Rough schemes for complex hyperbolic systems

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ABSTRACT This communication presents some attempts for the discretization of complex hyperbolic systems (with, possibly, several space dimensions). The complexity of the systems considered requires the use of rather "rough" schemes, in order to allow feasibility and a reasonable computational cost. These rough schemes, however, show a somewhat surprisingly good behaviour.

Key Words : hyperbolic system, finite volume schemes.

1. Introduction

The aim of this communication is to present some discretization techniques for "complex" hyperbolic systems. The unknown of the system is a function $u = (u_1, \ldots, u_m)$, from $\Omega \times [0,T]$ to $\mathbb{R}^m$, where Ω is an open set of $\mathbb{R}^N$ (with $N \geq 1$ and $m \geq 1$), and u is a solution of the following system:

$$\begin{array}{c} \frac{\partial u_i(x,t)}{\partial t} + \sum_{j=1}^{N} \frac{\partial F_{i,j}(x,t,u(x,t))}{\partial x_j} = g_i(x,t,u(x,t)), \\ x = (x_1, \ldots, x_N) \in \Omega,\ t \in (0,T),\ i = 1, \ldots, m. \end{array} \tag{1}$$

The functions $F_j = (F_{1,j}, \ldots, F_{m,j})$ $(j = 1, \ldots, N)$ and $g = (g_1, \ldots, g_m)$ are given functions from $\Omega \times [0,T] \times \mathbb{R}^m$ (indeed, generally, a part of $\mathbb{R}^m$, instead of $\mathbb{R}^m$) to $\mathbb{R}^m$. One sets $F = (F_1, \ldots, F_N)$. One assumes the usual hyperbolicity condition, that is, for any (unit) vector of $\mathbb{R}^N$, $\mathbf{n}$, the derivative of $F \cdot \mathbf{n}$ with respect to its third argument (which can be considered as an $m \times m$ matrix) has only real eigenvalues and is diagonalizable.

Note that in real applications, diffusion terms may also be present in the equations, we shall omit them here.

To complete System (1), an initial condition for $t = 0$ and adequate boundary conditions for $x \in \partial\Omega$ must be specified.

In many applications, the functions g and F are rather "complex", and it is difficult or impossible to use classical schemes such as the 1D Godunov or Roe schemes, or their standard extensions, for multidimensional problems, using 1D solvers on the interfaces of the mesh. This is the case of gas dynamics (Euler equations) with real gas, for which the state law (pressure as a function of density and internal energy) is tabulated or given by some complex analytical expressions. In the modelization of multiphase flows in pipe-lines, the function F is difficult to handle and highly depends on x and u, because, for instance, of changes of the geometry and slope of the pipe, of changes of the friction law or, more generally, of the varying nature of the flow. Most of the attempts given below were developed for this last situation.

Other interesting cases of "complexity" are the treatment of boundary conditions (mathematical literature is rather scarce on this subject, see [FH 96]), and the way to handle the case where the eigenvalues (of the derivative of $F \cdot \mathbf{n}$ with respect to its third argument) are of very different magnitude, see Section **4**. Another case of complexity is the treatment of nonconservative terms in the equations. One refers, for instance, to [BHLU 96] and references therein, for this important case.

For the discretization of (1) with initial and boundary conditions, finite volume schemes are used, for which the general setting is now briefly described.

Let $\mathcal{T}$ be the space mesh, and k be the time step, the time step is assumed to be constant, the generalization to a variable time step is easy. The interface, $[K, K_v]$, between any 2 elements K and K_v of $\mathcal{T}$ is assumed to be included in a hyperplane of $\mathbb{R}^N$.

The discrete unknowns are the u_K^n, $K \in \mathcal{T}$, $n \in \{0, \ldots, M+1\}$ (with $(M+1)k = T$).

For $K \in \mathcal{T}$, let $N(K)$ be the set of its neighbours, that is the set of elements, K_v, of $\mathcal{T}$ such that the $N-1$ Lebesgue measure of $[K, K_v]$ is positive. For $K_v \in N(K)$, let $\mathbf{n}_{K,K_v}$ be the unit normal vector to $[K, K_v]$ oriented from K to K_v. Let $t^{(n)} = nk$, for $n \in \{0, \ldots, M+1\}$.

A finite volume scheme writes:

$$\begin{gathered} m(K)\frac{u_K^{n+1} - u_K^n}{k} + \sum_{K_v \in N(K)} m_{N-1}([K, K_v]) F_{K,K_v}^n = m(K) g_K^n, \\ K \in \mathcal{T},\ n \in \{0, \ldots, M\}, \end{gathered} \tag{2}$$

where:

• $m(K)$ (resp. $m_{N-1}([K,K_v])$) denotes the N (resp. $N-1$) Lebesgue measure of K (resp. $[K,K_v]$),

• the quantity g_K^n, which depends on u_K^n (or u_K^{n+1} or on u_K^n and u_K^{n+1}), for $K \in \mathcal{T}$, is some "consistent" approximation of g on element K, between times $t^{(n)}$ and $t^{(n+1)}$. This approximation is, generally, not too difficult to handle...

• the quantity F_{K,K_v}^n, which depends on u_K^n (or u_K^{n+1} or on u_K^n and u_K^{n+1}), for $K \in \mathcal{T}$, is an approximation of $F \cdot \mathbf{n}_{K,K_v}$ on $[K,K_v]$ between times $t^{(n)}$ and $t^{(n+1)}$. In order to obtain a "good" scheme, this approximation has to be "consistent" (this is easy to understand), conservative (that is F_{K,K_v}^n=-$F_{K_v,K}^n$) and has to give some stability properties on the approximate solution given by the scheme (indeed, one also needs some consistency w.r.t. entropies, when there are entropies...). Except in the scalar case, it is not so easy to see what kind of stability properties one needs...Indeed, in the scalar case, that is $m=1$, taking $g=0$ and $\Omega = \mathbb{R}^N$ (for simplicity), it is essentially sufficient to have an L^∞ estimate (that is a bound on u_K^n independent of K, n, and of the time and space discretizations) and a "touch" of "BV estimate" (see, for instance, [CCH 96] for more precise assumptions). In the case $m>1$, it is not possible, in general, to give stability properties in order to have a convergent scheme, but one may require some stability properties such as the positivity of some quantities depending on the unknowns; in the case of flows, they are positivity of density, energy, pressure...; for the computation of $F(u)$ or for its hyperbolicity, the positivity of these quantities may be essential.

Generally, the computation of F_{K,K_v}^n is performed, for each interface, by a 1D solver for the equation $u_t + \mathrm{div}(F \cdot \mathbf{n}_{K,K_v}) = 0$, which gives consistency, conservativity (and, hopefully, stability) of the final scheme (that is (2)). Therefore, in the following, we shall mainly keep to the case $N=1$.

In Sections **2** and **3**, possible modifications of Godunov and Roe schemes (including "classical" improvements to avoid excessive artificial diffusion) are described to handle "complex" systems. Section **4** deals with the treatment of eigenvalues of different magnitude.

Because of the complexity of the models, the justification of the schemes presented here is rather numerical than mathematical. Several of these schemes are adaptations of already existing methods. Many variations have also been developed, not presented here. I apologize for numerous missing references in the bibliography.

2. Godunov, Roe and "VFRoe" schemes

Let us describe these schemes in the case $N=1$, $\Omega = \mathbb{R}$, $F(x,t,u) = F(u)$ and $g=0$ (but $m \geq 1$). Then System (1) becomes:

$$\frac{\partial u(x,t)}{\partial t} + \frac{\partial F(u(x,t))}{\partial x} = 0, x \in \mathbb{R},\, t \in (0,T). \tag{3}$$

To complete this system, an initial condition must be specified, the discretization of which is standard.

Let $\mathcal{T} = (M_i)_{i\in\mathbb{Z}}$, with $M_i = [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}]$ $(x_{i-\frac{1}{2}} < x_{i+\frac{1}{2}})$, $i \in \mathbb{Z}$. One sets $h_i = x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}$, $i \in \mathbb{Z}$. The discrete unknowns are u_i^n, $i \in \mathbb{Z}$, $n \in \{0, \ldots, M+1\}$.

The scheme (2) becomes:

$$h_i \frac{u_i^{n+1} - u_i^n}{k} + F^n_{i+\frac{1}{2}} - F^n_{i-\frac{1}{2}} = 0, i \in \mathbb{Z}, \, n \in \{0, \ldots, M\}. \tag{4}$$

Note that this scheme is conservative (in the sense defined above).

Let us consider only explicit schemes, so that $F^n_{i+\frac{1}{2}}$ is a function of u_j^n, $j \in \mathbb{Z}$.

The principle of the Godunov scheme ([GO 76]) is to take $F^n_{i+\frac{1}{2}} = F(w)$, where w, depending on u_i^n, u_{i+1}^n and F, is the solution, for $x = 0$ (and any $t > 0$) of the following (Riemann) problem:

$$\frac{\partial u(x,t)}{\partial t} + \frac{\partial F(u(x,t))}{\partial x} = 0, x \in \mathbb{R}, \, t \in \mathbb{R}_+, \tag{5}$$

$$\begin{aligned} u(x,0) &= u_i^n, \text{ if } x < 0, \\ u(x,0) &= u_{i+1}^n, \text{ if } x > 0. \end{aligned} \tag{6}$$

The time step is limited by the so-called "CFL condition", which yields $k \le Lh_i$, for all $i \in \mathbb{Z}$, where L is given by F and the initial condition.

The quantity u_i^{n+1}, given by Godunov's scheme, see [GO 76], is, for all $i \in \mathbb{Z}$, the mean value on M_i of the exact solution at time k of (3) with the initial condition (at time $t = 0$), u_0, defined, for all $x \in \mathbb{R}$, by $u_0(x) = u_i^n$, if $x_{i-\frac{1}{2}} < x < x_{i+\frac{1}{2}}$.

The Godunov scheme is a good scheme (one has consistency, stability...), sometimes too diffusive (especially if k is far from Lh_i defined above), but easy improvements are possible, such as the MUSCL technique, see below. Its principal drawback is its difficult implementation for many problems, indeed the computation of $F(w)$ can be impossible or too expensive. For instance, this computation may need a non trivial parametrization of the non linear waves. Note also that F is generally not given directly as a function of u (the components of u are called "conservative unknowns") but as a function of some "physical" unknowns (for instance, pressure, velocity, energy...), and the passage from u to these physical unknowns (or the converse) is often not so easy...it may be the consequence of expensive and implicit calculations, using, for instance, Newton's algorithm.

Due to this difficulty of implementation of the Godunov scheme, some "Godunov type" schemes were developed (see [HLVL 83]). The idea is to take, for u_i^{n+1}, the mean value on M_i of an *approximate* solution at time k of (3) with the initial condition (at time $t = 0$), u_0, defined by $u_0(x) = u_i^n$, if $x_{i-\frac{1}{2}} < x < x_{i+\frac{1}{2}}$.

In order for the scheme to be written under the form (4), which yields conservativity, this approximate solution must satisfy some consistency relation (another relation is needed for the consistency with entropies).

One of the best known of this family of schemes is the Roe scheme (see [RO 80]), where this approximate solution is computed by the resolution of the following linearized Riemann problems:

$$\frac{\partial u(x,t)}{\partial t} + A(u_i^n, u_{i+1}^n)\frac{\partial u(x,t)}{\partial x} = 0,\ x \in \mathbb{R},\ t \in \mathbb{R}_+, \tag{7}$$

$$\begin{aligned} u(x,0) &= u_i^n, \text{ if } x < 0, \\ u(x,0) &= u_{i+1}^n, \text{ if } x > 0, \end{aligned} \tag{8}$$

where $A(\cdot,\cdot)$ is an $m \times m$ matrix, continuously depending on its two arguments, with only real eigenvalues, diagonalizable and satisfying the so-called "Roe condition":

$$A(u,v)(u-v) = F(u) - F(v), \forall u, v \in \mathbb{R}^m. \tag{9}$$

Thanks to (9), the Roe scheme can be written as (4) with

$$\begin{aligned} F_{i+\frac{1}{2}}^n &= F(u_i^n) + A^-(u_i^n, u_{i+1}^n)(u_i^n - u_{i+1}^n) \\ (&= F(u_{i+1}^n) + A^+(u_i^n, u_{i+1}^n)(u_i^n - u_{i+1}^n)), \end{aligned} \tag{10}$$

where $A^{\pm}$ are the classical nonnegative and nonpositive parts of the matrix A: let A be a matrix with only real eigenvalues, $(\lambda_p)_{p=1,\ldots,m}$, and diagonalizable, and $(\varphi_p)_{p=1,\ldots,m}$ a basis of $\mathbb{R}^m$ associated to these eigenvalues. Then, the matrix A^+ is the matrix which has the same eigenvectors as A and has $(\max(\lambda_p, 0))_{p=1,\ldots,m}$ as corresponding eigenvalues. The matrix A^- is $(-A)^+$.

Roe's scheme is a good scheme, often less expensive than Godunov's scheme, with, more or less, the same limitation on the time step, the same diffusion effect and some lack of entropy consistency, which can be corrected. It has some properties of consistency and stability. Its main drawback is the difficulty of the computation of a matrix $A(u,v)$ satisfying (9). For instance, when it is possible to compute and diagonalize the derivative of F, $DF(u)$, one can take $A(u,v) = DF(u^\star)$, but the difficulty is to find $u^\star$ such that (9) holds. In some difficult cases, there is some attempt to do it by using a "limited expansion" w.r.t. a some small parameter.

The Godunov and Roe schemes can both be written under the form (4) with $F_{i+\frac{1}{2}}^n$ computed as a function of u_i^n and u_{i+1}^n; both schemes are consistent (in the sense of Section **1**, i.e. consistency of the "fluxes") since $F_{i+\frac{1}{2}}^n = F(u)$ if $u_i^n = u_{i+1}^n = u$. Going further along this line of thought yields (among others possibilities, see Section **3**) the "VFRoe" scheme which is (4), with $F_{i+\frac{1}{2}}^n = F(w)$, where w is the solution of the linearized Riemann problem (7) (8), with $A(u_i^n, u_{i+1}^n) = DF(w^\star)$, where $w^\star$ is some value between u_i^n and u_{i+1}^n (for instance, $w^\star = \frac{1}{2}(u_i^n + u_{i+1}^n)$). In this scheme, Condition (9) is not required.

Then, the VFRoe scheme appears to be a simplified version of the Godunov and Roe schemes. The study of the scalar case ($m = 1$) shows that, in order to have some stability, at least as much as in "Roe's scheme", the choice of $w^\star$ is essential. In practice, the choice $w^\star = \frac{1}{2}(u_i^n + u_{i+1}^n)$ is often adequate, at least for regular meshes.

Numerical tests show the good behaviour of the VFRoe scheme. Its two main flaws are a lack of entropy consistency (as in Roe's scheme) and a large diffusion effect (as in the Godunov and Roe schemes). The first drawback can be corrected, as for Roe's scheme, with a nonparametric entropy correction inspired from [HHL 76] (see [MFG 96]). The second drawback can be corrected with a classical MUSCL technique, which consists in replacing, in (8), u_i^n and u_{i+1}^n by $u_{i+\frac{1}{2},-}^n$ and $u_{i+\frac{1}{2},+}^n$, which depend on $\{u_j^n,\, j = i-1, i, i+1, i+2\}$ (see, for instance [GR 91], [LE 90]). For stability reasons, the computation of the gradient of the unknown (mesh by mesh) and of the "limiters" is performed on some "physical" quantities (such as density, pressure, velocity for Euler equations) instead of u. The extension of the MUSCL technique to the case $N > 1$ is more or less straightforward.

This MUSCL technique improves the space accuracy (in the truncation error) and the numerical results are significantly better. However, stability is sometimes lost. Indeed, considering the linear scalar equation, one remarks that the scheme is antidiffusive when the limiters are not active, this might lead to a loss of stability. The time step must then be reduced (it is reduced by a factor 10 in severe situations...). In order to allow larger time steps, the time accuracy should be improved by using, for instance, an order 2 Runge-Kutta scheme (in the severe situations suggested above, the time step is then multiplied by a factor 4). Surprisingly, this improvement of time accuracy is used to gain stability rather than precision...

Several numerical experiments (see [MFG 96]) were performed which prove the efficiency of the VFRoe scheme, such as the classical Sod tests ([SO 78]). The shock velocities are exact, there are no oscillations.... For these tests, the treatment of the boundary conditions is straightforward. Throughout these experiments, the use of a MUSCL technique yields a significant improvement, while the use of a higher order time scheme is not necessary. In one of the Sod tests, the entropy correction is needed.

A comparison between the VFRoe scheme and the Godunov scheme was performed by J. M. Hérard (personal communication) for the Euler equations on a Van Der Wals gas, for which a matrix satisfying (9) seems difficult to find. The numerical results are better with the VFRoe scheme, which is also much cheaper computationally.

3. More complexity, "GROEMLF" scheme

The flux F may also highly, and not continuously, depend on the space variable, x. In the space discretization, it is "natural" to set the discontinuities of F w.r.t. x on the boundaries of the mesh. The function F may change drastically from M_i to M_{i+1}. In this case, the implementation of the VFRoe scheme yields two additional difficulties:

(i) The matrix $A(u_i^n, u_{i+1}^n)$ in the linearized Riemann problem (7) (8) now depends on x: $A(u_i^n, u_{i+1}^n) = D_u F(x, w^\star)$, where $w^\star$ is some value between u_i^n and u_{i+1}^n and $D_u F$ denotes the derivative of F w.r.t. its "u" argument.

(ii) once the solution, w, of the linearized problem (7) (8), for $x = 0$ and any $t > 0$, is calculated, the choice $F_{i+\frac{1}{2}}^n = F(x, w)$ again depends on x.

The choice of $F_{i+\frac{1}{2}}^n$ (point (ii)) may be solved by remarking that, in Roe's scheme, $F_{i+\frac{1}{2}}^n$may be written (thanks to (9)) as:

$$F_{i+\frac{1}{2}}^n = \frac{1}{2}(F(u_i^n) + F(u_{i+1}^n)) + \frac{1}{2} A_{i+\frac{1}{2}}^n (u_i^n - u_{i+1}^n), \tag{11}$$

where $A_{i+\frac{1}{2}}^n = |A(u_i^n, u_{i+1}^n)|$, and $|A| = A^+ + A^-$.

Under this form, the second term of the RHS of (11) appears to be a stabilization term, which does not affect the consistency. Indeed, in the scalar case ($m = 1$), one has $A_{i+\frac{1}{2}}^n = \frac{|F(u_i^n) - F(u_{i+1}^n)|}{|u_i^n - u_{i+1}^n|}$, which easily yields the L^∞ stability of the scheme (but not the consistency w.r.t. the entropies). Moreover, the scheme will be stable and consistent w.r.t. the entropies, under a CFL condition, if $F_{i+\frac{1}{2}}^n$ is nondecreasing w.r.t. u_i^n and nonincreasing w.r.t. u_{i+1}^n, which holds if $A_{i+\frac{1}{2}}^n \geq \sup\{|F'(s)|,\, s \in [u_i^n, u_{i+1}^n] \text{ or } [u_{i+1}^n, u_i^n]\}$.

A slightly different version of the VFRoe scheme (closer to Roe's scheme), is obtained by taking (11), instead of $F_{i+\frac{1}{2}}^n = F(w)$, with, as for the VFRoe scheme, $A_{i+\frac{1}{2}}^n = |DF(w^\star)|$. It is also possible, in (11), to take other convex combinations of u_i^n and u_{i+1}^n, without modifying the consistency of the scheme.

When F depends on x, the discontinuities of F being on the boundaries of the control volumes, the generalization of (11) is obvious, except for the choice of $A_{i+\frac{1}{2}}^n$. The quantity $F(u_i^n)$ is replaced by $F(x_i, u_i^n)$, where x_i is the center of M_i.

Let us now turn to the choice of a convenient matrix $A_{i+\frac{1}{2}}^n$ for this modified VFRoe scheme, when F highly depends on x. A first possible choice is $A_{i+\frac{1}{2}}^n = \frac{1}{2}(|D_u F(x_i, u_i^n)| + |D_u F(x_{i+1}, u_{i+1}^n)|)$. The following slightly different choice for $A_{i+\frac{1}{2}}^n$ seems to give better numerical results (see [FH 96]). For $i \in \mathbb{Z}$, let $A_i = D_u F(x_i, u_i^n)$ (n is fixed). Let $(\lambda_p^{(i)})_{p=1,\ldots,m}$ be the eigenvalues of A_i (with $\lambda_{p-1}^{(i)} \leq \lambda_p^{(i)}$, for all p) and $(\varphi_p^{(i)})_{p=1,\ldots,m}$ a basis of $\mathbb{R}^m$ associated to these eigenvalues. Then, the matrix $A_{i+\frac{1}{2}}^-$ [resp. $A_{i+\frac{1}{2}}^+$] is the matrix which has the same eigenvectors as A_i [resp. A_{i+1}] and has $(\max(|\lambda_p^{(i)}|, |\lambda_p^{(i+1)}|))_{p=1,\ldots,m}$ as corresponding eigenvalues. The choice of $A_{i+\frac{1}{2}}^n$ is:

$$A_{i+\frac{1}{2}}^n = \frac{K}{2}(A_{i+\frac{1}{2}}^- + A_{i+\frac{1}{2}}^+), \tag{12}$$

where K is a parameter, the "normal" value of which is 1. Numerically, larger values of K, say $K = 2$ or $K = 3$, are sometimes needed, in severe situations, to obtain enough stability. Too large values of K yield too much artificial diffusion.

The new scheme is (4) with (11), where $F(u_i^n)$ stands for $F(x_i, u_{i+1}^n)$ if F depends on x, and (12). It has, more or less, the same properties as the Roe and VFRoe schemes but allows the simulation of more complex systems. It needs a MUSCL technique to reduce diffusion effects and order 2 Runge-Kutta for stability. It was implemented for the simulation of multiphase flows in pipe lines (see [FH 96]). The other difficulties encountered in this case are the treatment of the boundary conditions and the different magnitude of the eigenvalues, which is discussed in the next (and last) section.

4. Partial implicitation, "Turbo" scheme

Eigenvalues of different magnitude often occur in the modelling of flows, where "propagation" phenomena, corresponding to "large'" eigenvalues (large is "far from 0", positive or negative) and "convection" phenomena, corresponding to "small" eigenvalues coexist. Large and small eigenvalues may differ by a factor 10 or 100.

With the explicit schemes described in the preceding sections, the time step is limited by the CFL condition corresponding to the large eigenvalues. Roughly speaking, with the notations of Section **2**, this condition is $k \leq |\lambda|^{-1} h_i$, where λ is the largest eigenvalue. In some cases, this limitation can be unsatisfactory for two reasons. Firstly, the time step is too small and implies a prohibitive computational cost. Secondly, the discontinuities in the solutions, associated to the small eigenvalues, are not sharp, even with a MUSCL (and order 2 Runge-Kutta) method, because the time step is far from the CFL condition of the small eigenvalues. This is in fact a major problem when the discontinuities associated to the small eigenvalues need to be computed precisely. It is the case of interest here.

A first method to avoid the time step limitation is to take a "fully implicit" version of the schemes developed in the preceding sections, that is $F_{i+\frac{1}{2}}^n$ function of u_j^{n+1}, $j \in \mathbb{Z}$, instead of u_j^n, $j \in \mathbb{Z}$ (the terminology "fully implicit" is by opposition to "linearly implicit", see below and [FE 89]). However, in order to be competitive w.r.t. explicit schemes, the fully implicit scheme is used with large time steps. In practice, this prohibits the use of a MUSCL technique in the computation of the solution at time $t^{(n+1)}$ by, for instance, a Newton algorithm. This implicit scheme is therefore very diffusive and leads to smear discontinuities.

A second method consists in splitting the system into two systems, the first one is associated with the "small" eigenvalues, and the second one with the "large" eigenvalues (in the case of the Euler equations, this splitting may correspond to a "convection" system and a "propagation" system). At each time step, the first system is solved with an explicit scheme and the second one with an implicit scheme. Both use the same time step, which is limited by the CFL condition of the small eigenvalues. Using a MUSCL technique and an order 2 Runge-Kutta method for the first system yields sharp discontinuities associated to the small eigenvalues. This method is often satisfactory, but is difficult to handle in the case of severe boundary conditions, since the convenient boundary conditions for each system may be difficult to determine.

Another method, developed by E. Turkel (see [TU 87]), in connexion with Roe's scheme, uses a change of variables in order to reduce the ratio between large and small eigenvalues.

Let us now describe a partial linearly implicit method ("turbo" scheme) which was successfully tested for multiphase flows in pipe lines (see [FH 96], other cases are in [FE 89]). For the sake of simplicity, the method is described for the last scheme of Section **3**, without MUSCL and order 2 Runge-Kutta improvements. This scheme corresponds to (4) with (F can depend on x):

$$F^n_{i+\frac{1}{2}} = \frac{1}{2}(F(x_i, u^n_i) + F(x_{i+1}, u^n_{i+1})) + \frac{1}{2}A^n_{i+\frac{1}{2}}(u^n_i - u^n_{i+1}), \tag{13}$$

where:

$$A^n_{i+\frac{1}{2}} = \frac{K}{2}(A^-_{i+\frac{1}{2}} + A^+_{i+\frac{1}{2}}). \tag{14}$$

See Section **3** for the definition of A_i, $A^-_{i+\frac{1}{2}}$ and $A^+_{i+\frac{1}{2}}$.

Assume that $I \subset \{1, \dots, m\}$ is the set of index of large eigenvalues (and does not depend on i). The objective is to "implicit" only the large eigenvalues, let $\tilde{A}_i$, $\tilde{A}^-_{i+\frac{1}{2}}$ and $\tilde{A}^+_{i+\frac{1}{2}}$ be the matrix having the same eigenvectors as A_i, $A^-_{i+\frac{1}{2}}$ and $A^+_{i+\frac{1}{2}}$, with the same large eigenvalues (i.e. corresponding to $p \in I$) and 0 as small eigenvalues. Let $\tilde{A}^n_{i+\frac{1}{2}} = \frac{K}{2}(\tilde{A}^-_{i+\frac{1}{2}} + \tilde{A}^+_{i+\frac{1}{2}})$. Then, the partial linearly implicit scheme is obtained by replacing $F^n_{i+\frac{1}{2}}$ in (4) by $\tilde{F}^n_{i+\frac{1}{2}}$ defined by:

$$\begin{aligned}\tilde{F}^n_{i+\frac{1}{2}} = F^n_{i+\frac{1}{2}} &+ \tfrac{1}{2}(\tilde{A}_i(u^{n+1}_i - u^n_i) + \tilde{A}_{i+1}(u^{n+1}_{i+1} - u^n_{i+1})) \\ &+ \tfrac{1}{2}\tilde{A}^n_{i+\frac{1}{2}}(u^{n+1}_i - u^n_i + u^n_{i+1} - u^{n+1}_{i+1}).\end{aligned}$$

In order to obtain sharp discontinuities corresponding to the small eigenvalues, a MUSCL technique is used for the computation of $F^n_{i+\frac{1}{2}}$. Then, again for stability reasons, it is preferable to add an order 2 Runge-Kutta method for the time discretization. Although it is not so easy to implement, the order 2 Runge-Kutta method is needed to enable the use of "large" time step. The time step is, in severe situations, very close to that given by the usual CFL condition corresponding to the small eigenvalues, and can be considerably larger than that given by the large eigenvalues (see [FH 96] for many tests).

In many simulations of real situations, the treatment of the boundary conditions is not easy (in particular, when an eigenvalue changes of sign). We refer to [FH 96] for a detailed description of the discretization of the boundary conditions.

5. References

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