,r}(\Omega). \end{aligned} \tag{3.11}$$

Remark 3.2

1. *Let $\mu \in \mathcal{M}(\Omega)$ and $1 \le r < \infty$. The measure μ belongs to $(W^{1,r}(\Omega))'$ if and only if there exists some real number C such that*

$$\int_\Omega \varphi(x)d\mu(x) \le C||\varphi||_{W^{1,r}(\Omega)},\ \forall\varphi \in W^{1,r}(\Omega)\cap C(\overline{\Omega}). \tag{3.12}$$

   Indeed, if the measure μ satisfies (3.12) (for some $r \in [1,\infty)$ and some $C \in \mathbb{R}$), μ can be uniquely extend to a linear continuous application, still denoted by μ, from $W^{1,r}(\Omega)$ to $\mathbb{R}$. Uniqueness of the extension follows from of the density of $W^{1,r}(\Omega)\cap C(\overline{\Omega})$ in $W^{1,r}(\Omega)$.

2. *Let the hypotheses of Proposition 3.2 hold ; from a density argument, the formulation (3.11) is equivalent to the following one:*

$$\begin{aligned} & p \in W^{1,q}(\Omega),\ \forall q \in [1,2),\ \int_\Omega p(x)dx = 0, \\ & \int_\Omega A(x)\nabla p(x)\cdot\nabla\varphi(x)dx = \int_\Omega A(x)F(x)\cdot\nabla\varphi(x)dx + \langle\mu,\varphi\rangle_{(W^{1,s})',(W^{1,s})}, \\ & \qquad \forall\varphi \in W^{1,s}(\Omega),\ \forall s > 2. \end{aligned} \tag{3.13}$$

   We shall directly prove below the existence and uniqueness of the solution of (3.13).

3. *Note that, for $d = 3$, the existence result in Proposition 3.2 is not directly given by Proposition 3.1 since in Proposition 3.1 the solution belongs to $W^{1,q}(\Omega)$ for all $q \in [1, 3/2)$ and in Proposition 3.2 the solution belongs to $W^{1,q}(\Omega)$ for all $q \in [1,2)$.*

Proof of Proposition 3.2

This is proven by using a modification of a regularity result due to Meyers [12] in the case of Dirichlet boundary conditions (and assuming that the boundary of Ω is smooth). We use here a variant with Neumann boundary condition that can be found in [13] and also in [9].

Define

$$H^1_\star(\Omega) = \{u \in H^1(\Omega);\ \int_\Omega u(x)dx = 0\}.$$

For $f \in (H^1(\Omega))'$, by the classical Lax-Milgram Lemma, there exists a unique u such that

$$
\begin{aligned}
&u \in H^1_\star(\Omega),\\
&\int_\Omega \nabla u(x)\cdot A(x)\nabla\varphi(x)dx = \langle f,\varphi\rangle_{(H^1)',H^1},\ \forall \varphi \in H^1_\star(\Omega)
\end{aligned}
\tag{3.14}
$$

and the operator T from $(H^1(\Omega))'$ to $H^1(\Omega)$ defined by $T(f) = u$ (where u is the unique solution to (3.14)) is linear and continuous.

The regularity result in [13] gives the existence of $r_0 > 2$, depending only on A and Ω, such that u belongs to $W^{1,r}(\Omega)$ if f belongs to $(W^{1,r'}(\Omega))'$ with $2 < r < r_0$ and $r' = r/(r-1)$ (note that $(W^{1,r'}(\Omega))' \subset (H^1(\Omega))'$). Furthermore, for any $2 < r < r_0$, the operator T_r (defined by $T_r(f) = u$ where u is the unique solution to (3.14)) is linearly continuous from $(W^{1,r'}(\Omega))'$ to $W^{1,r}(\Omega)$. It is important to notice that the range of T_r, denoted by $R(T_r)$, is $W^{1,r}_\star(\Omega)$ wherein

$$
W^{1,r}_\star(\Omega) = \{v \in W^{1,r}(\Omega);\ \int_\Omega v(x)dx = 0\}.
$$

Indeed, let $v \in W^{1,r}_\star(\Omega)$ and define $f \in (W^{1,r'}(\Omega))'$ by

$$
\langle f,\varphi\rangle_{(W^{1,r'})',W^{1,r'}} = \int_\Omega \nabla v(x)\cdot A(x)\nabla\varphi(x)dx,\ \forall\varphi \in W^{1,r'}(\Omega).
$$

Then, $u = T_r(f)$ is solution of (3.14) and taking $\varphi = u - v$ in (3.14) leads to $u = v$. This gives $R(T_r) = W^{1,r}_\star(\Omega)$.

For $2 < r < r_0$, one consider the adjoint operator $T_r^\star$. This operator is linear continuous from $(W^{1,r}(\Omega))'$ to $W^{1,r'}(\Omega)$. By definition, for any $g \in (W^{1,r}(\Omega))'$, $v = T_r^\star(g)$ is the unique element of $W^{1,r'}(\Omega)$ such that

$$
\langle f, v\rangle_{(W^{1,r'})',W^{1,r'}} = \langle g, T_r(f)\rangle_{(W^{1,r})',W^{1,r}},\ \forall f \in (W^{1,r'}(\Omega))'.
$$

Since $R(T_r) = W^{1,r}_\star(\Omega)$ and from the definition of $T_r(f)$, $v = T_r^\star(g)$ is also the unique solution of

$$
\begin{aligned}
&v \in W^{1,r'}(\Omega),\\
&\int_\Omega \nabla u(x)\cdot A(x)\nabla v(x)dx = \langle g,u\rangle_{(W^{1,r})',W^{1,r}},\ \forall u \in W^{1,r}_\star(\Omega).
\end{aligned}
\tag{3.15}
$$

Therefore (3.15) has a unique solution for any $g \in (W^{1,r}(\Omega))'$ and any $2 < r < r_0$.

Now, consider g in $(W^{1,r}(\Omega))'$ for all $r > 2$. Then, thanks to its uniqueness, the solution of (3.15) does not depend on r, as r spans $(2, r_0)$. Therefore, there exists a unique solution to

$$
\begin{aligned}
&v \in W^{1,q}(\Omega),\ \forall q < 2,\\
&\int_\Omega A(x)\nabla v(x)\cdot\nabla\varphi(x)dx = \langle g,\varphi\rangle_{(W^{1,r})',W^{1,r}},\ \forall\varphi \in W^{1,r}_\star(\Omega),\ \forall r > 2.
\end{aligned}
$$

In order to complete the proof of Proposition 3.2, it is sufficient to take g such that, for all $r > 2$,

$$
\langle g,\varphi\rangle_{(W^{1,r})',W^{1,r}} = \int_\Omega A(x)F(x)\cdot\nabla\varphi(x)dx + \langle\mu,\varphi\rangle_{(W^{1,r})',W^{1,r}},\ \forall\varphi \in W^{1,r}(\Omega).
$$

Note that (3.13) is satisfied for all $\varphi \in W^{1,r}(\Omega)$ and not only for all $\varphi \in W^{1,r}_\star(\Omega)$ since for a constant function φ, (3.13) follows from $\mu(\Omega) = 0$. ■

3.2 Stability

One now states a stability result on the solution of (3.4) or (3.11) with respect to A, F and μ.

Proposition 3.3 Stability for the elliptic equation
Let $(A_n)_{n\in\mathbb{N}} \subset M_d(L^\infty(\Omega))$, $(F_n)_{n\in\mathbb{N}} \subset (L^2(\Omega))^d$ and $(f_n)_{n\in\mathbb{N}} \subset L^2(\Omega)$. Let $A \in M_d(L^\infty(\Omega))$, $F \in (L^2(\Omega))^d$ and $\mu \in \mathcal{M}(\Omega)$. Assume that:

1. *$\exists \alpha > 0$ such that $A_n\xi\cdot\xi \geq \alpha\,\xi\cdot\xi$ for all $\xi \in \mathbb{R}^d$, for all $n \in \mathbb{N}$ and* a.e. *on Ω, $A_n = (a_{i,j}^{(n)})_{i,j=1,\ldots,d}$ and $A = (a_{i,j})_{i,j=1,\ldots,d}$ such that for all $i,j \in \{1,\ldots,d\}$ $(a_{i,j}^{(n)})_{n\in\mathbb{N}}$ is bounded in $L^\infty(\Omega)$ and $a_{i,j}^{(n)} \to a_{i,j}$* a.e. *on Ω as $n \to \infty$,*
2. *$F_n \to F$ in $L^2(\Omega)^d$ as $n \to \infty$,*
3. *$f_n \to \mu$ for the weak-$\star$ topology of $(C(\overline{\Omega}))'$ as $n \to \infty$ (i.e. $\int_\Omega f_n(x)\varphi(x)dx \to \int_\Omega \varphi(x)d\mu(x)$, as $n \to \infty$, for all $\varphi \in C(\overline{\Omega})$), $(f_n)_{n\in\mathbb{N}}$ is bounded in $L^1(\Omega)$ and $\int_\Omega f_n(x)dx = 0$ for all $n \in \mathbb{N}$.*

Let u_n be the solution to (3.5) with A_n, F_n and f_n instead of A, F and f.

Then, there exists a subsequence of $(u_n)_{n\in\mathbb{N}}$, still denoted by $(u_n)_{n\in\mathbb{N}}$, and there exists a solution u to (3.4) such that $u_n \to u$ in $W^{1,q}(\Omega)$, as $n \to \infty$, for all $1 \leq q < d/(d-1)$.

If $d = 2$, the whole sequence u_n converges to u, in $W^{1,q}(\Omega)$, as $n \to \infty$, for all $1 \leq q < 2$ and u is the unique solution to (3.4) (note that (3.11) is identical to (3.4)).

If $d = 3$ and if $(f_n)_{n\in\mathbb{N}}$ is bounded in $(W^{1,q}(\Omega))'$ for all $q > 2$, then the whole sequence u_n converges to u, in $W^{1,q}(\Omega)$, as $n \to \infty$, for all $1 \leq q < 2$ and u is the unique solution to (3.11) (note also that $\mu \in (W^{1,q}(\Omega))'$ and $f_n \to \mu$ for the weak-$\star$ topology of $(W^{1,q}(\Omega))'$, for all $q > 2$).

This stability result, the global convergence of the sequence, will be crucial in the proof of our main result (see the proof of Theorem 2.1 in Section 5).

Proof of Proposition 3.3

The proof relies in 3 steps. The first step gives the weak convergence in $W^{1,q}(\Omega)$ for a subsequence of the sequence $(u_n)_{n\in\mathbb{N}}$ towards a solution of (3.4), the second step yields the strong convergence and the third step the convergence of the whole sequence towards the unique solution of (3.11).

Step 1. (weak convergence for a subsequence)

From estimate (3.8), one deduces that the sequence $(u_n)_{n\in\mathbb{N}}$ is bounded in $W^{1,q}(\Omega)$ for any $1 \leq q < d/(d-1)$. Then, up to a subsequence, the sequence $(u_n)_{n\in\mathbb{N}}$ converges, as $n \to \infty$, weakly towards some u in $W^{1,q}(\Omega)$ for any $1 \leq q < d/(d-1)$. Passing to the limit, as $n \to \infty$, in (3.5) (with u_n, A_n, F_n and f_n instead of u, A, F and f) one obtains a solution u of (3.4).

Furthermore, the compactness embedding of $W^{1,q}(\Omega)$ in $L^q(\Omega)$ gives $u_n \to u$ into $L^q(\Omega)$, as $n \to \infty$, for any $1 \leq q < d/(d-1)$. Then, once again, up to a subsequence, $u_n \to u$ a.e. on Ω.

Estimate (3.6) also gives, for any $k \in \mathbb{R}_+$, a bound in $H^1(\Omega)$ of the sequence $(T_k(u_n))_{n\in\mathbb{N}}$ where T_k is the function from $\mathbb{R}$ to $\mathbb{R}$ defined by $T_k(s) = \min(k, \max(-k, s))$. Since $T_k(u_n) \to T_k(u)$ a.e. on Ω, we then deduce that $T_k(u_n) \to T_k(u)$ weakly in $H^1(\Omega)$ as $n \to \infty$.

Up to now, we prove the following assertions :

1. $u_n \to u$, as $n \to \infty$, a.e. in Ω, weakly in $W^{1,q}(\Omega)$ and in $L^q(\Omega)$ for any $1 \leq q < d/(d-1)$,
2. $T_k(u_n) \to T_k(u)$, as $n \to \infty$, weakly in $H^1(\Omega)$ for any $k \in \mathbb{R}_+$,
3. u is solution of (3.4).

Step 2. (strong convergence of a subsequence)

In this step, one proves that $\nabla u_n \to \nabla u$ in measure, as $n \to \infty$. This gives, up to a subsequence, $\nabla u_n \to \nabla u$ a.e. in Ω. Then, since (from Step 1) the sequence $(u_n)_{n\in\mathbb{N}}$ is bounded in $W^{1,q}(\Omega)$ for any $1 \leq q < d/(d-1)$ (and $u_n \to u$ in $L^q(\Omega)$ for any $1 \leq q < d/(d-1)$), one deduces that $u_n \to u$, as $n \to \infty$, in $W^{1,q}(\Omega)$ for any $1 \leq q < d/(d-1)$. This strong convergence is a consequence of a well known result which is recalled in Lemma 3.1 after the proof of Proposition 3.3. To be precise, one uses Lemma 3.1 with $(v_n)_{n\in\mathbb{N}} = (D_i(u_n)\ )_{n\in\mathbb{N}}$ for each $i \in \{1,\ldots,d\}$.

Let us now prove that $\nabla u_n \to \nabla u$ in measure. For a given function B from Ω to $\mathbb{R}$ and a real number b, one denotes $\{B \geq b\}$ the set $\{x \in \Omega;\ B(x) \geq b\}$. Let $\lambda > 0$, one has to prove that $\text{meas}(\{|\nabla u_n - \nabla u| \geq \lambda\}) \to 0$ as $n \to \infty$. Let $\varepsilon > 0$. For $k \in \mathbb{R}_+$ and $\delta \in R_+^\star$, one has

$$\{|\nabla u_n - \nabla u| \geq \lambda\} \subset \{|u| \geq k\} \cup \{|u_n - u| \geq \delta\} \cup E_{k,\delta,n},$$

where $E_{k,\delta,n} = \{|u| < k\} \cap \{|u_n - u| < \delta\} \cap \{|\nabla u_n - \nabla u| \geq \lambda\}$.

Let us choose $k \in \mathbb{R}_+$ large enough in order to have $\text{meas}(\{|u| \geq k\}) \leq \varepsilon$. Since $u_n \to u$ in $L^1(\Omega)$, for any fixed $\delta \in R_+^\star$ there exists some n_1 (depending on δ) such that $\text{meas}(\{|u_n - u| \geq \delta\}) \leq \varepsilon$ for $n \geq n_1$. Then, it remains to choose $\delta \in \mathbb{R}_+^\star$ such that $\text{meas}(E_{k,\delta,n}) \leq \varepsilon$ for n large enough. In order to bound $\text{meas}(E_{k,\delta,n})$, let us take $\varphi = T_\delta(u_n - T_k(u))$ in (3.5) (with u_n, A_n, F_n and f_n instead of u, A, F and f). This yields

$$\begin{aligned} &\alpha\lambda^2 \text{meas}(E_{k,\delta,n}) \leq \int_\Omega A_n(x)\nabla(u_n(x) - T_k(u(x))) \cdot \nabla T_\delta(u_n(x) - T_k(u(x)))dx \\ &\leq \int_\Omega A_n(x)\left(F_n(x) - \nabla T_k(u(x))\right) \cdot \nabla T_\delta\left(u_n(x) - T_k(u(x))\right)dx + \delta\|f_n\|_{L^1(\Omega)}. \end{aligned} \tag{3.16}$$

One considers now the two terms on the right hand side of (3.16).

Since $(f_n)_{n\in\mathbb{N}}$ is bounded in $L^1(\Omega)$, the second one goes to 0, as $\delta \to 0$, uniformly with respect to n. Then there exists $\delta_1 > 0$ such that

$$\delta \leq \delta_1 \Rightarrow \delta\|f_n\|_{L^1(\Omega)} \leq \varepsilon\alpha\lambda^2. \tag{3.17}$$

Assuming $\delta < 1$, the first term in the right hand side of (3.16) is equal to

$$\int_\Omega A_n(x)(F_n(x) - \nabla T_k(u(x))) \cdot \nabla T_\delta(T_{k+1}(u_n(x)) - T_k(u(x)))dx$$

which converges, as $n \to \infty$, towards

$$\int_\Omega A(x)(F(x) - \nabla T_k(u(x))) \cdot \nabla T_\delta(T_{k+1}(u(x)) - T_k(u(x)))dx.$$

This quantity can also be expressed as

$$\int_{\{k<|u|<k+\delta\}} A(x)(F(x) - \nabla T_k(u(x))) \cdot \nabla T_{k+1}(u(x))dx,$$

which goes to 0 as $\delta \to 0$ since $A(F - \nabla T_k(u)) \cdot \nabla T_{k+1}(u) \in L^1(\Omega)$. Then there exists $\delta_2 > 0$ and $n_2 \in \mathbb{N}$ such that

$$\delta \leq \delta_2 \text{ and } n \geq n_2 \Rightarrow \int_\Omega A_n(x)(F_n(x) - \nabla T_k(u(x))) \cdot \nabla T_\delta(u_n(x) - T_k(u(x)))dx \leq \varepsilon\alpha\lambda^2. \tag{3.18}$$

Then, choosing $\delta = \min(\delta_1, \delta_2)$ (and $\delta < 1$) leads to $\text{meas}(E_{k,\delta,n}) \leq 2\varepsilon$ if $n \geq n_2$. Therefore $n \geq \max(n_1, n_2)$ (recall that n_1 is given by δ) implies $\text{meas}(\{|\nabla u_n - \nabla u| \geq \lambda\}) \leq 4\varepsilon$. This proves the convergence in measure of ∇u_n towards ∇u as $n \to \infty$. From Step 1, one deduces that, up to a subsequence, $u_n \to u$, as $n \to \infty$, in $W^{1,q}(\Omega)$ for any $1 \leq q < d/(d-1)$. This concludes Step 2.

Step 3. (convergence for the whole sequence)

When $d = 2$, formulations (3.4) and (3.11) are identical. Then, uniqueness of the solution of (3.11) gives the convergence of the whole sequence $(u_n)_{n\in\mathbb{N}}$ towards the unique solution of (3.4). Recall that, by Step 2, this convergence holds in $W^{1,q}(\Omega)$ for any $1 \leq q < 2$.

In the case $d = 3$, one uses once again the results of [13]. Following [13], the quantity r_0 (appearing in Proposition 3.2) associated to A_n can be chosen independently of n (indeed r_0 is bounded from below by some quantity only depending on α and on the L^∞ bound for A_n) and the norm of the operator T_r associated to A_n is also bounded by some quantity independent of n (it depends only on r, α and on the L^∞ bound for A_n). Then the norm of $T_r^\star$ is also bounded by some quantity independent of n (it depends only on r, α and on the L^∞ bound on the component of A_n). Therefore the sequence $(u_n)_{n\in\mathbb{N}}$ is bounded in $W^{1,q}(\Omega)$ for any $1 \leq q < 2$. Then the limit of a convergent subsequence of $(u_n)_{n\in\mathbb{N}}$ is indeed a solution of (3.11) and not only a solution of (3.4). As for the case $d = 2$, the uniqueness of the solution of (3.11) leads to the convergence of the whole sequence $(u_n)_{n\in\mathbb{N}}$ towards the unique solution to (3.11). This convergence holds in $W^{1,q}(\Omega)$ for any $1 \leq q < 2$. This completes the proof of Proposition 3.3. ■

Lemma 3.1 ($L^p - L^q$ **"compactness"**) *Let Ω be a borelian set of $\mathbb{R}^d$ with a finite Lebesgue measure. Let $1 \le q < p \le \infty$ and $(v_n)_{n\in\mathbb{N}}$ be a bounded sequence in $L^p(\Omega)$. Assume that $v_n \to v$ a.e. on Ω, as $n \to \infty$. Then $v \in L^p(\Omega)$ and $v_n \to v$ in $L^q(\Omega)$, as $n \to \infty$.*

Proof of Lemma 3.1

Let $M \in \mathbb{R}$ be such that $||v_n||_{L^p(\Omega)} \le M$ for all $n \in \mathbb{N}$. By Fatou's Lemma, $v \in L^p(\Omega)$ and $||v||_{L^p(\Omega)} \le M$. In order to prove that $v_n \to v$ in $L^q(\Omega)$, let $\varepsilon > 0$. By Egorov's Theorem, there exists a borelian set $A \subset \Omega$ such that $\text{meas}(A) \le \varepsilon$ and $v_n \to v$ uniformly on $\Omega \setminus A$. Then, there exists $n_0 \in \mathbb{N}$ such that $|v_n(x) - v(x)| \le \varepsilon$ if $x \in \Omega \setminus A$ and $n \ge n_0$. This yields, for $n \ge n_0$,

$$\begin{aligned}\int_\Omega |v_n(x) - v(x)|^q dx &\le \varepsilon^q \text{meas}(\Omega) + \int_A |v_n(x) - v(x)|^q dx,\\ &\le \varepsilon^q \text{meas}(\Omega) + ||v_n - v||^q_{L^p(\Omega)} (\text{meas}(A))^{1-\frac{q}{p}},\\ &\le \varepsilon^q \text{meas}(\Omega) + 2^q M^q \varepsilon^{1-\frac{q}{p}}\end{aligned}$$

and proves that $v_n \to v$ in $L^q(\Omega)$, as $n \to \infty$. ■

4 Existence for the regularized problem

We first recall the definition of a solution to the regularized problem.

Definition 4.1 *let (2.6)-(2.14) hold and assume μ in $L^2(\Omega)$. A pair (u,p) is solution of the regularized problem if*

$$\begin{aligned}&u \in L^2((0,T); H^1(\Omega)) \cap C([0,T]; L^2(\Omega)),\ 0 \le u \le 1,\\ &u_t \in L^2((0,T); (H^1(\Omega))')\end{aligned} \tag{4.1}$$

$$u(.,0) = u_0 \ (in\ L^2(\Omega)), \tag{4.2}$$

$$p \in L^\infty((0,T); H^1(\Omega)),\ \mathbf{v} \in L^\infty((0,T); (L^2(\Omega))^d), \tag{4.3}$$

$$\begin{aligned}&\langle u_t(.,t), \varphi\rangle_{(H^1)',H^1} + \int_\Omega \mathbf{D}(\mathbf{v}(x,t))\,\nabla u(x,t)\cdot\nabla\varphi(x)dx\\ &-\int_\Omega \mathbf{v}(x,t)\cdot\nabla\varphi(x) f(u(x,t))dx + \int_\Omega \Big(\ u(x,t)b(x,t) - c(x,t)a(x,t)\Big)\,\mu(x)\varphi(x)dx = 0,\\ &\qquad\qquad \forall \varphi \in H^1(\Omega),\ for\ a.e.\ t \in (0,T),\end{aligned} \tag{4.4}$$

$$\mathbf{v} = -A(x,t,u)\,(\nabla p(x,t) - \mathbf{g}(x,t,u)) \tag{4.5}$$

$$\begin{aligned}-\int_\Omega \mathbf{v}\cdot\nabla\psi(x)dx &= \int_\Omega \psi(x)a(x,t)\mu(x)dx - \int_\Omega \psi(x)b(x,t)\mu(x)dx,\\ &\qquad \forall\psi \in H^1(\Omega),\ for\ a.e.\ t \in (0,T).\end{aligned} \tag{4.6}$$

4.1 Fixed point method

In order to analyse the previous system, we introduce a map S from $L^2((0,T); L^2(\Omega))$ to $L^2((0,T); L^2(\Omega))$ (a solution of the regularized problem will appear as a fixed point of S).

For $\bar{u} \in L^2((0,T); L^2(\Omega))$, let us define $u = S(\bar{u})$. First, there exists a unique solution p of

$$
\begin{aligned}
&p \in L^\infty((0,T); H^1(\Omega)), \int_\Omega p(x,t)dx = 0, \text{ for a.e. } t \in (0,T), \\
&\int_\Omega A(x,t,\bar{u}(x,t)) \left(\nabla p(x,t) - \mathbf{g}(x,t,\bar{u}(x,t))\right) \cdot \nabla \psi(x) dx \\
&= \int_\Omega \psi(x) a(x,t)\mu(x) dx - \int_\Omega \psi(x) b(x,t)\mu(x)dx, \forall \psi \in H^1(\Omega), \text{ for a.e. } t \in (0,T).
\end{aligned} \tag{4.7}
$$

The proof of the existence and uniqueness of solution p of (4.7) is an immediate consequence of the Lax-Milgram Lemma.

Then, there exists a unique u (and we set $u = S(\bar{u})$) solution of

$$
\begin{aligned}
&u \in L^2((0,T)\,;\, H^1(\Omega)) \cap C([0,T]; L^2(\Omega)),\ 0 \le u \le 1, \\
&u_t \in L^2((0,T); (H^1(\Omega))'), \\
&u(\cdot, 0) = u_0, \\
&\langle u_t(.,t), \varphi \rangle_{(H^1)', H^1} + \int_\Omega \mathbf{D}(\mathbf{v}(x,t)) \nabla u(x,t) \cdot \nabla \varphi(x) dx \\
&- \int_\Omega \mathbf{v}(x,t) \cdot \nabla \varphi(x) f(u(x,t)) dx + \int_\Omega u(x,t)\varphi(x) b(x,t)\mu(x) dx \\
&- \int_\Omega c(x,t)\varphi(x) a(x,t)\mu(x) dx = 0, \forall \varphi \in H^1(\Omega), \text{ for a.e. } t \in (0,T),
\end{aligned} \tag{4.8}
$$

where

$$
\mathbf{v}(x,t) = -A(x,t,\bar{u}(x,t)) \left(\nabla p(x,t) - \mathbf{g}(x,t,\bar{u}(x,t))\right), \text{ for a.e. } (x,t) \in \Omega \times (0,T). \tag{4.9}
$$

The existence and uniqueness of a solution u of (4.8) being classical, see [10], we omit its proof.

In order to prove the existence of a fixed point to S, let us first prove, in Section 4.2, some *a priori* estimates on u, $\mathbf{v}$ and p independent of $\bar{u}$. Some of them will be rather usefull for our main result (proved in Section 5). Then, we will prove in Section 4.3 the continuity and the compactness of S (from $L^2((0,T); L^2(\Omega))$ to $L^2((0,T); L^2(\Omega)))$ and we will conclude with the Schauder fixed point Theorem.

4.2 Estimates

Let us recall that f is assumed to be Lipschitz continuous on $[0,1]$ with

$$
\begin{aligned}
f(y) = 0 \quad &\text{for} \quad y \le 0 \\
f(y) = 1 \quad &\text{for} \quad y \ge 1
\end{aligned} \tag{4.10}
$$

4.2.1 L^∞ estimates

For $u = S(\bar{u})$ with $\bar{u} \in L^2((0,T); L^2(\Omega))$, one proves here that $0 \le u \le 1$.

Proposition 4.1
let (2.6)-(2.14) hold and assume μ in $L^2(\Omega)$. Setting $u = S(\bar{u})$ with $\bar{u} \in L^2((0,T); L^2(\Omega)$ (where S is defined with (4.7)-(4.8)), one has

$$
0 \le u \le 1. \tag{4.11}
$$

Proof of Proposition 4.1
Taking $\varphi = -u^-(\cdot,t)$ in (4.8), one has, for a.e. $t \in (0,T)$,

$$
\begin{aligned}
&-\langle u_t(\cdot,t), u^-(\cdot,t) \rangle_{(H^1)',H^1} + \int_\Omega \mathbf{D}(\mathbf{v}(x,t)) \nabla u^-(x,t) \cdot \nabla u^-(x,t) dx + \int_\Omega f(u(x,t)) \mathbf{v}(x,t) \cdot \nabla u^-(x,t) dx \\
&+ \int_\Omega (u^-(x,t))^2 b(x,t)\mu(x) dx + \int_\Omega c(x,t)\, u^-(x,t) a(x,t)\mu(x) dx = 0.
\end{aligned}
$$

Since $f(s) = 0$ for $s \le 0$ and since $a(\cdot,t)$, $b(\cdot,t)$ and μ are nonnegative functions, one gets (using also (2.11))

$$-\langle u_t(\cdot,t), u^-(\cdot,t)\rangle_{(H^1)',H^1} + \gamma \int_\Omega |\nabla u^-(x,t)|^2 dx \le 0.$$

Integrating this last inequality over $(0,s)$ and using

$$\frac{1}{2}(\|u^-(\cdot,s)\|^2_{L^2(\Omega)} - \|u_0^-\|^2_{L^2(\Omega)}) = -\int_0^s \langle u_t(\cdot,t), u^-(\cdot,t)\rangle_{(H^1)',H^1} dt,$$

one deduces that $\|u^-(\cdot,s)\|^2_{L^2(\Omega)} \le \|u_0^-\|^2_{L^2(\Omega)}$ for all $s \in [0,T]$ and therefore, since $u_0 \ge 0$ a.e., that $u(\cdot,s) \ge 0$ a.e., for all $s \in [0,T]$.

Taking $\varphi = (u-1)^+(\cdot,t)$ in (4.8), one has, for a.e. $t \in (0,T)$,

$$\begin{aligned} &\langle u_t(\cdot,t), (u-1)^+(\cdot,t)\rangle_{(H^1)',H^1} + \int_\Omega \mathbf{D}(\mathbf{v}(x,t))\nabla(u-1)^+(x,t)\cdot\nabla(u-1)^+(x,t)dx \\ &-\int_\Omega f(u(x,t))\,\mathbf{v}(x,t)\cdot\nabla(u(x,t)-1)^+dx \\ &+\int_\Omega u(x,t)(u(x,t)-1)^+ b(x,t)\mu(x)dx - \int_\Omega c(x,t)(u(x,t)-1)^+ a(x,t)\mu(x)dx = 0. \end{aligned} \tag{4.12}$$

Since $f(s)=1$ for $s \ge 1$, $\mathrm{div}\mathbf{v}(\cdot,t) = a(\cdot,t)\mu - b(\cdot,t)\mu$ on Ω and $\mathbf{v}(\cdot,t)\cdot\mathbf{n} = 0$ on $\partial\Omega$, the following holds

$$\begin{aligned} &-\int_\Omega f(u(x,t))\,\mathbf{v}(x,t)\cdot\nabla(u(x,t)-1)^+dx = -\int_\Omega \mathbf{v}(x,t)\cdot\nabla(u(x,t)-1)^+dx = \\ &\int_\Omega \mathrm{div}\mathbf{v}(x,t)(u(x,t)-1)^+dx = \int_\Omega (a(x,t)\mu(x) - b(x,t)\mu(x))(u(x,t)-1)^+dx. \end{aligned}$$

Then, (4.12) becomes

$$\begin{aligned} &\langle u_t(\cdot,t), (u-1)^+(\cdot,t)\rangle_{(H^1)',H^1} + \int_\Omega \mathbf{D}(\mathbf{v}(x,t))\nabla(u-1)^+(x,t)\cdot\nabla(u-1)^+(x,t)dx = \\ &\int_\Omega (c(x,t)-1)a(x,t)\mu(x)(u(x,t)-1)^+dx - \int_\Omega (u(x,t)-1)b(x,t)\mu(x)(u(x,t)-1)^+dx. \end{aligned}$$

Since $c(x,t) \le 1$ and since $a(\cdot,t)$, $b(\cdot,t)$ and μ are nonnegative functions, this yields

$$\langle u_t(\cdot,t), (u-1)^+(\cdot,t)\rangle_{(H^1)',H^1} + \int_\Omega \mathbf{D}(\mathbf{v}(x,t))\nabla(u-1)^+(x,t)\cdot\nabla(u-1)^+(x,t)dx \le 0.$$

Integrating this last inequality over $(0,s)$ and using

$$\frac{1}{2}(\|(u-1)^+(\cdot,s)\|^2_{L^2(\Omega)} - \|(u_0-1)^+\|^2_{L^2(\Omega)}) = \int_0^s \langle u_t(\cdot,t), (u-1)^+(\cdot,t)\rangle_{(H^1)',H^1} dt,$$

one obtains $\|(u-1)^+(\cdot,s)\|^2_{L^2(\Omega)} \le \|(u_0-1)^+\|^2_{L^2(\Omega)}$ and then, since $u_0 \le 1$ a.e., $u(\cdot,s) \le 1$ a.e., for all $s \in [0,T]$. This completes the proof of Proposition 4.1. ■

4.2.2 $L^2((0,T);H^1(\Omega))$ estimate

We are now able to establish a $L^2((0,T);H^1(\Omega))$ bound which depends only on the norm of $a\mu$ and $b\mu$ in $L^1((0,T)\,;\,L^1(\Omega))$. This estimate, as the above L^∞-bound, will be very usefull in proving of our main result (Section 5).

Proposition 4.2
Let (2.6)-(2.14) hold and assume μ in $L^2(\Omega)$. Setting $u = S(\bar u)$ with $\bar u \in L^2((0,T);L^2(\Omega))$ (where S is defined with (4.7)-(4.8)), there exists C depending only on f such that, for all $t \in [0,T]$,

$$\|u(t)\|^2_{L^2(\Omega)} + 2\gamma \int_0^t \int_\Omega |\nabla u(x,\tau)|^2 dx d\tau \le \|u_0\|^2_{L^2(\Omega)} + C\int_0^t (\|a(\cdot,\tau)\,\mu\|_{L^1(\Omega)} + \|b(\cdot,\tau)\,\mu\|_{L^1(\Omega)})d\tau. \tag{4.13}$$

Proof of Proposition 4.2

Taking φ equal to $u(\cdot,t)$ as a test function in (4.8), one obtains, for a.e. $t \in (0,T)$,

$$\langle u_t(\cdot,t), u(\cdot,t)\rangle_{(H^1)',H^1} + \int_\Omega \mathbf{D}(\mathbf{v}(x,t))\nabla u(x,t)\cdot\nabla u(x,t)dx - \int_\Omega f(u(x,t))\,\mathbf{v}(x,t)\cdot\nabla u(x,t)dx$$
$$+\int_\Omega (u(x,t))^2 b(x,t)\mu(x)dx - \int_\Omega c(x,t)\,u(x,t)a(x,t)\mu(x)dx = 0,$$

which gives, using $\mathrm{div}\mathbf{v}(\cdot,t) = a(\cdot,t)\mu - b(\cdot,t)\mu$ on Ω and $\mathbf{v}(\cdot,t)\cdot\mathbf{n} = 0$ on $\partial\Omega$,

$$\langle u_t(\cdot,t), u(\cdot,t)\rangle_{(H^1)',H^1} + \int_\Omega \mathbf{D}(\mathbf{v}(x,t))\nabla u(x,t)\cdot\nabla u(x,t)dx + \int_\Omega \left(u(x,t)^2 - F(u(x,t))\right) b(x,t)\mu(x)dx$$
$$-\int_\Omega \left(c(x,t)\,u(x,t) - F(u(x,t))\right) a(x,t)\mu(x)dx = 0,$$

where $F(u) = \displaystyle\int_0^u f(\tau)d\tau$.

Then, estimate (4.13) follows easily from an integration over time of the previous inequality (4.11).

■

For sake of completeness, let us give now some bounds on p and $\mathbf{v}$.

Proposition 4.3

Let (2.6)-(2.14) hold and let $\mu \in L^2(\Omega)$. Setting $u = S(\bar u)$ with $\bar u \in L^2((0,T);L^2(\Omega))$ (where S is defined with (4.7)-(4.8)). Then,

1. *There exists C_1 depending only on the norms of a and b in $L^\infty((0,T);C(\overline\Omega))$, $||\mu||_{L^2(\Omega))}$, β, $\bar g$ and α such that*

$$||\nabla p||_{L^\infty((0,T);(L^2(\Omega))^d)} \le C_1,\ ||\mathbf{v}||_{L^\infty((0,T);(L^2(\Omega))^d)} \le C_1, \tag{4.14}$$

2. *For any $q < d/(d-1)$, there exists C_2 depending only on the norms of a and b in $L^\infty((0,T);C(\overline\Omega))$, $||\mu||_{L^1(\Omega))}$, β, $\bar g$, Ω, q and α such that*

$$||\nabla p||_{L^\infty((0,T);(L^q(\Omega))^d)} \le C_2,\ ||\mathbf{v}||_{L^\infty((0,T);(L^q(\Omega))^d)} \le C_2. \tag{4.15}$$

Note that p and $\mathbf{v}$ are given by (4.7)-(4.8).

Proof of Proposition 4.3

The proof of (4.14) follows from the choice $\psi = p(\cdot,t)$ in (4.7).

The proof of (4.15) is given in Proposition 3.1, see Estimate (3.8). ■

To obtain compactness properties it remains to establish some bound on u_t.

Proposition 4.4

Assume (2.6)-(2.14) and $\mu \in L^2(\Omega))$. Let $u = S(\bar u)$ with $\bar u \in L^2((0,T);L^2(\Omega))$ (where S is defined by (4.7)-(4.8)). Then,

1. *There exists C_3 depending only on T, $||a||_{L^\infty((0,T);C(\overline\Omega))}$, $||b||_{L^\infty((0,T);C(\overline\Omega))}$, $||\mu||_{L^2(\Omega)}$, β, $\bar g$, $\mathbf{D}$, Ω and α such that*

$$||u_t||_{L^2((0,T);(H^1(\Omega))')} \le C_3, \tag{4.16}$$

2. *For any $q > d$, there exists C_4 depending only on T, $||a||_{L^\infty((0,T);C(\overline\Omega))}$, $||b||_{L^\infty((0,T);C(\overline\Omega))}$, $||\mu||_{L^1(\Omega)}$, β, $\bar g$, α, $\mathbf{D}$, Ω and q such that*

$$||u_t||_{L^2((0,T);(W^{1,q}(\Omega))')} \le C_4. \tag{4.17}$$

Proof of Proposition 4.4
From (4.8), one has for all $\phi \in L^2((0,T)\,;\,H^1(\Omega))$,

$$\Big|\langle u_t\,,\,\phi\rangle_{\{L^2((0,T)\,;\,H^1(\Omega))\}'\,,\,\{L^2((0,T)\,;\,H^1(\Omega))\}}\Big| \le$$

$$C\big(\|\nabla\, u\|_{L^2((0,T)\,;\,L^2(\Omega))} + \|\mathbf{v}\|_{L^2((0,T)\,;\,L^2(\Omega))}\big)\|\phi\|_{L^2((0,T)\,;\,H^1(\Omega))}$$
$$+C\big(\|a\mu\|_{L^2((0,T)\,;\,L^2(\Omega))} + \|b\mu\|_{L^2((0,T)\,;\,L^2(\Omega))}\big)\|\phi\|_{L^2((0,T)\,;\,H^1(\Omega))},$$

for some suitable constant C.
Then, the result follows from (4.14) and (4.13).

Estimate (4.17) is obtained in a similar way, using (4.15), (4.13) and the fact that $W^{1,q}(\Omega)$ is continuously imbedded in $L^\infty(\Omega)$ for $q > d$. ■

Estimate (4.16) is used in the following section (Section 4.3) and estimate (4.17) will be usefull for the proof of the main result (Section 5).

4.3 Existence of a solution

In this section, we first prove that the application S is continuous from $L^2((0,T);L^2(\Omega))$ to itself. Then, we shall prove that the range of S (namely $R(S) = \{S(\bar u),\ \bar u \in L^2((0,T);L^2(\Omega))\}$) is relatively compact in $L^2((0,T);L^2(\Omega))$. Therefore the Schauder fixed point theorem will give the existence of u such that $u = S(u)$.

Proposition 4.5
Let (2.6)-(2.14) hold, and assume μ in $L^2(\Omega)$. Then the application S defined by (4.7)-(4.8) is a continuous mapping from $L^2((0,T);L^2(\Omega))$ to $L^2((0,T);L^2(\Omega))$.

Proof of Proposition 4.5
Let $(\bar u_n)_n$ and $\bar u$ such that $\bar u_n \to \bar u$ in $L^2((0,T)\,;\,L^2(\Omega))$ as $n\to\infty$. Let $(p,\mathbf{v},u)$ be the solution of (4.7)-(4.9) and $(p_n,\mathbf{v}_n,u_n)$ be the solution of (4.7)-(4.9) with $\bar u_n$ instead of $\bar u$ (so that $u_n = S(\bar u_n)$ and $u = S(\bar u)$). One has to prove that $u_n \to u$ in $L^2((0,T);L^2(\Omega))$. The proof relies on two steps.

Step 1. One first proves that the sequence $(\mathbf{v}_n)_n$ converges to $\mathbf{v}$ in $L^2((0,T)\,;\,(L^2(\Omega))^d)$.
The difference $p_n - p$ satisfies, for all $\psi \in L^2((0,T);H^1(\Omega))$,

$$\int_0^T\int_\Omega A(x,\tau,\bar u_n(x,\tau))\nabla\Big(p_n(x,\tau) - p(x,\tau)\Big)\cdot\nabla\psi(x,\tau)dxd\tau$$
$$+\int_0^T\int_\Omega\Big(A(x,\tau,\bar u_n(x,\tau)) - A(x,\tau,\bar u(x,\tau))\Big)\nabla p(x,\tau)\cdot\nabla\psi(x,\tau)dxd\tau$$
$$-\int_0^T\int_\Omega\Big(A(x,\tau,\bar u_n(x,\tau))\mathbf{g}(x,\tau,\bar u_n(x,\tau)) - A(x,\tau,\bar u(x,\tau))\mathbf{g}(x,\tau,\bar u(x,\tau))\Big)\cdot\nabla\psi(x,\tau)dxd\tau = 0.$$

Then, taking the as a test function ψ the difference $p_n - p = \pi_n$ it follows

$$\begin{aligned}\|\nabla\pi_n\|^2_{L^2((0,T);(L^2(\Omega))^d)} &\le C_1\,\|(A(\cdot,\cdot,\bar u_n) - A(\cdot,\cdot,\bar u))\nabla\,p\|^2_{L^2((0,T);(L^2(\Omega))^d)}\\ &+C_2\,\|A(\cdot,\cdot,\bar u_n)\mathbf{g}(\cdot,\cdot,\bar u_n) - A(\cdot,\cdot,\bar u)\mathbf{g}(\cdot,\cdot,\bar u)\|^2_{L^2((0,T);(L^2(\Omega))^d)}\,,\end{aligned}\tag{4.18}$$

for some convenient C_1 and C_2. Since the sequence $(\bar u_n)_n$ converges in $L^2((0,T)\,;\,L^2(\Omega))$ (which can be viewed as $L^2(\Omega\times(0,T))$) to u and since the functions A and $\mathbf{g}$ are of Caratheodory type, the right hand side of (4.18) goes to zero. This proves that $\nabla p_n \to \nabla p$ in $L^2((0,T);(L^2(\Omega))^d)$ as $n\to\infty$. Now, since

$$\mathbf{v}_n(x,t) = A(x,t,\bar{u}_n(x,t))\,(\nabla p_n(x,t) - \mathbf{g}(x,t,\bar{u}_n(x,t)))\,,$$

$(\mathbf{v}_n)_n$ converges to $\mathbf{v}$ in $L^2((0,T)\,;\,(L^2(\Omega))^d)$. This achieves the proof of Step 1.

Step 2. One now proves now the convergence of u_n towards u.

Step 1 gives the convergence of $(\mathbf{v}_n)_n$ towards $\mathbf{v}$ in $L^2((0,T)\,;\,(L^2(\Omega))^d)$. Using classical compactness results [11], [18], it follows from the estimates (4.13) and (4.16), that, up to a subsequence, we can assume that

$$\begin{array}{l} u_n \to \tilde{u} \text{ weakly in } L^2((0,T)\,;\,H^1(\Omega)), \\ (u_n)_t \to \tilde{u}_t \text{ weakly in } L^2((0,T)\,;\,(H^1(\Omega))'), \\ u_n \to \tilde{u} \text{ in } C([0,T]\,;\,(H^1(\Omega))') \\ u_n \to \tilde{u} \text{ in } L^2((0,T)\,;\,L^2(\Omega)), \\ u_n \to \tilde{u} \text{ a.e. in } \Omega \times (0,T). \end{array}$$

Then we can pass to the limit in (4.8) (with u_n and $\mathbf{v}_n$ instead of u and $\mathbf{v}$)as $n \to \infty$ and prove that $\tilde{u}$ is solution of (4.8). Therefore, the uniqueness of the solution of (4.8) implies that $\tilde{u} = u$ and that the whole sequence u_n converges towards u in $L^2((0,T);L^2(\Omega))$. This ends the proof of Proposition 4.5. ∎

It is now possible to prove Proposition 2.1, which gives existence of a solution of the regularized problem (that is assuming (2.6)-(2.14) and μ in $L^2(\Omega)$)) in the sense of Definition 4.1, using the Schauder fixed point Theorem.

Proof of Proposition 2.1

In order to prove Proposition 2.1, it is sufficient to prove the existence of a fixed point for the mapping S defined by (4.7)-(4.8). Proposition 4.5 gives the continuity of S (from $L^2((0,T);L^2(\Omega))$ to $L^2((0,T);L^2(\Omega))$. Let $R(S)$ be the range of S. The Schauder fixed point theorem gives the existence of a fixed point for S provided that $R(S)$ is relatively compact in $L^2((0,T);L^2(\Omega))$. Let us show this last property holds.

The energy estimate, namely Estimate (4.13), gives that $R(S)$ is bounded in $L^2((0,T);H^1(\Omega))$, the estimate (4.16) gives that $\{u_t,\, u \in R(S)\}$ is bounded in $L^2((0,T),(H^1(\Omega))'$. Since $H^1(\Omega)$ is compactely embedded in $L^2(\Omega)$ and since $H^1(\Omega)$ is dense in $L^2(\Omega)$, a classical compactness result [11], [18] gives the relative compactness of $R(S)$ in $L^2((0,T);L^2(\Omega))$ (and also in $C([0,T];(H^1(\Omega))')$. This concludes the proof of Proposition 2.1. ∎

5 Proof of the main result

We now study the case where $\mu \in \mathcal{M}_+(\Omega)$, which corresponds to Theorem 2.1.

Proof of Theorem 2.1

Step 1. (Construction of the sequence $(\mu_n)_{n\in\mathbb{N}}$)

Let $(\rho_n)_{n\in\mathbb{N}}$ be a sequence of mollifiers, that is $\rho_n(x) = n^d\rho(nx)$, for all $x \in \mathbb{R}^d$ and all $n \in \mathbb{N}^\star$, with $\rho \in C^\infty(\mathbb{R}^d, \mathbb{R}_+)$ such that

$$\int \rho(x)dx = 1 \text{ and } (|x| \geq 1 \Rightarrow \rho(x) = 0).$$

Let $\mu_n = (\rho_n \star \tilde{\mu})|_\Omega$, where $\tilde{\mu}$ is the extension by 0 of μ on $\mathbb{R}^d$.

It is easy to prove that, for all $n \in \mathbb{N}$, $\mu_n \in L^2(\Omega)$, $\mu_n \geq 0$ a.e. and $||\mu_n||_{L^1(\Omega)} \leq \mu(\Omega)$. We claim that $\mu_n \to \mu$, as $n \to \infty$, for the weak-$\star$ topology of $C(\overline{\Omega})$ and that the sequence $(\mu_n)_{n\in\mathbb{N}}$ is bounded in $(W^{1,q}(\Omega))'$ for any $q > 2$.

First, one proves that $\mu_n \to \mu$ for the weak-$\star$ topology of $C(\overline{\Omega})$.

Let $\varphi \in C_c(\Omega)$ (that is φ continuous from Ω to $\mathbb{R}$ with a compact support in Ω). Then,

$$\int_\Omega \varphi(x)\mu_n(x)dx = \int_{\mathbb{R}^d} \tilde\varphi(x)\rho_n \star \tilde\mu(x)dx,$$

where $\tilde\varphi(x) = \varphi(x)$ if $x \in \Omega$ and $\tilde\varphi(x) = 0$ if $x \notin \Omega$.
Since, as $n \to \infty$,

$$\int_{\mathbb{R}^d} \tilde\varphi(x)\rho_n \star \tilde\mu(x)dx \to \int_{\mathbb{R}^d} \tilde\varphi(x)d\tilde\mu(x) = \int_\Omega \varphi(x)d\mu(x),$$

we conclude that

$$\int_\Omega \varphi(x)\mu_n(x)dx \to \int_\Omega \varphi(x)d\mu(x),\ \forall \varphi \in C_c(\Omega). \tag{5.1}$$

One also has

$$\int_{\mathbb{R}^d} \rho_n \star \tilde\mu(x)dx = \mu(\Omega) \tag{5.2}$$

and

$$\int_{\mathbb{R}^d\setminus\Omega} \rho_n \star \tilde\mu(x)dx \le \mu(A_n), \tag{5.3}$$

where $A_n = \{x \in \Omega,\ \mathrm{dist}(x,\Omega^c) \le 1/n\}$ and $\mathrm{dist}(x,\Omega^c) = \inf\{|x-y|,\ y \notin \Omega\}$.
Since $\cap_{n\in\mathbb{N}} A_n = \emptyset$, one has $\mu(A_n) \to 0$ as $n \to \infty$. Therefore (5.2) and (5.3) lead to

$$\int_\Omega \mu_n(x)dx \to \mu(\Omega),$$

which, together with (5.1), gives that $\mu_n \to \mu$ for the weak-$\star$ topology of $C(\overline\Omega)$ (that is (5.1) with $C(\overline\Omega)$ instead of $C_c(\Omega)$).

Now, we prove that the sequence $(\mu_n)_{n\in\mathbb{N}}$ is bounded in $(W^{1,q}(\Omega))'$ for any $q > 2$.

Let $q \in (2,\infty)$. Since $\mu \in (W^{1,q}(\Omega))'$, one has $\tilde\mu \in (W^{1,q}(\mathbb{R}^d))'$ (and $\|\tilde\mu\|_{(W^{1,q}(\mathbb{R}^d))'} \le \|\mu\|_{(W^{1,q}(\Omega))'}$). Then, by a classical characterisation of $(W^{1,q}(\mathbb{R}^d))'$, there exists $g \in L^{q'}(\mathbb{R}^d)$ and $G \in (L^{q'}(\mathbb{R}^d))^d$, with $q' = q/(q-1)$, such that $\tilde\mu = g + \mathrm{div}G$ and

$$\|\tilde\mu\|_{(W^{1,q}(\mathbb{R}^d))'} = \|g\|_{L^{q'}(\mathbb{R}^d)} + \|G\|_{(L^{q'}(\mathbb{R}^d))^d},$$

with

$$\|G\|_{(L^{q'}(\mathbb{R}^d))^d} = \left(\int_{\mathbb{R}^d} |G(x)|^{q'} dx\right)^{\frac{1}{q'}}$$

and

$$\|\varphi\|_{W^{1,q}(\mathbb{R}^d)} = \left(\int_{\mathbb{R}^d} |\varphi(x)|^q dx\right)^{\frac{1}{q}} + \left(\int_{\mathbb{R}^d} |\nabla\varphi(x)|^q dx\right)^{\frac{1}{q}},$$

the norm $\|\cdot\|_{(W^{1,q}(\mathbb{R}^d))'}$ is the corresponding dual norm. (Note that g and $\mathrm{div}(G)$ are not necessarily some nonnegatives measures.)

Then, with $\nu_n = \rho_n \star \tilde\mu = \rho_n \star g + \mathrm{div}(\rho_n \star G)$,

$$\begin{aligned}\|\nu_n\|_{(W^{1,q}(\mathbb{R}^d))'} &\le \|\rho_n \star g\|_{L^{q'}(\mathbb{R}^d)} + \|\rho_n \star G\|_{(L^{q'}(\mathbb{R}^d))^d}\\ &\le \|g\|_{L^{q'}(\mathbb{R}^d)} + \|G\|_{(L^{q'}(\mathbb{R}^d))^d}\\ &\le \|\tilde\mu\|_{(W^{1,q}(\mathbb{R}^d))'}.\end{aligned} \tag{5.4}$$

The measure μ_n is the restriction to Ω of the measure ν_n. We shall deduce from (5.4) that the sequence $(\mu_n)_{n\in\mathbb{N}}$ is bounded in $(W^{1,q}(\Omega))'$. Indeed, there exists a linear continuous mapping P from $W^{1,q}(\Omega)$

to $W^{1,q}(\mathbb{R}^d)$ such that $P\varphi = \varphi$ a.e. on Ω. Furthermore, P can be chosen such that $P\varphi \in C(\mathbb{R}^d)$ if $\varphi \in C(\overline{\Omega})$ and $P\varphi \geq 0$ a.e. on $\mathbb{R}^d$ if $\varphi \geq 0$ a.e. on Ω. Let $\varphi \in C(\overline{\Omega}) \cap W^{1,q}(\Omega)$, one has

$$\begin{aligned} &\int_\Omega \varphi(x)\mu_n(x)dx \leq \int_\Omega |\varphi(x)|\mu_n(x)dx \leq \int_{\mathbb{R}^d} P(|\varphi|)(x)\nu_n(x)dx \\ &\leq \|P(|\varphi|)\|_{W^{1,q}(\mathbb{R}^d)}\|\nu_n\|_{(W^{1,q}(\mathbb{R}^d))'} \leq C\|(|\varphi|)\|_{W^{1,q}(\Omega)}\|\nu_n\|_{(W^{1,q}(\mathbb{R}^d))'} \\ &\qquad\qquad\qquad\qquad\qquad\qquad \leq C\|\varphi\|_{W^{1,q}(\Omega)}\|\nu_n\|_{(W^{1,q}(\mathbb{R}^d))'}, \end{aligned} \tag{5.5}$$

where C is the norm of P (as a linear continuous mapping from $W^{1,q}(\Omega)$ to $W^{1,q}(\mathbb{R}^d)$), it depends only on Ω and q. We used also a classical result of Stampacchia [19] which gives $\|(|\varphi|)\|_{W^{1,q}(\Omega)} = \|\varphi\|_{W^{1,q}(\Omega)}$. Inequalities (5.4) and (5.5) give that the sequence $(\mu_n)_{n\in\mathbb{N}}$ is bounded in $(W^{1,q}(\Omega))'$. Note also that one has $\mu_n \to \mu$, as $n \to \infty$, for the weak-$\star$ topology of $(W^{1,q}(\Omega))'$ for any $q > 2$, thanks to the separability of $W^{1,q}(\Omega)$ for $q \in (2,\infty)$ and the uniqueness of the limites of the converging subsequences of $(\mu_n)_{n\in\mathbb{N}}$ in $(W^{1,q}(\Omega))'$ for the weak-$\star$ topology. This concludes Step 1.

Step 2. (Approximate solutions and estimates)

For $n \in \mathbb{N}^\star$, μ_n is defined in the preceding step, let us also define $\varepsilon_n(t)$ and $a_n(\cdot,t)$ for a.e. $t \in (0,T)$ by

$$\begin{aligned} &\varepsilon_n(t) = 0,\ a_n(\cdot,t) = a(\cdot,t),\ \text{if } \textstyle\int_\Omega a d\mu_n = \int_\Omega b d\mu_n, \\ &\varepsilon_n(t) = \frac{\int_\Omega (b(x,t) - a(x,t))\mu_n(x)dx}{\int_\Omega \mu_n(x)dx},\ a_n(\cdot,t) = a(\cdot,t) + \varepsilon_n(t),\ \text{if } (0 \leq) \int_\Omega a d\mu_n < \int_\Omega b d\mu_n, \\ &\varepsilon_n(t) = \frac{\int_\Omega (a(x,t) - b(x,t))\mu_n(x)dx}{\int_\Omega a(x,t)\mu_n(x)dx},\ a_n(\cdot,t) = a(\cdot,t)(1-\varepsilon_n(t)),\ \text{if } \int_\Omega a d\mu_n > \int_\Omega b d\mu_n (\geq 0). \end{aligned}$$

Note that, in the second case, $\varepsilon_n(t) > 0$ and, in the third case, $\varepsilon_n(t) \in (0,1]$. Then, the function a_n belongs to $L^\infty((0,T);C(\overline{\Omega}))$ and $a_n(\cdot,t) \geq 0$ on $\overline{\Omega}$ for a.e. $t \in (0,T)$. Furthermore, one has

$$\int_\Omega a_n(x,t)\mu_n(x)dx = \int_\Omega b(x,t)\mu_n(x)dx, \quad \text{for a.e. } t \in (0,T),$$

and, since $b - a \in L^\infty((0,T);C(\overline{\Omega}))$, the sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ is bounded in $L^\infty((0,T))$ and (from (2.14)) $\varepsilon_n \to 0$ a.e. on $(0,T)$ as $n \to \infty$.

For $n \in \mathbb{N}^\star$, let $(u_n, \mathbf{v}_n, p_n)$ be the solution of the approximated problem (2.22)-(2.27) with μ_n instead of μ and a_n instead of a (existence of such a solution is given by Proposition 2.1). Then, $(u_n, \mathbf{v}_n, p_n)$ satisfies:

$$\begin{aligned} &u_n \in L^2((0,T);H^1(\Omega)) \cap C([0,T];L^2(\Omega)),\ 0 \leq u_n(\cdot,t) \leq 1, \\ &(u_n)_t \in L^2((0,T);(H^1(\Omega))'), \end{aligned} \tag{5.6}$$

$$u_n(.,0) = u_0 \ \text{ in } L^2(\Omega), \tag{5.7}$$

$$p_n \in L^\infty((0,T);H^1(\Omega)),\ \mathbf{v}_n \in L^\infty((0,T);(L^2(\Omega))^d), \tag{5.8}$$

$$\begin{aligned} &\langle (u_n)_t(.,t), \varphi\rangle_{(H^1)',H^1} + \int_\Omega \mathbf{D}(\mathbf{v}_n(x,t))\nabla u_n(x,t)\cdot\nabla\varphi(x)dx \\ &- \int_\Omega \mathbf{v}_n(x,t)\cdot\nabla\varphi(x)f(u_n(x,t))dx + \int_\Omega u_n(x,t)\varphi(x)b(x,t)\mu_n(x)dx \\ &- \int_\Omega c(x,t)\varphi(x)a_n(x,t)\mu_n(x)dx = 0, \\ &\qquad\qquad\qquad\qquad\qquad \forall\varphi \in H^1(\Omega),\ \text{for a.e. } t \in (0,T), \end{aligned} \tag{5.9}$$

$$\mathbf{v}_n(x,t) = -A(x,t,u_n(x,t))\left\{\nabla p_n(x,t) - \mathbf{g}(x,t,u_n(x,t))\right\},\ \text{ for a.e. } (x,t) \in \Omega\times(0,T), \tag{5.10}$$

$$\begin{aligned} -\int_\Omega \mathbf{v}_n(x,t)\cdot\nabla\psi(x)dx = \int_\Omega \psi(x)a_n(x,t)\mu_n(x)dx - \int_\Omega \psi(x)b(x,t)\mu_n(x)dx, \\ \forall\psi \in H^1(\Omega),\ \text{for a.e. } t \in (0,T). \end{aligned} \tag{5.11}$$

Estimate (4.13) gives a bound in $L^2((0,T);H^1(\Omega))$ for the sequence $(u_n)_n$ and Estimate (4.17) gives a bound in $L^2((0,T);(W^{1,q})')$ (for any $q>d$) for the sequence $((u_n)_t)_n$. Then, classical compactness [11], [18] result leads to the relative compactness of the sequence $(u_n)_n$ in $L^2((0,T);L^2(\Omega))$ and in $C([0,T];(W^{1,q})')$ (for any $q>d$). Then, up to a subsequence, one can assume, as $n\to\infty$,

$$\begin{array}{l} u_n(\cdot,t)\to u(\cdot,t) \text{ in } (W^{1,q})',\ \forall q>d, \text{ uniformly with respect to } t\in[0,T], \\ u_n\to u \text{ weakly in } L^2((0,T);H^1(\Omega)), \\ (u_n)_t\to u_t \text{ weakly in } L^2((0,T);(W^{1,q})'),\ \forall q>d. \end{array} \tag{5.12}$$

Note that $u\in C([0,T];(W^{1,q})')$ (for any $q>d$), $u\in L^2((0,T);H^1(\Omega))$ and $u_t\in L^2((0,T);(W^{1,q})')$ for any $q>d$.

Furthermore, since $L^2((0,T);L^2(\Omega))$ can be identify to $L^2(\Omega\times(0,T))$, one can also assume (up to a subsequence) that $u_n\to u$ a.e on $\Omega\times(0,T)$ (which is equivalent to $u_n(\cdot,t)\to u(\cdot,t)$ a.e. on Ω for a.e. $t\in(0,T)$). Using the bound on u_n given in (5.6) (namely $0\le u_n\le 1$ a.e. on $\Omega\times(0,T)$), this gives, in particular, $0\le u\le 1$ a.e. on $\Omega\times(0,T)$.

The function u satisfies (2.17), (2.18) and the first part of (2.16). Indeed, using (for $d=3$) more precise estimates on both sequences $(\mu_n)_{n\in\mathbb{N}}$ and $(\mathbf{v}_n)_{n\in\mathbb{N}}$ (namely, the fact that the sequence $(\mu_n)_{n\in\mathbb{N}}$ is bounded in $(W^{1,q}(\Omega))'$ for any $q>2$ and therefore the sequence $(\mathbf{v}_n)_{n\in\mathbb{N}}$ is bounded in $L^\infty((0,T);(L^q(\Omega))^d)$ for any $q<2$), the sequence $((u_n)_t)_{n\in\mathbb{N}}$ is bounded in $L^2((0,T);(W^{1,q}(\Omega))')$ for any $q>2$ which gives $u_t\in L^2((0,T);(W^{1,q}(\Omega))')$, for any $q>2$, and therefore $u\in C([0,T];(W^{1,q}(\Omega))')$ for any $q>2$. This proves the assertion of Remark 2.2.

It remains to prove (2.19)-(2.21) and the second part of (2.16). This will be done in the next step.

Step 3. (Passing to the limit in (5.9)-(5.11))

We first prove the convergence of the whole sequences $(p_n)_n$ and $(\mathbf{v}_n)_n$ and we prove (2.19), (2.21). The sequence $(p_n)_n$ verifies, for almost every $t\in(0,T)$,

$$\begin{aligned} &\int_\Omega A(x,t,u_n(x,t))\Big(\nabla p_n(x,t)-\mathbf{g}(x,t,u_n(x,t))\Big)\cdot\nabla\psi(x)\,dx \\ &=\int_\Omega (a_n(x,t)-b(x,t))\psi(x)\mu_n(x)dx,\ \forall\psi\in H^1(\Omega). \end{aligned}$$

In order to prove the convergence of the sequences $(p_n)_n$ and $(\mathbf{v}_n)_n$, we shall apply the result of Proposition 3.3.

First, we remark that for almost every t in $(0,T)$, one has, as $n\to\infty$,

$$\begin{array}{l} A(\cdot,t,u_n(\cdot,t))\to A(\cdot,t,u(\cdot,t)) \text{ a.e. in } \Omega, \\ \mathbf{g}(\cdot,t,u_n(\cdot,t))\to \mathbf{g}(\cdot,t,u(\cdot,t)) \text{ a.e. in } \Omega. \end{array}$$

Next, since $\mu_n\to\mu$ for the weak-$\star$ topology of $C(\overline{\Omega})$ and $\varepsilon_n(t)\to 0$ for a.e. $t\in(0,T)$, one also has, as $n\to\infty$, for a.e. $t\in(0,T)$,

$$(a_n(\cdot,t)-b(\cdot,t))\mu_n\to(a(\cdot,t)-b(\cdot,t))\mu \text{ for the weak-}\star\text{ topology of } C(\overline{\Omega}).$$

In order to apply the result of Proposition 3.3, it remains to show that the sequence $((a_n(\cdot,t)-b(\cdot,t))\mu_n)_{n\in\mathbb{N}}$ is bounded in $(W^{1,q}(\Omega))'$ for any $q>2$ and a.e. $t\in(0,T)$ (then, one also has $(a(\cdot,t)-b(\cdot,t))\mu\in(W^{1,q}(\Omega))'$ and $(a_n(\cdot,t)-b(\cdot,t))\mu_n\to(a(\cdot,t)-b(\cdot,t))\mu$ for the weak-$\star$ topology of $(W^{1,q}(\Omega))'$ for any $q>2$ and a.e. $t\in(0,T)$). This is a consequence of the fact that $(\mu_n)_{n\in\mathbb{N}}$ is a sequence of nonnegative measures on Ω, bounded in $(W^{1,q}(\Omega))'$ for any $q>2$. Indeed, let $q>2$ and $\varphi\in W^{1,q}(\Omega)$, one has

$$\begin{aligned} \int_\Omega\varphi(x)(a_n(x,t)-b(x,t))\mu_n(x)dx &\le \max\{|a_n(x,t)-b(x,t)|,\ x\in\Omega\}\int_\Omega|\varphi(x)|\mu_n(x)dx \\ &\le \max\{|a_n(x,t)-b(x,t)|,\ x\in\Omega\}\|(|\varphi|)\|_{W^{1,q}(\Omega)}\|\mu_n\|_{(W^{1,q}(\Omega))'}. \end{aligned} \tag{5.13}$$

From a well known result of Stampacchia (see [19]), $\|(|\varphi|)\|_{W^{1,q}(\Omega)}=\|\varphi\|_{W^{1,q}(\Omega)}$. Then, since $a,\ b\in L^\infty((0,T);C(\overline{\Omega}))$ and $(\varepsilon_n)_{n\in\mathbb{N}}$ is bounded in $L^\infty((0,T))$, (5.13) gives that the sequence $((a_n(\cdot,t)-b(\cdot,t))\mu_n)_{n\in\mathbb{N}}$ is bounded in $(W^{1,q}(\Omega))'$ for any $q>2$ and a.e. $t\in(0,T)$.

Then, we can applied the stability result given in Proposition 3.3 to conclude that there exists a unique p in $L^\infty((0,T)\,;\,W^{1,q}(\Omega))$ (for any $q<2$) such that, as $n\to\infty$ (without extraction of subsequences),

$$p_n(\cdot,t)\to p(\cdot,t) \ \text{ in } W^{1,q}(\Omega),\ \forall q<2,\ \text{ for a.e. } t\in(0,T). \tag{5.14}$$

Then, defining $\mathbf{v}$ with the first line of (2.21), the functions $\mathbf{v}$ and p satisfy (2.19) and (2.21). Note also that the sequences $(\nabla p_n(\cdot,t))_{n\in\mathbb{N}}$ and $(\mathbf{v}_n(\cdot,t))_{n\in\mathbb{N}}$ are bounded in $(L^q(\Omega))^d$ for any $q<2$, uniformly with respect to $t\in(0,T)$, except on a neligeable set.

It remains to prove (2.20) and the second part of (2.16). Using (5.14) and the definition of $\mathbf{v}$, one has $\mathbf{v}_n(\cdot,t)\to\mathbf{v}(\cdot,t)$ in $(L^q(\Omega))^d$, for all $q<2$. Therefore, $\mathbf{D}(\mathbf{v}_n(\cdot,t))\to\mathbf{D}(\mathbf{v}(\cdot,t))$ in $M_d(L^q(\Omega))$, for any $q<\infty$ and for a.e. $t\in(0,T)$. Thanks to the a.e. convergence of u_n towards u and the L^∞-bound on $(u_n)_{n\in\mathbb{N}}$, we also have $\mathbf{v}_n(\cdot,t)f(u_n(\cdot,t))\to\mathbf{v}(\cdot,t)f(u(\cdot,t))$ in $(L^q(\Omega))^d$, for any $q<2$ and for a.e. $t\in(0,T)$.

By a classical decomposition result, the measure μ can be expressed as $\mu=g\lambda+\bar\mu$, where λ is the Lebesgue measure on Ω, $g\in L^1(\Omega)$ (which is $L^1(\Omega,\lambda)$) and $\bar\mu\perp\lambda$. Then, $\mu_n=g_n+\bar\mu_n$, where $g_n=(\rho_n\star\tilde g)|_\Omega\to g$ in $L^1(\Omega)$ and $\bar\mu_n=(\rho_n\star\tilde{\bar\mu})|_\Omega\to\bar\mu$ for the weak-$\star$ topology of $C(\overline{\Omega})$ as $n\to\infty$ (where $\tilde g$ and $\tilde{\bar\mu}$ are the extensions by 0 of g and $\bar\mu$ on $\mathbb{R}^d$).

Let $q>d$ and $\varphi\in L^2((0,T);W^{1,q}(\Omega))$. Equation (5.9) gives

$$\begin{aligned}
&\int_0^T\langle(u_n)_t(\cdot,t),\varphi(\cdot,t)\rangle_{(W^{1,q})',W^{1,q}}dt+\int_0^T\int_\Omega\mathbf{D}(\mathbf{v}_n(x,t))\nabla u_n(x,t)\cdot\nabla\varphi(x,t)dxdt\\
&-\int_0^T\int_\Omega\mathbf{v}_n(x,t)\cdot\nabla\varphi(x,t)f(u_n(x,t))dxdt\\
&-\int_0^T\int_\Omega c(x,t)\varphi(x,t)a_n(x,t)g_n(x)dxdt-\int_0^T\int_\Omega c(x,t)\varphi(x,t)a_n(x,t)\bar\mu_n(x)dxdt\\
&+\int_0^T\int_\Omega u_n(x,t)\varphi(x,t)b(x,t)g_n(x)dxdt+\int_0^T\int_\Omega u_n(x,t)\varphi(x,t)b(x,t)\bar\mu_n(x)dxdt=0.
\end{aligned}\tag{5.15}$$

It is quite easy to pass to limit, as $n\to\infty$, on each term of (5.15) (using, for some terms, the dominated convergence theorem) except for the last term. This proves that the last term has also a limit which we call $L(\varphi)$. Then, one has

$$\begin{aligned}
&\int_0^T\langle u_t(\cdot,t),\varphi(\cdot,t)\rangle_{(W^{1,q})',W^{1,q}}dt+\int_0^T\int_\Omega\mathbf{D}(\mathbf{v}(x,t))\nabla u(x,t)\cdot\nabla\varphi(x,t)dxdt\\
&-\int_0^T\int_\Omega\mathbf{v}(x,t)\cdot\nabla\varphi(x,t)f(u(x,t))dxdt\\
&-\int_0^T\int_\Omega c(x,t)\varphi(x,t)a(x,t)g(x)dxdt-\int_0^T\int_\Omega c(x,t)\varphi(x,t)a(x,t)\bar\mu(x)dxdt\\
&+\int_0^T\int_\Omega u(x,t)\varphi(x,t)b(x,t)g(x)dxdt+L(\varphi)=0
\end{aligned}\tag{5.16}$$

and

$$L(\varphi)=\lim_{n\to\infty}\int_0^T\int_\Omega u_n(x,t)\varphi(x,t)b(x,t)\bar\mu_n(x)dxdt.$$

Using Lemma 5.1 given below and noting that $\bar\mu\perp\lambda$, it is possible to choose u such that $u\in L^\infty((0,T);$ $L^1(\Omega,\bar\mu))$, $0\le u(\cdot,t)\le 1$ $\bar\mu$-a.e. for a.e. $t\in(0,T)$ and

$$L(\varphi)=\int_0^T\int_\Omega u(x,t)\varphi(x,t)b(x,t)d\bar\mu(x)dt.$$

Then, (5.16) becomes

$$\int_0^T \langle u_t(\cdot,t), \varphi(\cdot,t)\rangle_{(W^{1,q})',W^{1,q}} dt + \int_0^T \int_\Omega \mathbf{D}(\mathbf{v}(x,t))\nabla u(x,t)\cdot \nabla\varphi(x,t) dx dt$$
$$- \int_0^T \int_\Omega \mathbf{v}(x,t)\cdot\nabla\varphi(x,t) f(u(x,t)) dx dt$$
$$- \int_0^T \int_\Omega c(x,t)\varphi(x,t)a(x,t)g(x) dx dt, - \int_0^T \int_\Omega c(x,t)\varphi(x,t)a(x,t)\bar{\mu}(x) dx dt,$$
$$+ \int_0^T \int_\Omega u(x,t)\varphi(x,t)b(x,t)g(x) dx dt + \int_0^T \int_\Omega u(x,t)\varphi(x,t)b(x,t) d\bar{\mu}(x) dt = 0.$$

Since φ is arbitrary in $L^2((0,T); W^{1,q}(\Omega))$ (and q arbitrary in (d,∞)), this yields (2.20). Note that, since $\mu = g\lambda + \bar{\mu}$, one also has $u \in L^\infty((0,T); L^1(\Omega,\mu))$ and $0 \le u(\cdot,t) \le 1$ μ-a.e. for a.e. $t \in (0,T)$. Therefore u satisfies the second part of (2.16). This concludes the proof of Theorem 2.1. ■

Lemma 5.1 *Let Ω be an open bounded set of $\mathbb{R}^d$ $(d \ge 1)$ and $T > 0$. Let $(\mu_n)_{n\in\mathbb{N}}$ be a bounded sequence of nonnegative finite measures on Ω and μ be a nonnegative finite measure on Ω such that $\mu_n \to \mu$, as $n \to \infty$, for the weak-$\star$ topology of $(C(\bar{\Omega}))'$. Let $(v_n)_{n\in\mathbb{N}}$ be a sequence of functions from $\Omega\times(0,T)$ to $\mathbb{R}$ such that*

1. *There exists two nonnegative functions α, $\beta \in L^\infty((0,T); C(\overline{\Omega}))$ such that $\alpha(\cdot,t) \le v_n(\cdot,t) \le \beta(\cdot,t)$ μ_n-a.e., for a.e. $t \in (0,T)$ and for all $n \in \mathbb{N}$,*
2. *$\int_0^T \int_\Omega \varphi(x,t)v_n(x,t) d\mu_n(x) dt$ has a limit (in $\mathbb{R}$), as $n\to\infty$, for all $\varphi \in C(\bar{\Omega}\times[0,T])$.*

Then, there exists $v \in L^\infty((0,T); L^1(\Omega,\mu))$ such that

$$\int_0^T \int_\Omega \varphi(x,t)v_n(x,t) d\mu_n(x) dt \to \int_0^T \int_\Omega \varphi(x,t)v(x,t) d\mu(x) dt,\ \forall\varphi \in C(\bar{\Omega}\times[0,T]).$$

Furthermore, one has $\alpha(\cdot,t) \le v(\cdot,t) \le \beta(\cdot,t)$ μ-a.e., for a.e. $t \in (0,T)$.

Proof of Lemma 5.1

Let $Q = \Omega\times(0,T)$. For $n\in\mathbb{N}$, let ν_n be the measure on Q defined by $d\nu_n(x,t) = d\mu_n(x)dt$ and let ν be the measure on Q defined by $d\nu(x,t) = d\mu(x)dt$. The sequence $(\nu_n)_{n\in\mathbb{N}}$ is a bounded sequence of nonnegative measures on Q and $\nu_n \to \nu$, as $n\to\infty$, for the weak-$\star$ topology of $(C(\bar{Q}))'$. From the first item of the hypotheses on $(v_n)_{n\in\mathbb{N}}$, one deduces that $v_n \in L^1(Q,\nu_n)$, for all $n\in\mathbb{N}$, and that the sequence $(v_n\nu_n)_{n\in\mathbb{N}}$ is a bounded sequence of measures on Q.

Using to the second item of the hypotheses on $(v_n)_{n\in\mathbb{N}}$, the sequence $(v_n\nu_n)_{n\in\mathbb{N}}$ converges towards a measure $\bar{\nu}$ on $\bar{Q}$ (for the weak-$\star$ topology of $C(\bar{Q})$), that is (with $y = (x,t)$)

$$\int_Q \varphi(y)v_n(y) d\nu_n(y) \to \int_{\bar{Q}} \varphi(y) d\bar{\nu}(y),\ \forall\varphi\in C(\bar{Q}).$$

The existence of $v \in L^1(Q,\nu)$ such that $d\bar{\nu}(y) = v(y)d\nu(y)$ follows of the absolute continuity of $\bar{\nu}$ with respect to ν (that is $\bar{\nu} << \nu$, where ν is considered as a measure on $\bar{Q}$ with $\nu(\bar{Q}\setminus Q) = 0$) that we prove now.

In order to prove that $\bar{\nu} << \nu$, let A be a borelian set of $\bar{Q}$ such that $\nu(A) = 0$. One has to prove that $\bar{\nu}(A) = 0$. Let $K \subset A$ be a compact set. Let $\varepsilon > 0$. From the regularity of the measure ν, There is an open set $\mathcal{O}$ of $\bar{Q}$ (that is $\mathcal{O} = \bar{Q}\cap\omega$ where ω is an open set of $\mathbb{R}^d$) verifying $A\subset\mathcal{O}$ and $\nu(\mathcal{O}) \le \varepsilon$. There exists a continuous function φ from $\bar{Q}$ to $\mathbb{R}_+$ such that

$$\varphi(y) = 1,\ \forall y\in K,$$
$$\varphi(y) = 0,\ \forall y\in\mathcal{O}^c,$$
$$\varphi(y) \in [0,1],\ \forall y\in\bar{Q}.$$

Then, one gets

$$\begin{aligned}\bar{\nu}(K) \leq \int_{\bar{Q}} \varphi(y) d\bar{\nu}(y) &\leq \lim_{n\to\infty} \int_Q \varphi(y) v_n(y) d\nu_n(y)\\ &\leq \lim_{n\to\infty} \int_Q \varphi(y)\beta(y) d\nu_n(y)\\ &\leq \int_Q \varphi(y)\beta(y) d\nu(y) \leq M\nu(\mathcal{O}) \leq M\varepsilon,\end{aligned}$$

where $M = ||\varphi||_{C(\bar{Q})} ||\beta||_{L^\infty((0,T);C(\overline{\Omega}))}$. This gives, since $\varepsilon > 0$ is arbitrary, $\bar{\nu}(K) = 0$. Then, $\bar{\nu}(A) = 0$ follows from the regularity of the measure $\bar{\nu}$ (and the fact that K is any compact subset of A). This concludes the proof of $\bar{\nu} << \nu$ and gives the existence of $v \in L^1(Q,\nu)$ such that $\bar{\nu} = v\nu$, that is

$$\int_{\bar{Q}} \varphi(y) d\bar{\nu}(y) = \int_0^T \int_\Omega \varphi(x,t) v(x,t) d\mu(x) dt,\ \forall \varphi \in C(\bar{Q}).$$

It remains to prove that $\alpha(\cdot,t) \leq v(\cdot,t) \leq \beta(\cdot,t)$ μ-a.e., for a.e. $t \in (0,T)$.

Since $\alpha(\cdot,t) \leq v_n(\cdot,t) \leq \beta(\cdot,t)$ μ_n-a.e., for a.e. $t \in (0,T)$ and for all $n \in \mathbb{N}$, one has, for all $\varphi \in C(\bar{Q})$, φ nonnegative,

$$\int_0^T \int_\Omega \alpha(x,t)\varphi(x,t) d\mu_n(x) dt \leq \int_0^T \int_\Omega v_n(x,t)\varphi(x,t) d\mu_n(x) dt \leq \int_0^T \int_\Omega \beta(x,t)\varphi(x,t) d\mu_n(x) dt,$$

which yields, upon passing to the limit as $n \to \infty$ and using the dominated convergence theorem for the integration in time in the first and third integrals,

$$\int_0^T \int_\Omega \alpha(x,t)\varphi(x,t) d\mu(x) dt \leq \int_0^T \int_\Omega v(x,t)\varphi(x,t) d\mu(x) dt \leq \int_0^T \int_\Omega \beta(x,t)\varphi(x,t) d\mu(x) dt,$$

It follows that $\alpha \leq v \leq \beta$ ν-a.e., which is equivalent to $\alpha(\cdot,t) \leq v(\cdot,t) \leq \beta(\cdot,t)$ μ-a.e. for a.e. $t \in (0,T)$. This concludes the proof of Lemma 5.1. ■

Let us gives now the sketch of the proof of Theorem 2.2 and Theorem 2.3.

Sketch of the proof of Theorem 2.2

As for the proof of the Proposition 2.1, we build, for μ given in $L^2(\Omega)$, a regular solution through the Schauder fixed point Theorem with the application $S(c) = c^\star$ defined as follow :

For any given c, there exists a unique solution $(p, \mathbf{v})$ of

$$\begin{aligned}&\mathbf{v}(x,t) + \frac{\mathbf{K}(x)}{\nu(c(x,t))}\left(\nabla p(x,t) - \rho(c(x,t))\mathbf{g}\right) = 0\\ &\mathrm{div}\mathbf{v}(x,t) - a(x,t)\mu(x) + b(x,t)\mu(x) = 0\\ &\mathbf{v}(x,t)\cdot\mathbf{n} = 0,\ \int_\Omega p(x,t)dx = 0 \text{ for a.e. } t \in (0,T).\end{aligned}$$

Then, there exists a unique $c^\star$ solution of

$$\begin{aligned}&c_t^\star(x,t) + \mathrm{div}(\mathbf{v}\, c^\star)(x,t) - \mathrm{div}\big((\lambda(c) + \mathbf{D}(\mathbf{v}))\nabla c^\star\big)(x,t)\\ &+c^\star(x,t)a(x,t)\mu(x) - \bar{c}(x,t)b(x,t)\mu(x) = 0,\\ \\ &\left(\lambda(c(x,t)) + \mathbf{D}(\mathbf{v}(x,t))\right)\nabla c^\star(x,t)\cdot\mathbf{n} = 0\\ &c(x,0) = c_0.\end{aligned}$$

Then, the general case considered in Theorem 2.2 is obtained by passing to the limit on smooth approximations of the measure μ. There are no additional difficulties to the proof of Theorem 2.1. ■

Sketch of proof Theorem 2.3

- Parabolic regularisation

In order of avoid the degeneracy of the function h we introduce a modified problem where h is replaced in (2.38) by $h_\varepsilon = h + \varepsilon\ , \varepsilon > 0$. This kind of regularization reduces the problem of immiscible flow to the model problem (2.1)-(2.2), for which there is at least one (regular) solution for μ given in $L^2(\Omega)$.

• Uniform estimates in the regular case

By taking $H(s) = \int_0^s h(\sigma)d\sigma$ we prove without any difficulty that there exist two continuous functions $\alpha_1(t)$ and $\alpha_2(t)$, independent of ε such that the solution s_ε of the regularized problem verifies

$$\int_0^t ||\nabla H(s_\varepsilon(\cdot,\tau))||^2_{L^2(\Omega)} d\tau \le \alpha_1(t), \tag{5.17}$$

$$\int_0^t ||\partial_t s_\varepsilon(\tau)||^2_{(H^1(\Omega))'} d\tau \le \alpha_2(t).$$

Then assuming that H^{-1} is an Hölder continuous function with modulus ρ, we deduce from the previous estimate (see e.g. [17]) that s_ε is bounded in $L^{2/\rho}\left((0,T);W^{\theta\rho,2/\rho}(\Omega)\right)$ for any $\theta, 0 < \theta < 1$.
These estimates are sufficient to get a solution s of (2.38)-(2.42) provided that the equation (2.38) is written on the following form

$$\begin{aligned} &s_t(x,t) + \mathrm{div}\,(\mathbf{v}_T f(s))(x,t) - \mathrm{div}(\nabla H(s))(x,t)) - \mathrm{div}(\mathbf{k}(s))(x,t) \\ &+ s(x,t)a(x,t)\mu(x) - b(x,t)\mu(x) = 0. \end{aligned}$$

Moreover as s is uniformly bounded in $L^\infty((0,T)\times\Omega)$, the estimate (5.17) remains valid. The only difference is the estimate on the time derivative which becomes

$$\int_0^t ||\partial_t s_\varepsilon(\cdot,\tau)||^2_{(W^{1,q}(\Omega))'} d\tau \le \alpha_3(t), \quad \forall q > d.$$

This ends the proof of Theorem 2.3. ■

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