Convergence of a finite volume scheme for a nonlinear hyperbolic equation

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Abstract
This paper is concerned with the study of the following nonlinear hyperbolic equation: $u_t + div(vf(u)) = 0$ in $IR^2 \times [0,T]$, with initial data $u(.,0) = u_0(.)$ in IR^2. where $u_0 \in L^\infty(IR^2)$, v is a divergence-free bounded function of class C^1 from $IR^2 \times [0,T]$ to IR^2, and f is a function of class C^1 from IR to IR. A finite volume scheme for the discretization of this problem using a triangular mesh (in the space variable) is given. Passing to the limit when the mesh size and the time step go to 0, the existence and the uniqueness of an entropy weak solution of the above nonlinear hyperbolic equation with initial data is proven, together with the convergence (in $L^p_{loc}(IR^2 \times]0,T[)$, for any finite p) of the approximate solution towards the entropy weak solution. In fact, we first show the existence of a "process" solution (which generalizes the concept of entropy weak solution, and is obtained by passing to the limit of solutions of the numerical scheme). The uniqueness of this entropy process solution is then proved; it is also proven that the entropy process solution is in fact an entropy weak solution, hence the existence and the uniqueness of the entropy weak solution, and the convergence of the numerical scheme.

Keywords: Nonlinear hyperbolic equation, process solution, existence and uniqueness, convergence of finite volume schemes;
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1 Introduction.

Recently, several mathematical results of convergence of finite volume scheme, using, for two space variables, triangular meshes or quadrangular meshes, were obtained (generalizations to three space variables are possible). These include results for nonlinear hyperbolic equations (see, for instance, [3], [19], [2]), for elliptic equations (see, for instance, [11], [10], [1]), and also for systems coupling elliptic and hyperbolic equations ([7] and [14]). A general description of finite volume schemes may be founded in [15] or [8].
In this paper, we study, as an example, the following nonlinear hyperbolic equation with initial data:

$$u_t(x,t) + div(vf(u(x,t))) = 0, \quad x \in IR^2, \quad t \in [0,T] \tag{1}$$
$$u(x,0) = u_0(x), \quad x \in IR^2. \tag{2}$$

where $T > 0$, u_t denotes the partial derivative of u with respect to the time variable t, div denotes the divergence operator with respect to the space variable $x = (x_1, x_2)$, and under the following assumptions on the data T, v, f, u_0:

$$v = (v_1, v_2) \in C^1(IR^2 \times [0,T], IR^2), \tag{3}$$

$$\sup_{(x,t)\in IR^2\times[0,T]} |v(x,t)| = V < +\infty, \tag{4}$$

$$divv(x,t) = \sum_{i=1}^{2} \partial_{x_i} v_i(x,t) = 0, \text{ for all } (x,t) \in IR^2 \times [0,T], \tag{5}$$

$$u_0 \in L^\infty(IR^2), \tag{6}$$

$$f = f_1 + f_2,\ f_1, f_2 \in C^1(IR, IR),\ f_1'(s) \geq 0,\ f_2'(s) \leq 0,\ \forall s \in IR. \tag{7}$$

Remark 1 *Note that Assumption (4) is crucial in the sequel: it ensures the property of "propagation in finite time" which is needed for the uniqueness of the solution, and for the stability (under a "CFL" condition) of the (explicit in time) numerical scheme. Assumption (5), on the other hand, is only considered for the sake of simplicity. The results of existence and uniqueness of the entropy weak solution and convergence of the numerical scheme presented below are extendable to the case $divv$ not equal to 0 and to the case of three space dimensions. Note also that any function $f \in C^1(IR, IR)$ can be written $f = f_1 + f_2$ as in (7).*

For this example, a finite volume scheme using a triangular mesh (in the space variable) will be given. Passing to the limit when the mesh size and the time step go to 0, the existence and the uniqueness of an entropy weak solution of the above nonlinear hyperbolic equation with initial data will be proven, together with the convergence (in $L^p_{loc}(IR^2\times]0,T[)$, for any finite p) of the approximate solution towards the entropy weak solution.
First, recall that a function $u \in L^\infty(IR^2\times]0,T[)$ is said to be an entropy weak solution to Problem (1)-(2) if it satisfies:

$$\begin{array}{l} \displaystyle\int_{IR^N}\int_0^T \Big(\eta(u(x,t))\varphi_t(x,t) + \Phi(u(x,t))v(x,t).grad\varphi(x,t)\Big)dtdx \ + \\ \displaystyle\int_{IR^N} \eta(u_0(x))\varphi(x,0)dx \geq 0, \\ \forall\ \ \varphi \in C_c^1(IR^2 \times [0,T[, IR_+), \end{array} \tag{8}$$

for any convex function $\eta \in C^1(IR, IR)$, and $\Phi \in C^1(IR, IR)$ such that $\Phi' = f'\eta'$. ($C_c^1(E,F)$ denotes the set of functions C^1 from E to F, with compact support in E.)
The existence and uniqueness results are already known for the entropy weak solution of problem (1)-(2) under different assumptions than assumptions (3)-(7) (see e.g. [16], [20]). In particular, these results were obtained with a nonlinearity F (in our case $F = vf$) of class C^3. The methods which were used in [16] require a regularization of the function u_0, in order to take advantadge of compactness properties in a space smaller than $L^\infty(IR^2\times]0,T[)$ (namely the "BV space" of functions having locally bounded variation in the sense of Tonelli-Cesaro; indeed bounded sets of $L^\infty \cap BV$ are relatively compact in L^1_{loc}).
The existence of solutions to problems similar to (1)-(2) was already proven by passing to the limit on solutions of an appropriate numerical scheme. This is done in the work

of [17] in the case of a single space variable, which was generalized to several space dimensions [5]. The work of [5] uses a finite difference scheme on a uniform rectangular grid, and requires that the initial condition u_0 (an thus, the solution to Problem (1)-(2)) have locally bounded variation in the sense of Tonelli-Cesaro. Here we only assume $u_0 \in L^\infty(IR^2 \times]0,T[)$ and we may also work with more general meshes. We may use, for instance, a triangular mesh (for the sake of simplicity we limit this paper to the case of two space dimensions). For each of these reasons, the BV framework may not be used. To prove existence of solution to (1)-(2) by passing to the limit with finite volume scheme on triangular meshes, the lack of compactness forces us to work with the weak star convergence in L^∞ of a family of approximate solutions. Then, passing to the limit with the solutions given by a finite volume scheme gives the existence of a so-called "process solution" (which is defined below) to Problem (1)-(2).
Uniqueness results for "generalized" solutions (namely measure valued solutions) to (1)-(2) have recently been proven (see [6], [18], [13]). The proofs of these results rely on one hand on the concept of measure valued solutions and on the other hand on the existence of an entropy weak solution. The direct proof of the uniqueness of a measure valued solution (i.e. without assuming any existence result of entropy weak solutions) leads to a difficult problem involving the application of the theorem of continuity in means. This difficulty is overcome by the introduction of this new concept of solution (namely entropy process solution), which generalizes the concept of measure valued-solution. The proof of the uniqueness of the entropy process solution strongly uses the properties related to the weak star convergence in the space $L^\infty(IR^2 \times]0,T[)$.
This uniqueness result allows us to conclude to the convergence of the approximate solution given by the numerical sheme ((10)-(11)) towards the entropy weak solution to (1)-(2). Other convergence results and error estimates may be found in [2], [3], [19].
In this paper, we first decribe, in section 2, a finite volume scheme using a triangular mesh for the discretization of (1)-(2). In section 3 some estimates on the approximate solution given by the numerical scheme are given. Using these estimates, we prove, in section 4, passing to the limit when the mesh size and the time step go to 0, the convergence of convenient subsequences of sequences of approximate solutions towards an entropy process solution , and the existence of an entropy process solution to(1)-(2), that is a function $\mu \in L^\infty(IR^2 \times]0,T[\times]0,1[, IR)$ which satisfies:

$$\begin{aligned}
&\int_{IR^2}\int_0^T\int_0^1 \Big(\eta(\mu(x,t,\alpha))\varphi_t(x,t) + \Phi(\mu(x,t,\alpha))v(x,t).grad\varphi(x,t)\Big)d\alpha dt dx \;\; + \\
&\int_{IR^2}\eta(u_0(x))\varphi(x,0)dx \geq 0, \\
&\forall \;\; \varphi \in C_c^1(IR^2 \times [0,T[, IR_+),
\end{aligned} \tag{9}$$

for any convex function $\eta \in C^1(IR, IR)$, and $\Phi \in C^1(IR, IR)$ a function such that $\Phi' = f'\eta'$.

Remark 2 *From an entropy weak solution $u(x,t)$ to problem (1)-(2), one may easily construct an entropy process solution to problem (1)-(2) by setting $\mu(x,t,\alpha) = u(x,t)$ for any $\alpha \in [0,1]$. Reciprocally, if μ is an entropy process solution to problem (1)-(2) such that there exists $u \in L^\infty(IR^2 \times]0,T[)$ such that $\mu(x,t,\alpha) = u(x,t)$, for a.e. $(x,t,\alpha) \in IR^2 \times]0,T[\times]0,1[$, then u is an entropy weak solution to problem (1)-(2).*

The name "entropy process solution" was derived from the notion of bounded measurable process, that is a measurable mapping from a probability space into a space of bounded measurable functions. Here, the probability space consists in the interval $]0,1[$, with the borelian σ-algebra and the Lebesgue measure, and the set of bounded measurable functions is the bounded subset of $L^\infty(IR^2\times]0,T[)$ defined by $\{\mu(.,.,\alpha), \alpha \in]0,1[; \|\mu(.,.,\alpha)\|_\infty \leq C\}$, where $C>0$ is independent of α.
In section 5 the uniqueness of the entropy process solution to problem (1)-(2) is proven. We then obtain existence and uniqueness of the entropy weak solution and also (section 6) L^p_{loc} convergence for any finite p of the approximate solution towards the entropy weak solution.
In this paper, the proofs are only sketched. More complete proofs may be found, for instance, in [9] and [2].

2 The numerical scheme.

Let $\mathcal{T}$ be a triangular mesh and k be a constant time step (the generalization to variable time steps is straightforward), let $N \in IN$ be such that $(N-1)k < T \leq Nk$. Let us discretise equation (1)-(2) using the explicit Euler scheme for the time discretisation and a finite volume scheme with upstream weighting for the spatial discretisation. The discrete unknowns are the values u^n_K, $n \in \{0,\ldots,N\}$, $K \in \mathcal{T}$, given by the numerical scheme (10)-(11) (described below)Ê; the value u^n_K is expected to be an approximation of the mean value of the solution to (1)-(2) in the triangle $K \in \mathcal{T}$ at time $t_n = nk$. In order to describe the scheme, some notations are required. Let $K \in \mathcal{T}$, we denote by $S(K)$ the area of the triangle K, by $\mathcal{T}_K$ the three triangles of $\mathcal{T}$ having a common edge with K, a_{K,K_v} the common edge to K and K_v (when $K_v \in \mathcal{T}_K$), and n_K the normal to the boundary of K, outward to K. Let $\mathcal{A}$ denote the set of edges of the meshÊ; for $a \in \mathcal{A}$, $l(a)$ is the length of the edge a, n_a a unit normal vector to a, K^1_a and K^2_a the two triangles of $\mathcal{T}$ having a as edge. We may then define the upstream finite volume scheme by :

$$\left\{\begin{array}{l} \frac{u^{n+1}_K - u^n_K}{k} S(K) + \sum_{K_v \in \mathcal{T}_K} (\int_{a_{K,K_v}} (v.n_K)^+ d\sigma)(f_1(u^n_K) + f_2(u^n_{K_v})) \\ -(\int_{a_{K,K_v}} (v.n_K)^- d\sigma)(f_2(u^n_K) + f_1(u^n_{K_v})) = 0, \\ \qquad\qquad \forall K \in \mathcal{T}, \forall n \in \{0,\ldots,N-1\}, \end{array}\right. \tag{10}$$

$$u^0_K = \frac{1}{S(K)} \int_K u_0(x)dx, \qquad \forall K \in \mathcal{T}. \tag{11}$$

In (10) the usual 1-dimensional measure on the boundary of K is denoted by $d\sigma$.
Let the approximate solution at time t_n be defined by : $u^n(x) = u^n_K$ if $x \in K$, so that, for a given mesh $\mathcal{T}$ and a time step k, the approximate solution is defined everywhere by :

$$u_{\mathcal{T},k}(x,t) = u^n(x), \quad for \quad t \in [nk,(n+1)k[, \quad x \in IR^2. \tag{12}$$

3 Estimates on the approximate solution

In this section we give some estimates on the approximate solution defined in the preceding section. To this purpose, we shall need some nondegenerescence assumptions on the mesh (very much like those used within the framework of finite element methods) and a classical stability assumption on the time step. More precisely, let α, β be two fixed positive real numbers, we consider meshes $\mathcal{T}$ such that there exists $h \in I\!R_+^*$, (the "space step") satisfying :

$$\begin{cases} \alpha h^2 & \leq & S(K) & \leq \beta h^2, & \forall \ K \in \mathcal{T}, \\ \\ \alpha h & \leq & l(a) & \leq \beta h, & \forall \ a \in \mathcal{A}. \end{cases} \tag{13}$$

We assume the following stability condition on the time step :

$$k \leq \frac{\alpha h}{2\beta \ \ V \ \ 2M}(1-\zeta), \tag{14}$$

where $\zeta > 0$ is a given number, $M = \sup(M_1, M_2)$, $M_1 = \sup(f_1'(s), s \in [A,B])$, $M_2 = \sup(-f_2'(s), s \in [A,B])$, with A, B such that $A \leq u_0 \leq B$ a.e. in $I\!R^2$.
Note that (14) is a linear condition between h and k. (It may be somehow weakened by writing it with a "local" form.)
In this section, we prove two stability results stated in Proposition 1 and Proposition 2.

Proposition 1 *Let v, f and u_0 satisfy (3)-(7), and consider the space and time discretisations $\mathcal{T}$ and k satisfying (13) and the stability condition (14). Let u_K^n and $u_{\mathcal{T},k}$ be defined by (10)-(11) and (12), thenÊ:*

$$\min_{K \in \mathcal{T}} u_K^n \leq \min_{K \in \mathcal{T}} u_K^{n+1} \leq \max_{K \in \mathcal{T}} u_K^{n+1} \leq \max_{K \in \mathcal{T}} u_K^n, \quad \forall \ \ n \in \{0, \ldots, N-1\}, \tag{15}$$

and therefore :

$$\|u_{\mathcal{T},k}\|_\infty \leq \|u_0\|_\infty. \tag{16}$$

We denote by $\|.\|_p$ the norm on $L^p(I\!R^2)$ or $L^p(I\!R^2 \times I\!R_+)$, for $1 \leq p \leq \infty$.
The result of Proposition 1 still holds with $\zeta = 0$ in condition (14). The proof of Proposition 1 is classical. It uses the fact that $divv = 0$ and the upstream discretisation of $vf(u)$ in (10).

Remark 3 *Under the assumptions of Proposition 1, we can also prove the following L^1 estimate (useful if $u_0 \in L^1(I\!R^2)$) :*

$$\sum_{K \in \mathcal{T}} S(K)|u_K^{n+1}| \leq \sum_{K \in \mathcal{T}} S(K)|u_K^n| \ \ \forall \ \ n \in \{0, \ldots, N-1\}, \tag{17}$$

and therefore :

$$\|u_{\mathcal{T},k}(.,t)\|_1 \leq \|u_0\|_1 \ \ \forall \ \ t \in I\!R_+. \tag{18}$$

Actually, these estimates will not be needed in the sequel.

We now prove the following estimate :

Proposition 2 *Let v, f and u_0 satisfy (3)-(7), and consider the space and time discretisations $\mathcal{T}$ and k satisfying (13) and the stability condition (14). Let u_K^n and $u_{\mathcal{T},k}$ be defined by (10)-(11) and (12). Let $r \geq 0$. Then there exists $C_r > 0$, depending only on f, u_0, ζ and r such that :*

$$\sum_{n=0}^{N} k \sum_{a \in \mathcal{A}_r} (\int_a |v.n_a| d\sigma)((f_1(u^n_{K_a^1}) - f_1(u^n_{K_a^2}))^2 + (f_2(u^n_{K_a^1}) - f_2(u^n_{K_a^2}))^2) \leq C_r, \tag{19}$$

where $\mathcal{A}_r = \{a \in \mathcal{A}$, such that $a \subset B(0,r)\}$, $B(0,r) = \{x \in I\!R^2, \|x\| \leq r\}$.

Remark 4 *Note that, in order to obtain the estimate of Proposition 2, the fact that $\zeta > 0$ is crucial. In fact (19) is a consequence of the numerical diffusion of the numerical scheme. This estimate is not an a priori estimate, since it is satisfied by the approximate solution (namely $u_{\mathcal{T},k}$) but not by a solution of (1)-(2).*

In order to prove this proposition, let $\varphi : I\!R^2 \mapsto I\!R$ be a function of class C^1, with compact support, $0 \leq \varphi \leq 1$, and $\varphi = 1$ on K if $K \cap B(0,r) \neq \emptyset$. We denote by φ_K the mean value of φ over K. Define $F_1(\xi) = \displaystyle\int_0^\xi s f_1'(s)ds, \ \ \forall \ \ \xi \in I\!R$, $F_2(\xi) = \displaystyle\int_0^\xi s f_2'(s)ds, \ \ \forall \ \ \xi \in I\!R$. Multiplying (10) by $u_K^n \varphi_K$ and using $divv = 0$ we obtain :

$$\left\{ \begin{array}{l} (-\frac{(u_K^{n+1}-u_K^n)^2}{2k} + \frac{(u_K^{n+1})^2}{2k} - \frac{(u_K^n)^2}{2k})S(K)\varphi_K \\ + \sum_{K_v \in \mathcal{T}_K} (\int_{a_{K,K_v}} (v.n_K)^+)\varphi_K(F_2(u_{K_v}^n) - F_2(u_K^n) + \int_{u_K^n}^{u_{K_v}^n} (u_K^n - s)f_2'(s)ds) \\ - \sum_{K_v \in \mathcal{T}_K} (\int_{a_{K,K_v}} (v.n_K)^-)\varphi_K(F_1(u_{K_v}^n) - F_1(u_K^n) + \int_{u_K^n}^{u_{K_v}^n} (u_K^n - s)f_1'(s)ds) \ \ = 0. \end{array} \right. \tag{20}$$

Using the fact that $\int_a^b f_1'(s)(b-s)ds \geq \frac{1}{2M_1}(f_1(b) - f_1(a))^2$, if $a, b \in [A, B]$ (and the analoguous property for f_2), we deduce Proposition 2.

4 Convergence towards an entropy process solution

Proposition 3 *Assume hypotheses (3)-(7). Let $(\mathcal{T}_i, k_i)_{i\in I\!N}$ be a sequence of meshes and time steps satisfying (13) and the stability condition (14), then there exists a subsequence, still denoted by $(\mathcal{T}_i, k_i)_{i\in I\!N}$, and a function $\mu \in L^\infty(I\!R^2\times]0,T[\times]0,1[)$ such that :*

1. *The approximate solution defined by (10)-(11) and (12), that is $u_{\mathcal{T}_i,k_i}$, converges towards μ in the sense :*

$$\lim_{i\to\infty} \int_{I\!R^2\times]0,T[} g(u_{\mathcal{T}_i,k_i}(x,t))\varphi(x,t)dxdt = \int_{I\!R^2\times]0,T[\times]0,1[} g(\mu(x,t,\alpha))\varphi(x,t)dxdtd\alpha, \tag{21}$$

   for any function $\varphi \in L^1(I\!R^2\times]0,T[)$.

2. *μ is an entropy process solution of (1)-(2).*

The first part of the preceding proposition (namely the convergence towards μ) is a consequence of Proposition 1, that is the so-called "L^∞-stability" of the scheme, and the following Proposition :

Proposition 4 *Let $(u_n)_{n\in I\!N}$ be a bounded sequence of $L^\infty(I\!R^2\times]0,T[)$. There exists a subsequence of $(u_n)_{n\in I\!N}$ (which will still be denoted by $(u_n)_{n\in I\!N}$) and there exists $\mu \in L^\infty(I\!R^2\times]0,T[\times]0,1[)$ such that for any function $g \in C(I\!R, I\!R)$:*

$$\lim_{n\to\infty} \int_{I\!R^2\times]0,T[} g(u_n(x,t))\varphi(x,t)dxdt = \int_{I\!R^2\times]0,T[\times]0,1[} g(\mu(x,t,\alpha))\varphi(x,t)dxdtd\alpha, \tag{22}$$

for any function $\varphi \in L^1(I\!R^2\times]0,T[)$.

In order to prove the second part of Proposition 3 we multiply (10) (with $k = k_i$, and $\mathcal{T} = \mathcal{T}_i$) by $\eta'(u_K^n)\varphi(x,t_n)$, with a given convex function $\eta \in C^1(IR, IR)$ and a given $\varphi \in C^1(IR^2 \times [0,T[, IR_+)$ and we pass to the limit as $i \to \infty$ to obtain (9). For this proof we make a crucial use of the estimate (19).

Remark 5 *Note that a consequence of Proposition 3 is the existence of an entropy process solution to problem (1)-(2).*

5 Uniqueness of the entropy process solution

Theorem 1 *Under Assumptions (3)-(7), there exists a unique entropy process solution μ of problem (1)-(2), as defined by relation (9). Moreover, there exists a function $u \in L^\infty(IR^2\times]0,T[)$ such that $u(x,t) = \mu(x,t,\alpha)$, for a.e. $(x,t,\alpha) \in IR^2\times]0,T[\times]0,1[$. The function u is thus the unique entropy weak solution to Problem (1)-(2).*

Then an immediate consequence of this theorem is :

Theorem 2 *Under Assumptions (3)-(7), there exists a unique entropy weak solution to Problem (1)-(2).*

The existence of an entropy process solution is given by Proposition 3. It remains to show the uniqueness (then the values of the entropy process solution $\mu(x,t,\alpha)$ do not depend on α, so that finally the entropy process solution is therefore an entropy weak solution to problem (1)-(2), and this gives the existence and uniqueness of the entropy weak solution).
The following proposition give a characterization of an entropy process solution (which is an adaptation of the well known Kruzkov entropies to process solutions):

Proposition 5 *A function $\mu \in L^\infty(IR^2\times]0,T[\times]0,1[, IR)$ is an entropy process solution of problem (1) if and only if for any $k \in IR$, one has :*

$$\begin{array}{l}\displaystyle\int_{IR^N}\int_0^T\int_0^1 \Big(|\mu(x,t,\alpha)-k|\varphi_t(x,t) + F(\mu(x,t,\alpha),k)v(x,t).grad\varphi(x,t)\Big)d\alpha dt dx \;\; + \\ \displaystyle\int_{IR^N}|u_0(x)-k|\varphi(x,0)dx \geq 0 \;\;\forall\;\; \varphi \in C_c^1(IR^2\times[0,T[,IR_+), \forall k,l \in IR\end{array} \tag{23}$$

where $F(k,l) = sign(l-k)(f(l)-f(k))$.

The characterization (23) can be obtained from (9), by using regularizations of the function $|.-k|$. Reciprocally, (9) may be obtained from (23) by approximating any convex function $\eta \in C^1(IR, IR)$ by functions of the form: $\eta_n(.) = \sum_{i=1}^{n}\alpha_i^{(n)}|.-k_i^{(n)}|$, with $\alpha_i^{(n)} \geq 0$. This characterization is essential for the proof of the uniqueness of the entropy process solution of problem (1)-(2). To prove it we follow the method which was introduced in [2]. Note, however, that in [2], the existence of an entropy weak solution was assumed; hence stronger assumptions were needed in order to use previous results of existence of an entropy weak solution, such as [16].

The proof can be decomposed into four steps which we sketch here and which are detailled in [9]. Denote by μ and ν two entropy process solutions to Problem (1)-(2), and r_μ, r_ν their respective $L^\infty(I\!R^2\times]0,T[\times]0,1[)$ norms.
In step 1, the initial condition is proven to be satisfied in the following sense:

$$\lim_{\tau\to 0}\frac{1}{\tau}\int_0^\tau\int_{B_{N,a}}\int_0^1|\mu(x,t,\alpha)-u_0(x)|d\alpha dx dt=0,\ \forall a>0, \tag{24}$$

where $B_{p,r} = \{x\in I\!R^p,\ |x|\le r\}$ for any $r>0$ and any $p\in I\!N$.
The same property is clearly also verified if μ is replaced with ν.
In step 2, it is shown that:

$$\begin{aligned}&\int_0^T\int_{I\!R^N}\int_0^1\int_0^1\Big(|\mu(x,t,\alpha)-\nu(x,t,\beta)|\varphi_t(x,t)+\\ &F(\mu(x,t,\alpha),\nu(x,t,\beta))v(x,t).grad\varphi(x,t)\ \Big)d\alpha d\beta dx dt\ge 0\\ &\forall\ \ \varphi\in C_c^1(I\!R^2\times]0,T[,I\!R_+).\end{aligned} \tag{25}$$

Then, for a given $a>0$, define $A(t)$ for $0<t<\inf(T,\dfrac{a}{\omega})$ by:

$$A(t)=\int_{B_{N,a-\omega t}}\int_0^1\int_0^1|\mu(x,t,\alpha)-\nu(x,t,\beta)|d\alpha d\beta\ \ dx. \tag{26}$$

where $\omega=VM_f$, avec $M_f=\sup\limits_{s\in[-b,b]}|f'(s)|$ and $b=\sup(r_\nu,r_\mu)$,
It is shown in step 3 that A is a.e. non-increasing, i.e.:

$$A(t_1)\le A(t_2)\text{ for a.e. } t_1,t_2\in[0,\inf(T,\frac{a}{\omega})],t_1\ge t_2. \tag{27}$$

In step 4, it is deduced from (27) and (24), that $\displaystyle\int_0^1\int_0^1|\mu(x,t,\alpha)-\nu(x,t,\beta)|d\alpha d\beta=0$, for a.e. $(x,t)\in I\!R^2\times]0,T[$, and we prove that one may define $u(x,t)$, such that $\mu(x,t,\alpha)=\nu(x,t,\beta)=u(x,t)$ for a.e. $(x,t)\in I\!R^2\times]0,T[$ and $(\alpha,\beta)\in]0,1[\times]0,1[$.

6 Convergence towards the entropy weak solution

Theorem 3 *Assume hypotheses (3)-(7). Let u be the entropy weak solution to (1)-(2). Let $(k,\mathcal{T})$ satisfy (13) and the stability condition (14) (α, β, and ζ are fixed). Define $u_{\mathcal{T},k}$ by (10)-(11) and (12), then $u_{\mathcal{T},k}\to u$, in $L^p_{loc}(I\!R^2\times I\!R_+)$ for any $p\in[1,\infty[$, as $h\to 0$.*

We prove this theorem by contradiction. Assume that there exists p_0, such that $1\le p_0<+\infty$, $\varepsilon>0$, K a compact subset of $I\!R^2\times I\!R_+$, and a sequence $((\mathcal{T}_i,k_i))_{i\in I\!N}$ such that, for any $i\in I\!N$, $(\mathcal{T}_i,k_i)$ satisfies assumptions (13) and (14) (with h_i associated to $\mathcal{T}_i$ in (14)), $h_i\to 0$, as $i\to\infty$, and

$$\int_K|u_{\mathcal{T}_i,k_i}-\overline{u}|^{p_0}dx dt\ge\varepsilon,\ \forall i\in I\!N. \tag{28}$$

Using Proposition 3, there exists a subsequence of the sequence $((\mathcal{T}_i,k_i))_{i\in I\!N}$, still denoted by $((\mathcal{T}_i,k_i))_{i\in I\!N}$, and a function $\mu\in L^\infty(I\!R^2\times]0,T[\times]0,1[)$ such that :

1. The approximate solution defined by (10)-(11) and (12), that is $u_{\mathcal{T}_i,k_i}$, converges towards μ in the following sense :

$$\lim_{i\to\infty} \int_{I\!R^2\times]0,T[} g(u_{\mathcal{T}_i,k_i}(x,t))\varphi(x,t)dxdt = \int_{I\!R^2\times]0,T[\times]0,1[} g(\mu(x,t,\alpha))\varphi(x,t)dxdtd\alpha, \tag{29}$$

for any function $\varphi \in L^1(I\!R^2\times]0,T[)$.

2. μ is an entropy process solution of (1)-(2).

By Theorem 1 one has $\mu(.,.,\alpha) = u$, for a.e. $\alpha \in [0,1]$ (and u is the entropy weak solution to (1)-(2)). Taking first $g(s) = s^2$ in (29) and then $g(s) = s$ and φu instead of φ in (29) one obtains :

$$\int_{I\!R^2\times]0,T[} (u_{\mathcal{T}_i,k_i}(x,t)) - u(x,t))^2\varphi(x,t)dxdt \to 0, \tag{30}$$

for any function $\varphi \in L^1(I\!R^2\times]0,T[)$. From (30), and thanks to the L^∞-bound on $(u_{\mathcal{T}_i,k_i})_{i\in I\!N}$, one deduces the convergence of $(u_{\mathcal{T}_i,k_i})_{i\in I\!N}$ towards u in $L^p_{loc}(I\!R^2 \times I\!R+)$ for any $p \in [1,\infty[$, which is in contradiction with (28).
This completes the proof of our convergence theorem.

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