Elliptic equations with measure data

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Abstract
In this paper, we describe some recent results for (linear and nonlinear) elliptic equations on a bounded open set of $I\!R^N$ with boundary Dirichlet condition and measure data. Existence and Uniqueness results are given.

Keywords: Nonlinear elliptic equation, existence and uniqueness, non variational framework.

1 Introduction.

Let Ω be a bounded open set of $I\!R^N$ $(N \geq 1)$. This paper is concerned by the study of an (eventually nonlinear) elliptic equation on Ω with boundary Dirichlet condition and measure data. Elliptic equations with measure data, or with L^1 data (which is similar to measure data for existence results but seems simpler for uniqueness results) appear in some engineering problems. It may appears naturally or for technical reasons (for proving, for instance, existence of solutions for complex systems). Examples can be found, for instance, in [9] (taking into account wells in porous media fluid flows), [10] (coupling electric conduction and thermal conduction with "joule effect"), [8] (electromagnetic induction heating), [13] ($k - \varepsilon$ model of turbulence).
From the mathematical point of view, the principal difficulty of this kind of problem is that it is not possible (for $N > 1$) to work with the classical variational framework for elliptic equations (even for linear equations in divergence form). Existence results are obtained by a regularization method (for linear equations, it is also possible to use duality methods, see [17]). Unfortunetly, Following [16], nonuniquenes results can be showed. Uniqueness results are obtained by considering, in the case of L^1 data (for instance), solutions satisfying additional so called "entropy conditions".
All the results presented here are detailled in some recent papers. see, for instance, [3], [4], [1], [15], [11], [5], [7].
Existence and uniqueness results in the case of a linear elliptic operator in divergence form are presented in section 2 and 3. In section 4 is given the generalization to a nonlinear elliptic operator, and in section 5 are presented some results for operators which are not in divergence form.
This paper is restricted to elliptic equations. Let us mention, however, that many similar results can be obtained for parabolic equations (see sections 4 and [3]).

2 Linear equations in divergence form, existence

Let Ω be a bounded open set of $I\!R^N$ ($N \geq 1$) and $(a_{i,j})_{i,j\in\{1,\ldots,N\}}$ satisfy :

$$a_{i,j} \in L^\infty(\Omega),\ \forall (i,j) \in \{1,\ldots,N\}, \tag{1}$$

$$\exists \alpha > 0;\ \sum_{i,j=1}^{N} a_{i,j}(x)\xi_i\xi_j \geq \alpha|\xi|^2,\ \forall \xi = (\xi_1,\ldots,\xi_N)^t \in I\!R^N, \text{ for a.e. } x \in \Omega, \tag{2}$$

where $|\xi|$ denotes the euclidian norm of ξ in $I\!R^N$.
We consider the following elliptic problem :

$$\left\{ \begin{array}{rcl} -\sum_{i,j=1}^{N} D_i(a_{i,j}(x)D_j u(x)) & = & f(x),\ x \in \Omega, \\ u(x) & = & 0,\ x \in \partial\Omega, \end{array} \right. \tag{3}$$

where $\partial\Omega$ denotes the boundary of Ω, D_i denotes the partial derivatives with respect to the variable x_i with $x = (x_1,\ldots,x_N)^t$, and f is a given datum.
If $f \in H^{-1}(\Omega)$ (the dual space of the sobolev space $H_0^1(\Omega)$), it is well known that (3) has a unique solution in the following sense (this is the so called "variational framework") :

$$\left\{ \begin{array}{rcl} & u \in H_0^1(\Omega), & \\ \sum_{i,j=1}^{N} \int_\Omega a_{i,j}(x)D_j u(x)D_i\varphi(x)dx & = & < f,\varphi >_{H^{-1}(\Omega),H_0^1(\Omega)},\ \forall \varphi \in H_0^1(\Omega). \end{array} \right. \tag{4}$$

Remark 1

1. *Recall that, if $f \in L^1_{loc}(\Omega)$, we identify f with the distribution defined by f, that is :*

   $$< f,\varphi >_{\mathcal{D}',\mathcal{D}} = \int_\Omega f(x)\varphi(x)dx,\ \forall \varphi \in C_c^\infty(\Omega),$$

   where $\mathcal{D} = C_c^\infty(\Omega) = \{\varphi \in C^\infty(\Omega, I\!R)$, with compact support in $\Omega\}$ (endowed with its usual topology), and $\mathcal{D}'$ is its dual space.

2. *If $f \in \mathcal{D}'$, then $f \in H^{-1}(\Omega)$ if and only if there exists $C \in I\!R$ such that :*

   $$< f,\varphi >_{\mathcal{D}',\mathcal{D}} \leq C\|\varphi\|_{H_0^1(\Omega)},\ \forall \varphi \in C_c^\infty(\Omega). \tag{5}$$

   In fact, if $f \in \mathcal{D}'$ satisfies (5), then (by the density of $C_c^\infty(\Omega)$ in $H_0^1(\Omega)$) the distribution f define a unique element of $H^{-1}(\Omega)$ which is also identify with f.

3. *Note that u is solution to (4) if and only if u is solution to (4) where $\varphi \in H_0^1(\Omega)$ is repaced by $\varphi \in C_c^\infty(\Omega)$.*

We are interested here by Problem (3) when $f \in L^p(\Omega)$, for some $p \in [1,\infty]$, or when f is a signed (bounded) measure on the Borel σ-algebra of Ω.

Remark 2 *If f is a signed measure on the Borel σ-algebra of Ω, then f define an element of $C_b(\Omega)'$ (that is the dual space of the space of bounded continuous functions from Ω to $I\!R$ endowed with the norm of $L^\infty(\Omega)$) and define an element of $C(\overline{\Omega})'$ (that is the dual space of the space of continuous functions from $\overline{\Omega}$ to $I\!R$ endowed with the norm of $L^\infty(\Omega)$). In the sequel we consider f as an element of $C(\overline{\Omega})'$, which is denoted by $\mathcal{M}(\Omega)$. This gives $< f, \varphi >_{\mathcal{M}(\Omega),C(\overline{\Omega})} = \int_\Omega \varphi(x) df(x)$, for any $\varphi \in C(\overline{\Omega})$.*
Note also that, with the usual identifications between functions and linear forms induced by them, one has $L^p(\Omega) \subset \mathcal{M}(\Omega)$, for any $p \in [1, \infty]$.

To study Problem (3) when $f \in L^p(\Omega)$ or $\mathcal{M}(\Omega)$, we have to distinguish different cases for N :

1. $N = 1$. One has $H_0^1(\Omega)$ continuously imbedded in $C(\overline{\Omega})$. Then, if $f \in \mathcal{M}(\Omega)$, the restriction of f to $H_0^1(\Omega)$, still denoted by f, is in $H^{-1}(\Omega)$. Therefore, for any $f \in \mathcal{M}(\Omega)$, Problem (3) has a unique solution in the sense given by (4).

2. $N = 2$. One has $H_0^1(\Omega)$ continuously imbedded in $L^p(\Omega)$ for any $p \in [1, \infty)$. Then, by duality, $L^p(\Omega)$ is continuously imbedded in $H^{-1}(\Omega)$, for any $p \in (1, \infty]$. Therefore, for any $f \in L^p(\Omega)$, $p \in (1, \infty]$, Problem (3) has a unique solution in the sense given by (4). But, if $f \in L^1(\Omega)$ or $\mathcal{M}(\Omega)$, one cannot use the formulation (4) of (3).

3. $N > 2$. One has $H_0^1(\Omega)$ continuously imbedded in $L^{2^\star}(\Omega)$, with $2^\star = \frac{2N}{N-2}$, then, by duality, $L^{\overline{m}}(\Omega)$ is continuously imbedded in $H^{-1}(\Omega)$, with $\overline{m} = \frac{2N}{N+2}$. Therefore, for any $f \in L^p(\Omega)$, $p \in [\overline{m}, \infty]$, Problem (3) has a unique solution in the sense given by (4). But, if $f \in L^p(\Omega)$, $p \in [1, \overline{m})$ or $\mathcal{M}(\Omega)$, one cannot use the formulation (4) of (3).

Then, the cases of interest for us are : $N = 2$, $f \in L^1(\Omega)$ or $\mathcal{M}(\Omega)$, and $N > 2$, $f \in L^p(\Omega)$, $1 \leq p < \overline{m}$, or $\mathcal{M}(\Omega)$. The first difficulty is to give sense to (3) (equation and boundary condition). The first possibility is, for $f \in \mathcal{M}(\Omega)$ (and $N \geq 2$) :

$$\left\{ \begin{array}{rcl} & u \in W_0^{1,1}(\Omega), & \\ \sum_{i,j=1}^N \int_\Omega a_{i,j}(x) D_j u(x) D_i \varphi(x) dx & = & < f, \varphi >_{\mathcal{M}(\Omega),C(\overline{\Omega})}, \forall \varphi \in C_c^\infty(\Omega). \end{array} \right. \tag{6}$$

Then, the natural questions are existence, uniqueness, regularity of the solution to (6). For the question of existence of solution, two methods were developped, the first one is due to Stampacchia (1965, see [17]), it is a duality method which gives existence result by using regularity results on the "adjoint" equation, the second one is a method using approximations of the function f (see, for instance, [3]), it works also for nonlinear equations (see section 4). Indeed, let $f \in \mathcal{M}(\Omega)$, then there esists $(f_n)n \in I\!N \in C_c^\infty(\Omega)$ such that $f_n \to f$ for the weak-$\star$ topology of $\mathcal{M}(\Omega)$, and $\|f_n\|_{L^1(\Omega)} \leq \|f\|_{\mathcal{M}(\Omega)}$, for all $n \in I\!N$. Calling u_n the solution to (4) with f_n instead of f, one proves the following crucial estimate, with $C = \frac{\|f\|_{\mathcal{M}(\Omega)}}{\alpha}$:

$$\int_{\{x \in \Omega;\, k \leq |u_n(x)| \leq k+1\}} |grad u_n(x)|^2 dx \leq C, \forall n \in I\!N, \forall k \in Z\!\!Z. \tag{7}$$

A consequence of Estimate (7) is given in the following lemma :

Lemma 1 *let Ω be a bounded open set of $I\!R^N$, $N > 1$, and $(u_n)_{n \in I\!N} \in H_0^1(\Omega)$ a sequence satisfying (7) for some $C \in I\!R$, then $(u_n)_{n \in I\!N}$ is bounded in $W_0^{1,p}(\Omega)$ for all $p \in [1, \frac{N}{N-1})$.*

Using this lemma, one can pass to the limit in the equation satisfied by u_n and one obtains the following theorem :

Theorem 1 *let Ω be a bounded open set of $I\!R^N$, $N>1$, $(a_{i,j})_{i,j\in\{1,\ldots,N\}}$ satisfy (1)-(2), and $f\in\mathcal{M}(\Omega)$. Then there exists u solution to (3) in the following sense :*

$$\left\{\begin{array}{rcl} & u\in\cap_{p\in[1,\frac{N}{N-1})}W_0^{1,p}(\Omega), & \\ \sum_{i,j=1}^N\int_\Omega a_{i,j}(x)D_ju(x)D_i\varphi(x)dx & = & <f,\varphi>_{\mathcal{M}(\Omega),C(\overline{\Omega})},\ \forall\varphi\in C_c^\infty(\Omega). \end{array}\right. \quad (8)$$

Remark 3

1. *The formulation (8) is stronger than the formulation (6).*
2. *A function u is solution to (8) if and only if it is solution to (8) where $\varphi\in C_c^\infty(\Omega)$ is replaced by $\varphi\in\cup_{r>N}W_0^{1,r}(\Omega)$.*
3. *The functional space (namely $\cap_{p\in[1,\frac{N}{N-1})}W_0^{1,p}(\Omega)$) obtained in Theorem 1 is optimal in the sense that there exists $f\in\mathcal{M}(\Omega)$ (for instance $f=\delta_a$ with $a\in\Omega$, Assuming that Ω is not the empty set!) such that (8) has no solution with $u\in W_0^{1,\frac{N}{N-1}}(\Omega)$.*
4. *It is possible to precise the regularity of u, solution to (8) obtained by the approximation technique described before (see [4]). If $N>2$ and $f\in L^p(\Omega)$, $1<p<\overline{m}$, then $u\in W_0^{1,p^\star}(\Omega)$, with $p^\star=\frac{pN}{N-p}$. If $N\geq 2$ and $f\ln(1+|f|)\in L^1(\Omega)$, then $u\in W_0^{1,\frac{N}{N-1}}(\Omega)$.*

The duality method of Stampacchia in order to obtain an existence (and a uniquenes) result for (3) is different. let Ω be a bounded open set of $I\!R^N$, $N>1$, $(a_{i,j})_{i,j\in\{1,\ldots,N\}}$ satisfy (1)-(2). First, Stampacchia proves a regularity result, namely, if $f\in W^{-1,p}(\Omega)$, with some $p>N$ (note that $W^{-1,p}(\Omega)\subset H^{-1}(\Omega)$), then, the solution, u, to (4) (with $a_{i,j}$ replaced by $a_{j,i}$) is in $C(\overline{\Omega})$. More precisely the application $T_p: f\mapsto u$ is in $\mathcal{L}(W^{-1,p}(\Omega),C(\overline{\Omega}))$ (with the usual norms). Then the adjoint operator, $T_p^\star$, is in $\mathcal{L}(\mathcal{M}(\Omega),W_0^{1,p'}(\Omega))$ (with $p'=\frac{p}{p-1}<\frac{N}{N-1}$). Let $f\in\mathcal{M}(\Omega)$, noting that $T_p^\star(f)=T_q^\star(f)$ if $p,q>N$, Stampacchia shows that $u=T_p^\star(f)$ is solution to (3) in the following sense :

$$\left\{\begin{array}{rcl} & u\in\cap_{p\in[1,\frac{N}{N-1})}W_0^{1,p}(\Omega), & \\ -\sum_{i,j=1}^N\int_\Omega u(x)D_j(a_{i,j}(x)D_i\varphi(x))dx & = & <f,\varphi>_{\mathcal{M}(\Omega),C(\overline{\Omega})}, \\ & \forall\varphi\in C(\overline{\Omega})\cap H_0^1(\Omega), \text{ s. t. } & D_j(a_{i,j}D_i\varphi)\in C(\overline{\Omega}),\forall i,j. \end{array}\right. \quad (9)$$

It is possible to prove that u solution to (9) implies u solution to (8) (see [15]), but the converse is false. Note however that the solution to (8) obtained by the approximation technique described before is solution to (9) (see [15]).

3 Linear equations in divergence form, uniqueness

In the case $N=2$ it is possible to prove the uniqueness of the solution to (8) :

Theorem 2 *let Ω be a bounded open set of $I\!R^N$, $N=2$, $(a_{i,j})_{i,j\in\{1,\ldots,N\}}$ satisfy (1)-(2), and $f\in\mathcal{M}(\Omega)$. Then there exists a unique u solution to (8).*

Theorem 2 is a consequence of a regularity result of Meyers (see [2]) : There exists $p > 2$ (depending on $(a_{i,j})_{i,j\in\{1,\ldots,N\}}$) such that, if $f \in W^{-1,p}(\Omega)$, then the solution, u, to (4) belongs to $W_0^{1,p}(\Omega)$. From this result (which is true for all $N \geq 1$) one deduces the uniqueness of the solution to (8) only for $N = 2$ (i.e. Theorem 2, see [10]). In fact, for $N = 2$, u is solution to (8) if and only if $u = T_p^\star(f)$, where $T_p^\star \in \mathcal{L}(W^{-1,p'}(\Omega), W_0^{1,p'}(\Omega))$ is the adjoint operator of an operator $T_p \in \mathcal{L}(W^{-1,p}(\Omega), W_0^{1,p}(\Omega))$, where $p > 2$ is given by Meyers's regularity result and $p' = \frac{p}{p-1}$. Note that, thanks to $N = 2$, one has $p' < \frac{N}{N-1}$, and then $f \in W^{-1,p'}(\Omega)$ (since $\mathcal{M}(\Omega) \subset W^{-1,q}(\Omega)$, for all $q < \frac{N}{N-1}$). For $N > 2$, f is not necessarily in $W^{-1,p'}(\Omega)$ if $\frac{N}{N-1} < p'(< 2)$.
If $N > 2$, the solution to (8) is not necessarily unique. A counter example is essentialy in [16] (for a complete description, see [15]). Indeed, one can prove the following result :

Theorem 3 *Let $N \geq 2$, $\Omega = B_1 = \{x \in I\!R^N; |x| < 1\}$ and $\varepsilon > 0$. Then there exists $(a_{i,j})_{i,j\in\{1,\ldots,N\}}$ satisfying (1)-(2) (with $\alpha = 1$), such that the following problem has at least two solutions (one of them is $u = 0$) :*

$$\left\{ \begin{array}{rcl} & u \in \cap_{p\in[1,\frac{2}{1+\varepsilon})} W_0^{1,p}(\Omega), & \\ \sum_{i,j=1}^N \int_\Omega a_{i,j}(x) D_j u(x) D_i \varphi(x) dx & = & 0, \ \forall \varphi \in C_c^\infty(\Omega). \end{array} \right. \tag{10}$$

Taking, in Theorem 3, $N > 2$ and ε such that $\frac{2}{1+\varepsilon} \geq \frac{N}{N-1}$ one deduces the following result :

Theorem 4 *Let $N > 2$, $\Omega = B_1 = \{x \in I\!R^N; |x| < 1\}$. Then there exists $(a_{i,j})_{i,j\in\{1,\ldots,N\}}$ satisfying (1)-(2) (with $\alpha = 1$), such that, for any $f \in \mathcal{M}(\Omega)$, (8) has at least two solutions.*

Remark 4 *In Theorems 3 and 4, the functions $(a_{i,j})_{i,j\in\{1,\ldots,N\}}$ are as follows : $a_{i,j}(x) = \delta_{i,j} + (\frac{1}{\varepsilon^2} - 1)\frac{x_i x_j}{x_1^2 + x_2^2}$, if $1 \leq i, j \leq 2$, and $a_{i,j} = \delta_{i,j}$, if i or j is not in $\{1, 2\}$.*

If $f \in \mathcal{M}(\Omega) \cap (L^1(\Omega) + H^{-1}(\Omega))$, It is possible to obtain existence and uniqueness of the solution to (3) by adding to the formulation (8) some so called "entropy conditions" (see [1] and [7], if $f \in L^1(\Omega)$ an other formulation to obtain uniqueness is given in [14]). Define $T_k \in C(I\!R, I\!R)$ by $T_k(s) = \max(-k, \min(s, k))$.

Theorem 5 *let Ω be a bounded open set of $I\!R^N$, $N > 1$, $(a_{i,j})_{i,j\in\{1,\ldots,N\}}$ satisfy (1)-(2), and $f \in \mathcal{M}(\Omega) \cap (L^1(\Omega) + H^{-1}(\Omega))$. Then there exists a unique u solution to (8) and (11) :*

$$\left\{ \begin{array}{rcl} & u \in L^1(\Omega),\ T_k(u) \in H_0^1(\Omega), & \forall k \in I\!R_+, \\ \sum_{i,j=1}^N \int_\Omega a_{i,j}(x) D_j u(x) D_i T_k(u - \varphi)(x) dx & \leq & < f, T_k(u - \varphi) >, \\ & \forall k \in I\!R_+, & \forall \varphi \in C_c^\infty(\Omega)). \end{array} \right. \tag{11}$$

In Theorem 5, one has $f \in (L^1(\Omega) + H^{-1}(\Omega))$, this gives $f = f_1 + f_2$, with $f_1 \in L^1(\Omega)$ and $f_2 \in H^{-1}(\Omega)$. Then, for $\psi \in L^\infty(\Omega) \cap H_0^1(\Omega)$, one sets $< f, \psi >= \int_\Omega f_1(x)\psi(x)dx + < f_2, \psi >_{H^{-1}(\Omega), H_0^1(\Omega)}$.
In order to prove Theorem 5, we first remark that the solution to (8), obtained by the approximation technique described in section 2, is also solution to (11). Then, we prove

that (11) has at most one solution. Indeed, we remark that one can replace, in (11), $C_c^\infty(\Omega)$ by $L^\infty(\Omega)\cap H_0^1(\Omega)$. Then one takes $\varphi = T_h(v)$ in (11) (where v is another solution to (11)), $\varphi = T_h(u)$ in (11) with v instead of u, and we let h and k go to ∞ to conclude $u = v$. This completes the proof of Theorem 5. Note that a consequence of this proof is that, if u is solution to (11), then, u is solution to (8).

Remark 5 *One can prove that, if u is solution to (11), then, u is solution to (11) where $\leq$ is replaced by $=$.*

4 Nonlinear equations in divergence form

The results of sections 2 and 3 can be generalized to nonlinear elliptic equations in divergence form. Let Ω be a bounded open set of $I\!R^N$, $N \geq 1$, and a satisfy the following classical hypotheses (for the so called "Leray-Lions operators").

$$\left\{\begin{array}{l} a\ :\Omega\times I\!R\times I\!R^N\mapsto I\!R^N \quad \text{is measurable w.r.t.} \quad x\in\Omega,\ \forall (s,\xi)\in I\!R\times I\!R^N,\\ \qquad\qquad \text{continuous w.r.t.} \quad (s,\xi)\in I\!R\times I\!R^N, \quad \text{for a.e. } x\in\Omega, \end{array}\right. \tag{12}$$

$$\exists p\in(1,\infty),\exists\alpha>0;\ a(.,s,\xi).\xi\geq\alpha|\xi|^p,\ a.e.,\ \forall(s,\xi)\in I\!R\times I\!R^N, \tag{13}$$

$$\exists K\in L^{\frac{p}{p-1}}(\Omega),\exists\beta>0;\ |a(.,s,\xi)|\leq\beta(K+|s|^{p-1}+|\xi|^{p-1}),\ a.e.,\ \forall(s,\xi)\in I\!R\times I\!R^N, \tag{14}$$

$$(a(.,s,\xi)-a(.,s,\eta).(\xi-\eta)>0,\ a.e.,\ \forall(s,\xi,\eta)\in I\!R\times I\!R^N\times I\!R^N,\ \xi\neq\eta, \tag{15}$$

where $\xi.\eta$ denotes the scalar product of $\xi(\in I\!R^N)$ with $\eta(\in I\!R^N)$.

Remark 6 *A classical example of function a satisfying (12)-(15) is $a(x,s,\xi)=\sigma(x,s)$ $|\xi|^{p-2}\ \xi$, with $p>1$, σ continuous w.r.t. s, measurable w.r.t. x, and $0<\alpha\leq\sigma\leq\beta$.*

We consider the following problem :

$$\left\{\begin{array}{rcl} -div(a(x,u(x),gradu(x)) &=& f(x),\ x\in\Omega,\\ u(x) &=& 0,\ x\in\partial\Omega, \end{array}\right. \tag{16}$$

If $f\in W^{-1,p'}(\Omega)$ (with $p'=\frac{p}{p-1}$), it is known (see [12]) that there exists u, solution to (16) in the following sense (this is the so called "variational framework") :

$$\left\{\begin{array}{rcl} & u\in W_0^{1,p}(\Omega), & \\ \int_\Omega a(x,u(x),gradu(x)).grad\varphi(x)dx &=& <f,\varphi>_{W^{-1,p'}(\Omega),W_0^{1,p}(\Omega)},\\ & \forall\varphi\in W_0^{1,p}(\Omega). & \end{array}\right. \tag{17}$$

Note that (17) has a unique solution if a does not depend on its second variable (denoted by s) or if a is lipschitz continuous w.r.t. its second variable and $1<p\leq 2$ (see [6] for this last result and a counter example if $p>2$).
As before we are interested by (16) if $f\in\mathcal{M}(\Omega)$ (or $f\in L^q(\Omega)$ for some $q\geq 1$). Since $\mathcal{M}(\Omega)\subset W^{-1,p'}(\Omega)$ if $p>N$, the only case of interest is $p\leq N$ (and therefore $N>1$). A first result of existence is given by the following theorem (see [3], [4]).

Theorem 6 *let Ω be a bounded open set of $I\!R^N$, $N > 1$, a satisfy (12)-(15), $2 - \frac{1}{N} < p \le N$, and $f \in \mathcal{M}(\Omega)$. Then there exists u solution to (16) in the following sense :*

$$\left\{ \begin{array}{rcl} & u \in \cap_{p\in[1,\frac{N}{N-1}(p-1))} W_0^{1,p}(\Omega), & \\ \int_\Omega a(x,u(x),gradu(x)).grad\varphi(x)dx & = & < f,\varphi >_{\mathcal{M}(\Omega),C(\overline{\Omega})}, \forall \varphi \in C_c^\infty(\Omega). \end{array} \right. \tag{18}$$

Remark 7

1. *Note that a function u is solution to (18) if and only if it is solution to (18) where $\varphi \in C_c^\infty(\Omega)$ is replaced by $\varphi \in \cup_{r>N} W_0^{1,r}(\Omega)$.*
2. *It is possible to give regularity results as in the linear case (see Remark 3) if $f \in L^q(\Omega)$ (see [4]).*

The proof of Theorem 6 is very similar to that of the linear case. There is a new difficulty, which is to prove the (strong relative) compactness of a sequence of approximate solutions, $(u_n)_{n\in I\!N}$, in $W_0^{1,q}(\Omega)$, for $1 \le q < \frac{N}{N-1}(p-1)$. This is achieved by proving the convergence in measure of $(gradu_n)_{n\in I\!N}$.
Note that, thanks to $2 - \frac{1}{N} < p$, one has $\frac{N}{N-1}(p-1) > 1$ (therefore, there exists some $q \in [1, \frac{N}{N-1}(p-1)))$. In the case $1 < p \le 2 - \frac{1}{N}$, one cannot expect solutions to (16) to be in $W^{1,1}(\Omega)$. Then, in order to make sense to the equation, one has to change the definition of $gradu$. In fact, if u is a (class of) measurable function(s) from Ω to $I\!R$ such that $T_k(u) \in W_0^{1,p}(\Omega)$, for all $k \in I\!R_+$, one *defines* $gradu$ by $gradu = gradT_k(u)$, on the set $\{x \in \Omega; |u(x)| \le k\}$ (and one proves that this definition is correct). Then, it is possible to obtain an existence theorem for (16) (see [1]).
In order to obtain a uniqueness theorem, we also have to add (as in the linear case) some "entropy conditions" and to limit ourselves to the case $f \in \mathcal{M}(\Omega)\cap(L^1(\Omega)+W^{-1,p'}(\Omega))$. In fact, using, as in the linear case, the technique of approximation, one proves the following existence result, more precise than Theorem 6 (see [1], [7]).

Theorem 7 *let Ω be a bounded open set of $I\!R^N$, $N > 1$, a satisfy (12)-(15), $2 - \frac{1}{N} < p \le N$, and $f \in \mathcal{M}(\Omega) \cap (L^1(\Omega) + W^{-1,p'}(\Omega))$. Then there exists u solution to (18) and (19) :*

$$\left\{ \begin{array}{rcl} u \in L^1(\Omega),\, T_k(u) \in W_0^{1,p}(\Omega), & & \forall k \in I\!R_+, \\ \int_\Omega a(x,u(x),gradu(x)).gradT_k(u-\varphi)(x)dx & \le & < f, T_k(u-\varphi) >, \\ \forall k \in I\!R_+, & & \forall \varphi \in C_c^\infty(\Omega)). \end{array} \right. \tag{19}$$

Remark 8

1. *In Theorem 7, one has $f \in (L^1(\Omega) + W^{-1,p'}(\Omega))$, this gives $f = f_1 + f_2$, with $f_1 \in L^1(\Omega)$ and $f_2 \in W^{-1,p'}(\Omega)$. Then, for $\psi \in L^\infty(\Omega) \cap W_0^{1,p}(\Omega)$, one set $< f,\psi >= \int_\Omega f_1(x)\psi(x)dx + < f_2,\psi >_{W^{-1,p'}(\Omega),W_0^{1,p}(\Omega)}$.*
2. *One can prove that one can replace, in (19), $C_c^\infty(\Omega)$ by $L^\infty(\Omega) \cap W_0^{1,p}(\Omega)$.*

Using Remark 8, a uniqueness theorem is obtained, similarily to the linear case, if a does not depend of its second argument (denoted by s), see [1], [7].

Theorem 8 *let Ω be a bounded open set of $I\!R^N$, $N > 1$, a satisfy (12)-(15), $2 - \frac{1}{N} < p \leq N$, and $f \in \mathcal{M}(\Omega) \cap (L^1(\Omega) + W^{-1,p'}(\Omega))$. Assume that a does not depend on its second argument. Then there exists a unique solution, u, to (19).*

Note that a consequence of Theorems 8 and 7 is that, under the hypotheses of Theorem 8, if u is solution to (19), then, u is solution to (18).

Remark 9

1. *One can prove that, under the hypotheses of Theorem 8, if u is solution to (19), then, u is solution to (19) where $\leq$ is replaced by $=$.*
2. *The question of uniqueness of the solution to (19) if a depends on its second argument (with, for instance, $p \leq 2$, and a Lipschitz continuous w.r.t. its second argument) and under the hypotheses of Theorem 7 is an open question.*

Using methods similar to those developped here, it is possible to prove existence results for some parabolic equations as below (see [3]).

$$\begin{aligned} u_t(x,t) - div(a(x,t,u(x,t),gradu(x,t))) &= f(x,t),\ x \in \Omega,\ t \in [0,T), && (20)\\ u(x,t) &= 0,\ x \in \partial\Omega,\ t \in (0,T), && (21)\\ u(x,0) &= u_0(x),\ x \in \Omega, && (22) \end{aligned}$$

where u_t denotes $\frac{\partial u}{\partial t}$.
Under Hypotheses on a similar to (12)-(15), and if f is a signed measure on the Borel σ-algebra of $\Omega \times [0,T)$ and u_0 is a signed measure on the Borel σ-algebra of Ω, one obtains, roughly speaking, a solution, u, to (20)-(22), which is, for a convenient q, in $L^q((0,T), W_0^{1,p}(\Omega))$ (this makes sense for (21)), and which satisfy (20)-(22) in the following sense :

$$\left\{ \begin{array}{l} \int_{\Omega\times(0,T)}(-u(x,t)\varphi_t(x,t) + a(x,t,u(x,t),gradu(x,t)).grad\varphi(x,t))dxdt \\ \qquad - \int_\Omega \varphi(x,0)du_0(x) = \int_{\Omega\times[0,T)} \varphi(x,t)df(x,t),\ \forall \varphi \in C_c^1(\Omega\times[0,T)), \end{array} \right. \tag{23}$$

where $C_c^1(\Omega \times [0,T)) = \{\varphi \in C^1(\Omega \times [0,T), I\!R)$, with compact support in $\Omega \times [0,T)\}$.
Note that, if $f \in L^1(\Omega \times (0,T))$ and $u_0 \in L^1(\Omega)$, then, one can prove that $u \in C([0,T), L^1(\Omega))$ and the initial condition is satisfied in the sense $u(.,t) \to u_0$, in $L^1(\Omega)$, as $t \to 0$.

5 Equations with nondivergence form

In this section, we only give two examples of results in the case of an elliptic operator in nondivergence form.
Let Ω be a bounded open set of $I\!R^N$, $N \geq 2$, and $1 \leq p < \infty$. Consider the following problem :

$$\left\{ \begin{array}{rcl} -\Delta u(x) + |u(x)|^{p-1}u(x) &=& f(x),\ x \in \Omega, \\ u(x) &=& 0,\ x \in \partial\Omega. \end{array} \right. \tag{24}$$

For $f \in \mathcal{M}(\Omega)$, we call "solution to (24)" a function u such that :

$$\left\{ \begin{array}{rcl} & u \in W_0^{1,1}(\Omega) \cap L^p(\Omega), & \\ \int_\Omega gradu(x).grad\varphi(x)dx + \int_\Omega |u(x)|^{p-1}u(x)\varphi(x)d(x) & = & < f, \varphi >_{\mathcal{M}(\Omega), C(\overline{\Omega})}, \\ & \forall \varphi \in C_c^\infty(\Omega). & \end{array} \right. \tag{25}$$

One has the following results (see [11] and, for generalizations, [3]) :

Theorem 9 *Let Ω be a bounded open set of $I\!R^N$, $N \geq 2$, $1 \leq p < \infty$, and $f \in \mathcal{M}(\Omega)$.*

1. *if $1 \leq p < \frac{N}{N-2}$, then there exists a solution to (25),*
2. *if $\frac{N}{N-2} \leq p < \infty$ (and $N > 2$). Then, (25) has a solution if and only if $f \in (L^1(\Omega) + W^{-2,p}(\Omega))$. In particular, (25) has no solution if $f = \delta_a$, where $a \in \Omega$.*

One can also study nonlinearities depending on $gradu$, and not in divergence form. A classical example is :

$$\left\{ \begin{array}{rcl} -\Delta u(x) + u(x)|gradu(x)|^2 & = & f(x),\ x \in \Omega, \\ u(x) & = & 0,\ x \in \partial\Omega. \end{array} \right. \tag{26}$$

Here again, for $f \in \mathcal{M}(\Omega)$, we call "solution to (26)" a function u such that :

$$\left\{ \begin{array}{rcl} u \in H_0^1(\Omega); u|gradu|^2 \in L^1(\Omega) & & \\ \int_\Omega gradu(x).grad\varphi(x)dx + \int_\Omega u(x)|gradu(x)|^2\varphi(x)d(x) & = & \\ < f, \varphi >_{\mathcal{M}(\Omega), C(\overline{\Omega})}, & & \forall \varphi \in C_c^\infty(\Omega). \end{array} \right. \tag{27}$$

One has the following result :

Theorem 10 *Let Ω be a bounded open set of $I\!R^N$, $N \geq 1$, and $f \in \mathcal{M}(\Omega)$. Then, (27) has a solution if and only if $f \in (L^1(\Omega) + H^{-1}(\Omega))$. In particular, if $N > 1$, (27) has no solution if $f = \delta_a$, with $a \in \Omega$.*

Remark 10

1. *Recall that, if $N = 1$, one has $\mathcal{M}(\Omega) \subset H^{-1}(\Omega)$, therefore (27) has a solution for any $f \in \mathcal{M}(\Omega)$.*
2. *It is possible, with the concept of "capacity", to caracterize the sets $L^1(\Omega)+W^{-2,p}(\Omega)$ and $L^1(\Omega) + H^{-1}(\Omega)$ appearing in Theorems 9 and 10 (see [11] and [7]).*

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