Error estimates for the approximate solutions of a nonlinear hyperbolic equation given by finite volume schemes

R. Eymard[1], T. Gallouët[2], M. Ghilani[3] and R. Herbin[4]

Abstract. This paper is devoted to the study of an error estimate of the finite volume approximation to the solution $u \in L^\infty(\mathbb{R}^N \times \mathbb{R})$ of the equation $u_t + \mathrm{div}(\mathbf{v} f(u)) = 0$, where $\mathbf{v}$ is a vector function depending on time and space. A "$h^{1/4}$" error estimate for an initial value in $BV(\mathbb{R}^N)$ is shown for a large variety of finite volume monotoneous flux schemes, with an explicit or implicit time discretization. For this purpose, the error estimate is given for the general setting of approximate entropy solutions, where the error is expressed in terms of measures in $\mathbb{R}^N$ and $\mathbb{R}^N \times \mathbb{R}$. The study of the implicit schemes involves the study of the existence and uniqueness of the approximate solution. The cases where an "$h^{1/2}$" error estimate can be achieved are also discussed.

1 Introduction and main result

1.1 Presentation of the problem

We consider here the following nonlinear hyperbolic equation in N space dimensions ($N \geq 1$), with initial condition:

$$u_t(x,t) + \mathrm{div}(\mathbf{v}(x,t) f(u(x,t))) = 0, \forall x \in \mathbb{R}^N,\ \forall t \in \mathbb{R}_+, \tag{1}$$

$$u(x,0) = u_0(x), \forall x \in \mathbb{R}^N, \tag{2}$$

where u_t denotes the time derivative of u ($t \in \mathbb{R}_+$), and div the divergence of u w.r.t. the space variable (which belongs to $\mathbb{R}^N$). One denotes by $|x|$ the euclidean norm of x in $\mathbb{R}^N$, and by $x \cdot y$ the usual scalar product of x and y in $\mathbb{R}^N$.
The following hypotheses are made on the data (see Remark 1.4 for some comments on these hypotheses):

$$\begin{array}{ll} (i) & u_0 \in L^\infty(\mathbb{R}^N),\ U_m, U_M \in \mathbb{R},\ U_m \leq u_0 \leq U_M \ a.e., \\ (ii) & \mathbf{v} \in C^1(\mathbb{R}^N \times \mathbb{R}_+, \mathbb{R}^N), \mathrm{div}\mathbf{v}(x,t) = 0, \forall (x,t) \in \mathbb{R}^N \times \mathbb{R}_+, \\ & \exists V < \infty \text{ s.t. } |\mathbf{v}(x,t)| \leq V,\ \forall (x,t) \in \mathbb{R}^N \times \mathbb{R}_+, \\ (iii) & f \in C^1(\mathbb{R}, \mathbb{R}). \end{array} \tag{3}$$

For any pair of real numbers a, b, we denote by $a \top b$ the maximum of a and b, and by $a \bot b$ the minimum of a and b.
Recall that problem (1)-(2) has a unique entropy solution [19]. Defining the Kruzkov entropy pairs, for all $\kappa \in \mathbb{R}$, as $(|\cdot - \kappa|, f(\cdot \top \kappa) - f(\cdot \bot \kappa))$, this solution is the unique solution to the following problem:

$$\begin{array}{l} u \in L^\infty(\mathbb{R}^N \times]0,\infty[), \\ \displaystyle\int_{\mathbb{R}^N \times \mathbb{R}_+} \Big[|u(x,t) - \kappa| \varphi_t(x,t) + \Big(f(u(x,t) \top \kappa) - f(u(x,t) \bot \kappa) \Big) \mathbf{v}(x,t) \cdot \nabla \varphi(x,t) \Big] dx dt + \\ \displaystyle\int_{\mathbb{R}^N} |u_0(x) - \kappa| \varphi(x,0) dx \geq 0,\ \forall \kappa \in \mathbb{R},\ \forall \varphi \in C_c^\infty(\mathbb{R}^N \times \mathbb{R}_+, \mathbb{R}_+), \end{array} \tag{4}$$

where $\nabla\varphi$ denotes the gradient of the function φ with respect to the space variable (which belongs to $\mathbb{R}^N$).

[1]Laboratoire Central des Ponts et Chaussées and Université Paris Nord, 75732 Paris, France
[2]E.N.S. Lyon, 69364 Lyon, France
[3]Université Moulay Ismail, Faculté des sciences, Meknès, Maroc and Université de Provence, CMI, 13453 Marseille, France
[4]Université de Provence, CMI, 13453 Marseille, France

1.2 Definition of the schemes

Let $\mathcal{T}$ be a mesh such that the common interface of any two elements (which are called control volumes in the following) of $\mathcal{T}$ is included in a hyperplane of $\mathbb{R}^N$. This last assumption is not necessary and is introduced to simplify the formulation. We assume that there exist $h > 0$, $\alpha > 0$ such that, for any $p \in \mathcal{T}$:

$$\begin{array}{lcl} \alpha h^N & \leq & m(p), \\ m(\partial p) & \leq & h^{N-1}, \\ \delta(p) & \leq & h, \end{array} \tag{5}$$

where $m(p)$ denotes the N-dimensional Lebesgue measure of p, $m(\partial p)$ denotes the $(N-1)$-dimensional Lebesgue measure of ∂p (∂p is the boundary of p) and $\delta(p)$ denotes the diameter of p.
Let $F \in C(\mathbb{R}^2, \mathbb{R})$ be such that:

$$\begin{array}{ll} (i) & F(a_1, a_2) \text{ is nondecreasing w.r.t. } a_1 \text{ and nonincreasing w.r.t. } a_2, \text{ for } a_1, a_2 \in [U_m, U_M], \\ (ii) & F(a_1, a_2) \text{ is Lipschitz continuous with respect to } a_1 \text{ and } a_2 \text{ in } [U_m, U_M], \\ (iii) & F(a, a) = f(a), \text{ for all } a \in [U_m, U_M]. \end{array} \tag{6}$$

Remark 1.1 F is the numerical flux defining the scheme. The first assumption on F will ensure some stability properties of the schemes defined below. In particular, in the case of the "explicit scheme" (see (10)), it yields the monotonicity of the scheme under a CFL condition (namely, condition (9) with $\xi = 0$). The third condition is essential since it ensures the consistency of the fluxes (cf. [11]). This framework includes the generalized 1D Godunov scheme obtained with a one-dimensional Godunov scheme for each edge (see e.g., for the explicit scheme, [6], [7], [24]):

$$F(a_1, a_2) = \left\{ \begin{array}{l} \sup\{f(a), a_2 \leq a \leq a_1\} \text{ if } a_2 \leq a_1 \\ \inf\{f(a), a_1 \leq a \leq a_2\} \text{ if } a_1 \leq a_2, \end{array} \right. \tag{7}$$

and many other schemes. It is possible to replace the assumptions (6) on F by some slightly more general assumptions, in particular to include the case of some "Lax-Friedrichs type" scheme (see Remark 1.2 below).

Let $k > 0$ be the time step. The discrete unknowns are u_p^n, $n \in \mathbb{N}^*$, $p \in \mathcal{T}$.
The set $\{u_p^0, p \in \mathcal{T}\}$, is given by the initial condition:

$$u_p^0 = \frac{1}{m(p)} \int_p u_0(x) dx, \forall\, p \in \mathcal{T}. \tag{8}$$

The equations satisfied by the discrete unknowns u_p^n, $n \in \mathbb{N}^*$, $p \in \mathcal{T}$ are obtained by discretising equation (1). Let us first describe the "explicit scheme" associated to the function F satisfying asumption (6). The time step k is chosen such that:

$$k \leq (1-\xi) \frac{\alpha h}{V(F_1 + F_2)}, \tag{9}$$

where $\xi \in (0,1)$ is a given real value and $F_1 > 0$, $F_2 > 0$ are the Lipschitz constants of F w.r.t. the first and second variables on the interval $[U_m, U_M]$ (recall that $U_m \leq u_0 \leq U_M$ a.e.).
Let us consider the following explicit numerical scheme:

$$m(p) \frac{u_p^{n+1} - u_p^n}{k} + \sum_{q \in \mathcal{N}(p)} (v_{p,q}^n\ F(u_p^n, u_q^n) - v_{q,p}^n\ F(u_q^n, u_p^n)) = 0, \forall p \in \mathcal{T}, \forall n \in \mathbb{N}, \tag{10}$$

where:

(i) $\mathcal{N}(p)$ denotes the set of neighbours of the control volume p; for $q \in \mathcal{N}(p)$, we denote by $\sigma_{p,q}$ the common interface between p and q, and by $\mathbf{n}_{p,q}$ the unit normal vector to $\sigma_{p,q}$ oriented from p to q.

(ii) $v^n_{p,q} = \frac{1}{k}\int_{nk}^{(n+1)k}\int_{\sigma_{p,q}}\Big[(\mathbf{v}(\gamma,t)\cdot\mathbf{n}_{p,q})\top 0\Big]d\gamma dt$ and

$$v^n_{q,p} = \frac{1}{k}\int_{nk}^{(n+1)k}\int_{\sigma_{q,p}}\Big[(\mathbf{v}(\gamma,t)\cdot\mathbf{n}_{q,p})\top 0\Big]d\gamma dt = -\frac{1}{k}\int_{nk}^{(n+1)k}\int_{\sigma_{p,q}}\Big[(\mathbf{v}(\gamma,t)\cdot\mathbf{n}_{p,q})\bot 0\Big]d\gamma dt.$$

The approximate solution, denoted by $u_{\mathcal{T},k}$, is defined from $\mathbb{R}^N \times \mathbb{R}_+$ to $\mathbb{R}$ by:

$$u_{\mathcal{T},k}(x,t) = u^n_p, \text{ if } x \in p,\ t \in [nk,(n+1)k[\ p \in \mathcal{T},\ n \in \mathbb{N}. \tag{11}$$

The following implicit numerical scheme (for which condition (9) is not necessary) will also be studied:

$$m(p)\frac{u^{n+1}_p - u^n_p}{k} + \sum_{q\in\mathcal{N}(p)}(v^n_{p,q}\ F(u^{n+1}_p,u^{n+1}_q) - v^n_{q,p}\ F(u^{n+1}_q,u^{n+1}_p)) = 0,\ \forall p \in \mathcal{T},\ \forall n \in \mathbb{N}, \tag{12}$$

The implicit approximate solution $u_{\mathcal{T},k}$, is defined now from $\mathbb{R}^N \times \mathbb{R}_+$ to $\mathbb{R}$ by:

$$u_{\mathcal{T},k}(x,t) = u^{n+1}_p, \text{ if } x \in p,\ t \in]nk,(n+1)k]\ p \in \mathcal{T},\ n \in \mathbb{N}. \tag{13}$$

1.3 Main results

Under assumptions (3), let u be the solution of (1)-(2). Assuming (5), (6), let $u_{\mathcal{T},k}$ be the solution of (10) (explicit scheme), (8), (11), with the condition (9), or $u_{\mathcal{T},k}$ be the solution of (12) (implicit scheme), (8), (13). Our aim is to give an estimate of the error between u and $u_{\mathcal{T},k}$.

In the case of a cartesian grid, the convergence and error analysis reduces to a one-dimensional discretization problem for which results were proven some time ago, see e.g. [20], [8], [22]. In the case of general triangulations, more recent work allowed several convergence results and error estimates for time-explicit finite volume schemes, see e.g. [7], [3], [6], [24]: following Szepessy's work on the convergence of the streamline diffusion method [23], most of these works use DiPerna's uniqueness theorem [10] (or an adaptation of it [14],[13]), and the error estimates generalize the work by Kuznetsov [20]. In [7], [24], error estimates of order $h^{\frac{1}{4}}$ are given for a conservation law of the form $u_t + \mathrm{div}F(u) = 0$ where F is a vector valued function; in [7], the numerical flux is assumed to satisfy an estimate (see (2.16) in [7]) which does not seem easy to prove for any numerical scheme, and which we do not require here; in fact, we prove directly that any monotonic scheme defined by (10) and (12) satisfies a "weak BV estimate" (see lemmas 2.2, 3.1 and 3.2 below) which we believe to be the key estimate for obtaining the error estimate.
In both [7] and [24], an "inverse" CFL condition of the type $C \le \frac{k}{h}$ is required, in particular because of the use of DiPerna's uniqueness theorem. We shall not need this restrictive condition here since we make use of an adaptation of Di Perna's theorem which was proven in [14].
Note that the originality of the present work also lies in the fact that the nonlinearity which we consider is of the form $\mathbf{v}(x,t)f(u)$. This kind of flux is often encountered in porous medium modelling, where the hyperbolic equation may then be coupled with an elliptic or parabolic equation (see e.g. [12], [25], [26], [16]). It adds an extra difficulty to the case $F(u)$ because of the dependency on x and t. Note that the method which we present here for a nonlinearity of the form $\mathbf{v}(x,t)f(u)$ also yields the results in the case of a nonlinearity of the form $F(x,t,u)$, see the recent work [2].
Last but not least, we give here an error estimate for a time implicit scheme (which is, to our knowledge, the first result for implicit schemes) which adds the extra difficulties of proving the existence of the approximate solution and proving a strong time BV estimate (see Lemma 3.2) in order to show that the error for the implicit scheme may still be, at least in particular cases, of order $h^{\frac{1}{4}}$ even if the time step k behaves as $\sqrt{h}$.
Let us now state our main results precisely: In the case of the explicit scheme, we prove, in the following sections, the following theorem.

Theorem 1.1 *Assume (3), (5), (6) and condition (9). Let u be the unique entropy weak solution of (1)-(2) and $u_{\mathcal{T},k}$ be given by (11), (10), (8). Assume $u_0 \in BV(\mathbb{R}^N)$. Then, for any compact set $E \subset \mathbb{R}^N \times \mathbb{R}_+$, there exists C_e, depending only on E, $\mathbf{v}$, f, F, u_0, α and ξ, such that the following inequality holds :*

$$\int_E |u_{\mathcal{T},k}(x,t) - u(x,t)| dxdt \leq C_e h^{\frac{1}{4}}. \tag{14}$$

In Theorem 1.1, u_0 is assumed to belong to $BV(\mathbb{R}^N)$ (recall that $u_0 \in BV(\mathbb{R}^N)$ if $\sup\{\int u_0(x)\text{div}\varphi(x)dx$, $\varphi \in C_c^\infty(\mathbb{R}^N, \mathbb{R}^N)$; $|\varphi(x)| \leq 1$, $\forall x \in \mathbb{R}^N\} < \infty$). This assumption allows us to obtain an $h^{\frac{1}{4}}$ estimate in (14). If $u_0 \notin BV(\mathbb{R}^N)$, one can also give an error estimate which depends on the functions $\varepsilon(r, K)$ and $\varepsilon_0(r, K)$ defined in (115) and (128).
In some cases, it is possible to obtain $h^{\frac{1}{2}}$, instead of $h^{\frac{1}{4}}$, in Theorem 1.1. This is the case, for instance, when the mesh, $\mathcal{T}$, is composed of rectangles ($N = 2$) and when $\mathbf{v}$ does not depend on (x, t), since, in this case, one obtains a "BV estimate" on $u_{\mathcal{T},k}$. In this case, the right hand sides of estimates (21) and (22), proven below, are changed from $\dfrac{C_w}{\sqrt{h}}$ to C_w, so that the right hand side of (68) becomes Ch instead of $C\sqrt{h}$, which in turn yields $Ch^{\frac{1}{2}}$ in (14) instead of $Ch^{\frac{1}{4}}$. It is, however, still an open problem whether it is possible to obtain an error estimate with $h^{\frac{1}{2}}$, instead of $h^{\frac{1}{4}}$, in Theorem 1.1 (under the hypotheses of Theorem 1.1), even in the case where $\mathbf{v}$ does not depend on (x, t) (see [4] for an attempt in this direction).

Remark 1.2 Theorem 1.1 remains true with some slightly more general assumptions on F, instead of (6), in order to allow F to depend on $\mathcal{T}$ and k. Indeed, in (10), one can replace $F(u_p^n, u_q^n)$ (and $F(u_q^n, u_p^n)$) by $F_{p,q}(u_p^n, u_q^n, \mathcal{T}, k)$ (and $F_{q,p}(u_q^n, u_p^n, \mathcal{T}, k)$). One assumes that, for all $p_1, p_2 \in \mathcal{T}$ (having a common interface), the function $F_{p_1,p_2}(a_1, a_2, \mathcal{T}, k)$ is nondecreasing w.r.t. a_1 and nonincreasing w.r.t. a_2, and Lipschitz continuous w.r.t. a_1 and a_2, uniformly w.r.t. p_1 and p_2 and that $F_{p_1,p_2}(a, a, \mathcal{T}, k) = f(a)$, for $a, a_1, a_2 \in [U_m, U_M]$. Then Theorem 1.1 remains true. Note that Condition (9) and C_e in the estimate (14) of Theorem 1.1 depend on the Lipschitz constant of F_{p_1,p_2} on $[U_m, U_M]$.
An interesting form for F_{p_1,p_2} is $F_{p_1,p_2}(a_1, a_2, \mathcal{T}, k) = c_{p_1,p_2}(\mathcal{T}, k)f(a_1) + (1 - c_{p_1,p_2}(\mathcal{T}, k))\ f(a_2) + D_{p_1,p_2}(\mathcal{T}, k)\ (a_1 - a_2)$, with some $c_{p_1,p_2}(\mathcal{T}, k) \in [0, 1]$ and $D_{p_1,p_2}(\mathcal{T}, k) \geq 0$. In order to obtain the desired properties on F_{p_1,p_2}, it suffices to take $\max\{|f'(s)|,\ s \in [U_m, U_M]\} \leq D_{p_1,p_2}(\mathcal{T}, k) \leq D$ (for all p_1, p_2), with some $D \in \mathbb{R}$. The Lipschitz constant of F_{p_1,p_2} on $[U_m, U_M]$ is then given by D.
For instance, a "Lax-Friedrichs type" scheme consists, roughly speaking, in taking $D_{p_1,p_2}(\mathcal{T}, k)$ of order "$\frac{h}{k}$". The desired properties on F_{p_1,p_2} are satisfied, provided that $\frac{k}{h} \leq C$, with some C depending on $\max\{|f'(s)|,\ s \in [U_m, U_M]\}$. Note, however, that the condition $\frac{k}{h} \leq C$ is not sufficient to give a real "$h^{\frac{1}{4}}$" estimate, since the coefficient, C_e, in (14) depends on D. Taking, for example, k of order "h^2" leads to an estimate "$C_e h^{\frac{1}{4}}$" which do not goes to 0 as h goes to 0 (indeed, it is known, in this case, that the approximate solution does not converge towards the exact solution). One obtains a real "$h^{\frac{1}{4}}$" estimate, in the case of this "Lax-Friedrichs type" scheme, by taking $C_1 \leq \frac{k}{h} \leq C_2$.

In the case of the implicit scheme, we shall prove the following theorem:

Theorem 1.2 *Assume (3), (5) and (6). Let u be the unique entropy weak solution of (1)-(2). Assume that $u_0 \in BV(\mathbb{R}^N)$ and that $\mathbf{v}$ does not depend on t. Then, there exists a unique solution $\{u_p^n,\ n \in \mathbb{N},\ p \in \mathcal{T}\}$ to (12), (8) and (13) such that $u_p^n \in [U_m, U_M]$ for all $p \in \mathcal{T}$ and $n \in \mathbb{N}$. Furthermore, for any compact set $E \subset \mathbb{R}^N \times \mathbb{R}_+$ there exists C_e, depending only on E, $\mathbf{v}$, f, F, u_0 and α, such that the following inequality holds :*

$$\int_E |u_{\mathcal{T},k}(x,t) - u(x,t)| dxdt \leq C_e (k + h^{\frac{1}{2}})^{\frac{1}{2}}. \tag{15}$$

Remark 1.3 Note that, in Theorem 1.2, there is no restriction on k (this is usual for an implicit scheme), and one obtains an "$h^{\frac{1}{4}}$" error estimate for "large" values of k, namely if $k \leq h^{\frac{1}{2}}$. In Theorem 1.2, if $\mathbf{v}$ depends on t, one can also give an error estimate, indeed one obtains:

$$\int_E |u_{\mathcal{T},k}(x,t) - u(x,t)| dx dt \leq C_e \left(\frac{k}{h^{\frac{1}{2}}} + h^{\frac{1}{2}}\right)^{\frac{1}{2}},$$

which gives an "$h^{\frac{1}{4}}$" error estimate if k is of order "h".

Remark 1.4 *About hypotheses (3), note that the existence of $V < \infty$ s.t. $|\mathbf{v}(x,t)| \leq V$ in part (ii) of (3) is crucial. It ensures the property of "propagation in finite time" which is needed for the uniqueness of the solution of (4), and for the stability (under a "CFL" condition) of the, explicit in time, numerical scheme. Hypothesis $\mathrm{div}\mathbf{v}(x,t) = 0$, in part (ii) of (3), is not necessary to obtain Theorem 1.1 and Theorem 1.2, but is natural in many "appications" and avoids many technical complications. Note, in particular, that, for instance, if $\mathrm{div}\mathbf{v} \neq 0$, the L^∞-bound on the solution of (4) and the L^∞ estimate (in Lemma 2.1 and Proposition 3.1) on the approximate solution depends on $\mathbf{v}$ and T.*

2 Stability results for the explicit scheme

2.1 L^∞ stability

Lemma 2.1 *Under Assumptions (3), (5), (6) and Condition (9), let $u_{\mathcal{T},k}$ be given by (11), (10), (8); then:*

$$U_m \leq u_p^n \leq U_M, \forall n \in \mathbb{N}, \forall p \in \mathcal{T}, \tag{16}$$

and

$$\|u_{\mathcal{T},k}\|_{L^\infty(\mathbb{R}^N \times \mathbb{R}_+)} \leq \|u_0\|_{L^\infty(\mathbb{R}^N)}. \tag{17}$$

Proof of Lemma 2.1:
Note that (17) is a straightforward consequence of (16), which will be proved by induction. For $n = 0$, since $U_m \leq u_0 \leq U_M$ a.e., (16) follows from (8).
Let $n \in \mathbb{N}$, assume that $U_m \leq u_p^n \leq U_M$ for all $p \in \mathcal{T}$. Using the fact that $\mathrm{div}\mathbf{v} = 0$, which may be expressed as $\sum_{q \in \mathcal{N}(p)} (v_{p,q}^n - v_{q,p}^n) = 0$, we can rewrite (10) as:

$$m(p)\frac{u_p^{n+1} - u_p^n}{k} + \sum_{q \in \mathcal{N}(p)} \Big(v_{p,q}^n (F(u_p^n, u_q^n) - f(u_p^n)) - v_{q,p}^n (F(u_q^n, u_p^n) - f(u_p^n)) \Big) = 0. \tag{18}$$

Set, for $u_p^n \neq u_q^n$:

$$\tau_{p,q}^n = v_{p,q}^n \frac{F(u_p^n, u_q^n) - f(u_p^n)}{u_p^n - u_q^n} - v_{q,p}^n \frac{F(u_q^n, u_p^n) - f(u_p^n)}{u_p^n - u_q^n}, \tag{19}$$

and $\tau_{p,q}^n = 0$ if $u_p^n = u_q^n$.
The monotonicity properties of the function F yields: $0 \leq \tau_{p,q}^n \leq V m(\sigma_{p,q})(F_1 + F_2)$. Using (18), we can write:

$$u_p^{n+1} = \Big(1 - \frac{k}{m(p)} \sum_{q \in \mathcal{N}(p)} \tau_{p,q}^n \Big) u_p^n + \frac{k}{m(p)} \sum_{q \in \mathcal{N}(p)} \tau_{p,q}^n u_q^n, \tag{20}$$

which gives, under condition (9), $\inf_{q \in \mathcal{T}} u_q^n \leq u_p^{n+1} \leq \sup_{q \in \mathcal{T}} u_q^n$, for all $p \in \mathcal{T}$. This concludes the proof of (16), which, in turn, yields (17). ■

Remark 2.1 *Note that, in fact, the stability result (17) holds even if $\xi = 0$ in (9). However, we shall need $\xi > 0$ for the following "weak BV" estimate.*

2.2 A "weak BV" estimate

Lemma 2.2 *Assume (3), (5), (6) and condition (9), let $u_{\mathcal{T},k}$ be given by (11), (10), (8); let $T > 0$, $R > 0$ and $N_T = \max\{n \in \mathbb{N}, n \le T/k\}$, $\mathbb{N}_T = \{0,\dots,N_T\}$, $\mathcal{T}_R = \{p \in \mathcal{T}, p \subset B(0,R)\}$, and $\mathcal{E}_R^n = \{(p,q) \in \mathcal{T}^2, q \in \mathcal{N}(p), \sigma_{p,q} \subset B(0,R)$ and $u_p^n > u_q^n\}$. Then there exists $C_w \in \mathbb{R}$, depending only on $\mathbf{v}$, F, u_0, α, ξ, R, T such that, for $h \le R$:*

$$\begin{aligned}\sum_{n=0}^{N_T} k \sum_{(p,q)\in\mathcal{E}_R^n} \Big[v_{p,q}^n \Big(&\max_{u_q^n \le c \le d \le u_p^n} (F(d,c) - f(d)) + \max_{u_q^n \le c \le d \le u_p^n} (F(d,c) - f(c)) \Big) + \\ & v_{q,p}^n \Big(\max_{u_q^n \le c \le d \le u_p^n} (f(d) - F(c,d)) + \max_{u_q^n \le c \le d \le u_p^n} (f(c) - F(c,d)) \Big) \Big] \\ & \le \frac{C_w}{\sqrt{h}},\end{aligned} \tag{21}$$

and

$$\sum_{n=0}^{N_T} \sum_{p\in\mathcal{T}_R} m(p) |u_p^{n+1} - u_p^n| \le \frac{C_w}{\sqrt{h}}. \tag{22}$$

Proof of Lemma 2.2:
In this proof, we shall denote by C_i ($i \in \mathbb{N}$) various real functions depending only on $\mathbf{v}$, F, u_0, α, ξ, R, T.
We multiply (18) by $k u_p^n$, and we sum the result over $p \in \mathcal{T}_R$, $n \in \mathbb{N}_T$. This yields:

$$B_1 + B_2 = 0, \tag{23}$$

with:

$$B_1 = \sum_{n=0}^{N_T} \sum_{p\in\mathcal{T}_R} m(p) u_p^n (u_p^{n+1} - u_p^n), \tag{24}$$

and

$$B_2 = \sum_{n=0}^{N_T} k \sum_{p\in\mathcal{T}_R} \sum_{q\in\mathcal{N}(p)} \Big(v_{p,q}^n (F(u_p^n, u_q^n) - f(u_p^n)) u_p^n - v_{q,p}^n (F(u_q^n, u_p^n) - f(u_p^n)) u_p^n \Big). \tag{25}$$

Let us define B_3 by:

$$\begin{aligned} B_3 = \sum_{n=0}^{N_T} k \sum_{(p,q)\in\mathcal{E}_R^n} \Big[\quad & v_{p,q}^n \Big(u_p^n (F(u_p^n, u_q^n) - f(u_p^n)) - u_q^n (F(u_p^n, u_q^n) - f(u_q^n)) \Big) - \\ & v_{q,p}^n \Big(u_p^n (F(u_q^n, u_p^n) - f(u_p^n)) - u_q^n (F(u_q^n, u_p^n) - f(u_q^n)) \Big) \Big]. \end{aligned} \tag{26}$$

The expression $|B_3 - B_2|$ can be reduced to a sum of terms which are each bounded by $C_1 h^{N-1}$, thanks to (17). Each of these terms correspond to the boundary of a control volume which is included in $B(0, R+h) \setminus B(0, R-h)$, the measure of which is less than $C_2 h$. Therefore, the number of such terms is lower than $C_3 h / (\alpha h^N)$ (indeed $c_3 = C_2$). We can then deduce that:

$$|B_3 - B_2| \le C_4. \tag{27}$$

Denoting by Φ a primitive of the function $(\cdot f'(\cdot))$, an integration by parts yields, for all $(a,b) \in \mathbb{R}^2$:

$$\Phi(b) - \Phi(a) = \int_a^b x f'(x) dx = b(f(b) - F(a,b)) - a(f(a) - F(a,b)) - \int_a^b (f(x) - F(a,b)) dx. \tag{28}$$

Using (28), the term B_3 may be decomposed as:

$$B_3 = B_4 - B_5, \tag{29}$$

where:

$$B_4 = \sum_{n=0}^{N_T} k \sum_{(p,q)\in\mathcal{E}_R^n} \left(v_{p,q}^n \int_{u_p^n}^{u_q^n} (f(x) - F(u_p^n, u_q^n))dx + v_{q,p}^n \int_{u_q^n}^{u_p^n} (f(x) - F(u_q^n, u_p^n))dx\right), \tag{30}$$

and

$$B_5 = \sum_{n=0}^{N_T} k \sum_{(p,q)\in\mathcal{E}_R^n} (v_{p,q}^n - v_{q,p}^n)\Big(\Phi(u_p^n) - \Phi(u_q^n)\Big). \tag{31}$$

The term B_5 is again reduced to a sum of terms corresponding to control volumes included in $B(0, R + h) \setminus B(0, R - h)$, thanks to $\mathrm{div}\mathbf{v} = 0$; therefore, as for (27), there exists C_5 such that:

$$B_5 \le C_5. \tag{32}$$

Let us now turn to an estimate of B_4. For this purpose, let $a, b \in \mathbb{R}$, define $\mathcal{C}(a, b) = \{(c, d) \in [a \perp b, a \top b]^2;\ (d - c)(b - a) \ge 0\}$. Thanks to the monotonicity properties of F, the following inequality holds, for any $(c, d) \in \mathcal{C}(a, b)$:

$$\int_a^b (f(x) - F(a, b))dx \ge \int_c^d (f(x) - F(a, b))dx \ge \int_c^d (f(x) - F(c, d))dx \ge 0. \tag{33}$$

We now use the following technical lemma, the proof of which is given after completion of the present proof:

Lemma 2.3 *Let $g : \mathbb{R} \to \mathbb{R}$ be a monotonic, Lipschitz continuous function, with a Lipschitz constant $G > 0$. Then:*

$$\left|\int_c^d (g(x) - g(c))dx\right| \ge \frac{1}{2G}(g(d) - g(c))^2, \quad \forall c, d \in \mathbb{R}. \tag{34}$$

From Lemma 2.3, we can notice that:

$$\int_c^d (f(x) - F(c, d))dx \ge \int_c^d (F(c, x) - F(c, d))dx \ge \frac{1}{2F_2}(f(c) - F(c, d))^2, \tag{35}$$

and

$$\int_c^d (f(x) - F(c, d))dx \ge \int_c^d (F(x, d) - F(c, d))dx \ge \frac{1}{2F_1}(f(d) - F(c, d))^2. \tag{36}$$

Multiplying (35) (resp. (36)) by $F_2/(F_1+F_2)$ (resp. $F_1/(F_1+F_2)$), taking the maximum for $(c, d) \in \mathcal{C}(a, b)$, and adding both equations yields:

$$\int_a^b (f(x) - F(a, b))dx \ge \frac{1}{2(F_1 + F_2)}\Big(\max_{(c,d)\in\mathcal{C}(a,b)}(f(c) - F(c, d))^2 + \max_{(c,d)\in\mathcal{C}(a,b)}(f(d) - F(c, d))^2\Big). \tag{37}$$

We can then deduce, from (37):

$$
\begin{aligned}
B_4 \geq \quad & \frac{1}{2(F_1+F_2)}\sum_{n=0}^{N_T} k \sum_{(p,q)\in\mathcal{E}_R^n} \Big[\\
& v_{p,q}^n \Big(\max_{u_q^n \leq c \leq d \leq u_p^n} (F(d,c)-f(d))^2 + \max_{u_q^n \leq c \leq d \leq u_p^n} (F(d,c)-f(c))^2 \Big) + \\
& v_{q,p}^n \Big(\max_{u_q^n \leq c \leq d \leq u_p^n} (f(d)-F(c,d))^2 + \max_{u_q^n \leq c \leq d \leq u_p^n} (f(c)-F(c,d))^2 \Big) \Big].
\end{aligned} \tag{38}
$$

This gives a bound on B_2, since (with $C_6 = C_4 + C_5$):

$$
B_2 \geq B_4 - C_6. \tag{39}
$$

Let us now turn to B_1. We have:

$$
B_1 = -\frac{1}{2}\sum_{n=0}^{N_T}\sum_{p\in\mathcal{T}_R} m(p)(u_p^{n+1}-u_p^n)^2 + \frac{1}{2}\sum_{p\in\mathcal{T}_R} m(p)\Big(u_p^{N_T+1}\Big)^2 - \frac{1}{2}\sum_{p\in\mathcal{T}_R} m(p)\Big(u_p^0\Big)^2. \tag{40}
$$

Using (18) and the Cauchy-Schwarz inequality yields the following inequality:

$$
\begin{aligned}
& (u_p^{n+1}-u_p^n)^2 \leq \\
& \frac{k^2}{m(p)^2}\sum_{q\in\mathcal{N}(p)} (v_{p,q}^n + v_{q,p}^n) \sum_{q\in\mathcal{N}(p)} \Big[v_{p,q}^n \Big(F(u_p^n,u_q^n)-f(u_p^n)\Big)^2 + v_{q,p}^n \Big(F(u_q^n,u_p^n)-f(u_p^n)\Big)^2 \Big].
\end{aligned} \tag{41}
$$

Using the CFL condition (9) in (41) gives:

$$
m(p)(u_p^{n+1}-u_p^n)^2 \leq k\frac{1-\xi}{F_1+F_2}\sum_{q\in\mathcal{N}(p)} \Big[v_{p,q}^n \Big(F(u_p^n,u_q^n)-f(u_p^n)\Big)^2 + v_{q,p}^n \Big(F(u_q^n,u_p^n)-f(u_p^n)\Big)^2 \Big]. \tag{42}
$$

Summing equation (42) over $p \in \mathcal{T}_R$ and over $n \in \mathbb{N}_T$, and reordering the summation leads to:

$$
\begin{aligned}
\frac{1}{2}\sum_{n=0}^{N_T}\sum_{p\in\mathcal{T}_R} m(p)(u_p^{n+1}-u_p^n)^2 \leq \quad & \frac{1-\xi}{2(F_1+F_2)}\sum_{n=0}^{N_T} k \sum_{(p,q)\in\mathcal{E}_R^n} \Big[\\
& v_{p,q}^n \Big((F(u_p^n,u_q^n)-f(u_p^n))^2 + (F(u_p^n,u_q^n)-f(u_q^n))^2 \Big) + \\
& v_{q,p}^n \Big((f(u_p^n)-F(u_q^n,u_p^n))^2 + (f(u_q^n)-F(u_q^n,u_p^n))^2 \Big) \Big] + C_7,
\end{aligned} \tag{43}
$$

where C_7 accounts for the edges $\sigma_{p,q}$, where $p \in \mathcal{T}_R$ and $q \in \mathcal{N}(p), q \notin \mathcal{T}_R$ (these are included in $B(0,R+h)\setminus B(0,R-h)$).
Note that the right hand side of (43) is bounded by $(1-\xi)B_4 + C_7$ (from (38)). Using (23), (39) and (40) gives:

$$
\begin{aligned}
\frac{\xi}{2(F_1+F_2)}\sum_{n=0}^{N_T} k \sum_{(p,q)\in\mathcal{E}_R^n} \Big[\quad & v_{p,q}^n \Big(\max_{u_q^n \leq c \leq d \leq u_p^n} (F(d,c)-f(d))^2 + \max_{u_q^n \leq c \leq d \leq u_p^n} (F(d,c)-f(c))^2 \Big) + \\
& v_{q,p}^n \Big(\max_{u_q^n \leq c \leq d \leq u_p^n} (f(d)-F(c,d))^2 + \max_{u_q^n \leq c \leq d \leq u_p^n} (f(c)-F(c,d))^2 \Big) \Big] \leq \\
& \frac{1}{2}\sum_{p\in\mathcal{T}_R} m(p)\Big(u_p^0\Big)^2 + C_6 + C_7 = C_8.
\end{aligned} \tag{44}
$$

Applying the Cauchy-Schwarz inequality to the left hand side of (21) and (44) yields:

$$\begin{aligned}\sum_{n=0}^{N_T} k \sum_{(p,q)\in\mathcal{E}_R^n} \Big[\quad & v_{p,q}^n\Big(\max_{u_q^n\le c\le d\le u_p^n}(F(d,c)-f(d)) + \max_{u_q^n\le c\le d\le u_p^n}(F(d,c)-f(c))\Big) + \\ & v_{q,p}^n\Big(\max_{u_q^n\le c\le d\le u_p^n}(f(d)-F(c,d)) + \max_{u_q^n\le c\le d\le u_p^n}(f(c)-F(c,d))\Big)\Big] \\ \le \quad & C_9\Big(\sum_{n=0}^{N_T} k \sum_{(p,q)\in\mathcal{E}_R^n}(v_{p,q}^n+v_{q,p}^n)\Big)^{\frac{1}{2}},\end{aligned} \tag{45}$$

Noting that card $(\mathcal{E}_R^n) \le C_{10}h^{-N}$, and $v_{p,q}^n + v_{q,p}^n \le C_{11}h^{N-1}$ for all $(p,q)\in\mathcal{E}_R^n$, one obtains (21) from (45).
Then (22) is directly obtained from (21), because (18) leads to:

$$m(p)|u_p^{n+1}-u_p^n| \le k\sum_{q\in\mathcal{N}(p)}\Big(v_{p,q}^n|F(u_p^n,u_q^n)-f(u_p^n)| + v_{q,p}^n|F(u_q^n,u_p^n)-f(u_p^n)|\Big). \tag{46}$$

This completes the proof of Lemma 2.2. ■

It remains to prove Lemma 2.3.
We assume, for instance, that g is nondecreasing and $c<d$ (the other cases are similar). Then, one has $g(s)\ge h(s)$, for all $s\in[c,d]$, where $h(s)=g(c)$ for $s\in[c,d-l]$ and $h(s)=g(c)+(s-d+l)G$ for $s\in[d-l,d]$, with $lG=g(d)-g(c)$, and therefore:

$$\int_c^d (g(s)-g(c))ds \ge \int_c^d (h(s)-g(c))ds = \frac{l}{2}(g(d)-g(c)) = \frac{1}{2G}(g(d)-g(c))^2, \tag{47}$$

this complete the proof of Lemma 2.3. ■

3 Existence of the solution and stability results for the implicit scheme

This section is devoted to the implicit scheme (given by (12)).
We first prove the existence and uniqueness of the solution $\{u_p^n, n\in\mathbb{N}, p\in\mathcal{T}\}$ of (8), (12) and that $u_p^n\in[U_m,U_M]$ for all $p\in\mathcal{T}$ and $n\in\mathbb{N}$. We then give a "weak space BV" estimate (this is equivalent to the estimate (21) for the explicit scheme) and a "(strong) time BV" estimate (estimate (58) below). This last estimate requires that $\mathbf{v}$ does not depend on t (and it leads to the term "k" in the right hand side of (15) in Theorem 1.2). In the case where $\mathbf{v}$ depends on t, an estimate error is given in Remark 1.3 and follows from an easy adaptation of the proofs given in this paper.

3.1 Existence, uniqueness and L^∞ stability

The following proposition gives an existence and uniqueness result of the solution to (8), (12). For this proposition, $\mathbf{v}$ can depend on t and one does not need to assume $u_0\in BV(\mathbb{R}^N)$.

Proposition 3.1 *Assume (3), (5), (6), then there exists a unique solution* $\{u_p^n, n\in\mathbb{N}, p\in\mathcal{T}\}\subset[U_m,U_M]$ *to (8), (12).*

Proof of Proposition 3.1 One proves Proposition 3.1 by induction. Indeed, $(u_p^0,\ p\in\mathcal{T})$ is uniquely defined by (8) and one has $u_p^0\in[U_m,U_M]$, for all $p\in\mathcal{T}$, since $U_m\le u_0\le U_M$ a.e.. Assuming that, for some $n\in\mathbb{N}$, the set $\{u_p^n,\ p\in\mathcal{T}\}$ is given and that $u_p^n\in[U_m,U_M]$, for all $p\in\mathcal{T}$, existence and uniqueness of $\{u_p^{n+1}\ p\in\mathcal{T}\}$ such that $u_p^{n+1}\in[U_m,U_M]$ solution of (12) must be shown.

1. Uniqueness of $\{u_p^{n+1},\, p\in\mathcal{T}\}$ such that $u_p^{n+1}\in[U_m,U_M]$ solves of (12).
Let $n\in\mathbb{N}$, and let $(u_p,\, p\in\mathcal{T})$ and $(w_p,\, p\in\mathcal{T})$, satisfying:

$$m(p)\frac{u_p-u_p^n}{k}+\sum_{q\in\mathcal{N}(p)}(v_{p,q}^n\ F(u_p,u_q)-v_{q,p}^n\ F(u_q,u_p))=0,\ \forall p\in\mathcal{T}, \tag{48}$$

and

$$m(p)\frac{w_p-u_p^n}{k}+\sum_{q\in\mathcal{N}(p)}(v_{p,q}^n\ F(w_p,w_q)-v_{q,p}^n\ F(w_q,w_p))=0,\ \forall p\in\mathcal{T}, \tag{49}$$

then, substracting (49) from (48), for all $p\in\mathcal{T}$:

$$\begin{aligned}&\frac{m(p)}{k}(u_p-w_p)+\sum_{q\in\mathcal{N}(p)}v_{p,q}^n(F(u_p,u_q)-F(w_p,u_q))+\sum_{q\in\mathcal{N}(p)}v_{p,q}^n(F(w_p,u_q)-F(w_p,w_q))\\&-\sum_{q\in\mathcal{N}(p)}(v_{q,p}^n(F(u_q,u_p)-F(w_q,u_p))-\sum_{q\in\mathcal{N}(p)}(v_{q,p}^n(F(w_q,u_p)-F(w_q,w_p))=0\end{aligned} \tag{50}$$

thanks to the monotonicity properties of F, (50) leads to

$$\begin{aligned}&\frac{m(p)}{k}|u_p-w_p|+\sum_{q\in\mathcal{N}(p)}v_{p,q}^n|F(u_p,u_q)-F(w_p,u_q)|+\sum_{q\in\mathcal{N}(p)}v_{q,p}^n|F(w_q,u_p)-F(w_q,w_p)|\\&\le\sum_{q\in\mathcal{N}(p)}v_{p,q}^n|F(w_p,u_q)-F(w_p,w_q)|+\sum_{q\in\mathcal{N}(p)}v_{q,p}^n|F(u_q,u_p)-F(w_q,u_p)|.\end{aligned} \tag{51}$$

Let $\varphi:\mathbb{R}^N\mapsto\mathbb{R}_+^\star$ be defined by $\varphi(x)=\exp(-\gamma|x|)$, for some positive γ which will be specified in the sequel. For $p\in\mathcal{T}$, let φ_p be the mean value of φ on p. Since φ is integrable over $\mathbb{R}^N$ (and thanks to (5)), one has $\sum_{p\in\mathcal{T}}\varphi_p<\infty$. Therefore the series $\sum_{p\in\mathcal{T}}\varphi_p(\sum_{q\in\mathcal{N}(p)}v_{p,q}^n|F(w_p,u_q)-F(w_p,w_q)|)$ and $\sum_{p\in\mathcal{T}}\varphi_p\sum_{q\in\mathcal{N}(p)}(v_{q,p}^n|F(u_q,u_p)-F(w_q,u_p)|)$ are convergent (thanks to (5), and the boundedness of $\mathbf{v}$ on $\mathbb{R}^N$ and F on $[U_m,U_M]^2$).
Multiplying (51) by φ_p and summing for $p\in\mathcal{T}$ yelds five convergent series which can be reordered in order to give:

$$\begin{aligned}\sum_{p\in\mathcal{T}}\frac{m(p)}{k}|u_p-w_p|\varphi_p\ &\le\ \sum_{p\in\mathcal{T}}\sum_{q\in\mathcal{N}(p)}v_{p,q}^n|F(w_p,u_q)-F(w_p,w_q)||\varphi_p-\varphi_q|\\&+\ \sum_{p\in\mathcal{T}}\sum_{q\in\mathcal{N}(p)}v_{q,p}^n|F(u_q,u_p)-F(w_q,u_p)||\varphi_p-\varphi_q|,\end{aligned} \tag{52}$$

from which one deduces

$$\sum_{p\in\mathcal{T}}a_p|u_p-w_p|\le\sum_{p\in\mathcal{T}}b_p|u_p-w_p|, \tag{53}$$

where, for all $p\in\mathcal{T}$, $a_p=\frac{m(p)}{k}\varphi_p$ and $b_p=\sum_{q\in\mathcal{N}(p)}(v_{p,q}^nF_1+v_{q,p}^nF_2)|\varphi_p-\varphi_q|$.
Then, one takes γ small enough in order to have $a_p>b_p$, for all $p\in\mathcal{T}$. Indeed it suffices to take γ such that $\inf_{y\in B(x,h)}\varphi(y)\ge C\sup_{y\in B(x,2h)}|\nabla\varphi(y)|$, for all $x\in\mathbb{R}^N$, with $C=\frac{4kV(F_1+F_2)}{\alpha}$. One concludes, from (53), $u_p=w_p$, for all $p\in\mathcal{T}$. ∎

2. Existence of $\{u_p^{n+1},\, p\in\mathcal{T}\}$ such that $u_p^{n+1}\in[U_m,U_M]$ solves of (12).
Recall that n and $(u_p^n)_{p\in\mathcal{T}}$ are known.

We use again the sets $\mathcal{T}_r = \{p \in \mathcal{T},\ p \subset B_r\}$, for $r \in \mathbb{N}^*$, and assume that r is large enough in order to have $\mathcal{T}_r \neq \emptyset$. If $p \in \mathcal{T} \setminus \mathcal{T}_r$, one sets $u_p^{(r)} = u_p^n$. In step 1 below, one proves that there exists $(u_p^{(r)})_{p\in\mathcal{T}_r} \subset [U_m, U_M]$ which solves

$$m(p)\frac{u_p^{(r)} - u_p^n}{k} + \sum_{q\in\mathcal{N}(p)} (v_{p,q}^n\ F(u_p^{(r)}, u_q^{(r)}) - v_{q,p}^n\ F(u_q^{(r)}, u_p^{(r)})) = 0,\ \forall p \in \mathcal{T}_r. \tag{54}$$

Then, in step 2, one proves that passing to the limit as $r \to \infty$ (up to a subsequence) leads to a solution to (12) $\{u_p^{n+1}\, p \in \mathcal{T}\}$ such that $u_p^{n+1} \in [U_m, U_M]$.

Step 1. Let $U_r = (u_p^{(r)},\ p \in \mathcal{T}_r)$ be a solution of (54), and let $U_r^n = (u_p^n,\ p \in \mathcal{T}_r)$. The vectors U_r and U_r^n may be viewed as vectors of $\mathbb{R}^d$, with $d = \mathrm{card}(\mathcal{T}_r)$. Equations (54) give,

$$u_p^{(r)} + \frac{k}{m(p)} \sum_{q\in\mathcal{N}(p)} (v_{p,q}^n\ F(u_p^{(r)}, u_q^{(r)}) - v_{q,p}^n\ F(u_q^{(r)}, u_p^{(r)})) = u_p^n,\ \forall p \in \mathcal{T}_r. \tag{55}$$

This can be written on the form

$$U_r - G_r(U_r) = U_r^n, \tag{56}$$

where G_r is a continuous map from $\mathbb{R}^d$ into $\mathbb{R}^d$. Since $u_p^n \in [U_m, U_M]$, for all $p \in \mathcal{T}$, and $u_p^{(r)} \in [U_m, U_M]$, for all $p \in \mathcal{T}\setminus\mathcal{T}_r$, it is easy to show (using $\mathrm{div}(v) = 0$) that if U_r satisfies (56), then one has $u_p^{(r)} \in [U_m, U_M]$, for all $p \in \mathcal{T}_r$. Then, if $\mathcal{C}_r$ is a ball of $\mathbb{R}^d$ of center 0 and of large enough radiuse, equation (56) has no solution on the boundary of $\mathcal{C}_r$, and one can define the topological degree of the mapping $Id - G_r$ on the set $\mathcal{C}_r$ associated to U_r^n, that is $d(Id - G_r, \mathcal{C}_r, U_r^n)$ (see, for instance, [9] for a presentation of the degree). Furthermore, if $\lambda \in [0, 1]$, the same argument allows us to define $d(Id - \lambda G_r, \mathcal{C}_r, U_r^n)$, and the property of invariance of the degree asserts that $d(Id - \lambda G_r, \mathcal{C}_r, U_r^n)$ does not depend on λ. Then, one has $d(Id - G_r, \mathcal{C}_r, U_r^n) = d(Id, \mathcal{C}_r, U_r^n)$, and, since $U_r^n \in \mathcal{C}_r$, $d(Id, \mathcal{C}_r, U_r^n) = 1$. One can conclude that $d(Id - G_r, \mathcal{C}_r, U_r^n) \neq 0$, and this proves that there exists a solution to (56), $U_r \in \mathcal{C}_r$. Note that the components on U_r are necessarily in $[U_m, U_M]$.

Step 2. For $r \in \mathbb{N}$, let $(u_p^{(r)},\ p \in \mathcal{T})$ be the solution of (54) given by the preceding step. Since $\{u_p^{(r)},\ r \in \mathbb{N}\}$ is included in $[U_m, U_M]$, for all $p \in \mathcal{T}$, one can find (using a "diagonal process") a sequence $(r_l,\ l \in \mathbb{N})$, with $r_l \to \infty$, as $l \to \infty$, such that $(u_p^{r_l},\ l \in \mathbb{N})$ is convergent (in $[U_m, U_M]$) for all $p \in \mathcal{T}$. One sets $u_p^{n+1} = \lim_{l\to\infty} u_p^{r_l}$. Passing to the limit in (54) (this is possible because for all $p \in \mathcal{T}$, this equation is satisfied for all $l \in \mathbb{N}$ large enough) shows that $(u_p^{n+1},\ p \in \mathcal{T})$ is solution to (12).

Indeed, using the uniqueness of the solution of (12), one can show that $u_p^{(r)} \to u_p^{n+1}$, as $r \to \infty$, for all $p \in \mathcal{T}$.

This completes the proof of Proposition 3.1. ■

3.2 "Weak space BV" estimate

One gives here the same estimate as for the explicit scheme (estimate (21)). This estimate does not make use of $u_0 \in BV(\mathbb{R}^N)$ and $\mathbf{v}$ can depend on t.

Lemma 3.1 *Assume (3), (5) and (6). Let $\{u_p^n, n \in \mathbb{N}, p \in \mathcal{T}\}$ such that $u_p^{n+1} \in [U_m, U_M]$ be the solution of (12), (8) (existence and uniqueness of such a solution is given by Proposition 3.1). Let $T > 0$, $R > 0$ and $N_T = \max\{n \in \mathbb{N}, n \leq T/k\}$, $\mathbb{N}_T = \{0, \ldots, N_T\}$, $\mathcal{T}_R = \{p \in \mathcal{T}, p \subset B(0, R)\}$ and $\mathcal{E}_R^n = \{(p, q) \in \mathcal{T}_R^2, q \in \mathcal{N}(p),$ and $u_p^n > u_q^n\}$. Then there exists $C_v \in \mathbb{R}$, depending only on $\mathbf{v}$, F, u_0, α, R, T such that, for $h \leq R$:*

$$
\sum_{n=0}^{N_T} k \sum_{(p,q)\in\mathcal{E}_R^{n+1}} \Big[v_{p,q}^n \Big(\max_{u_q^{n+1}\le c\le d\le u_p^{n+1}} (F(d,c)-f(d)) + \max_{u_q^{n+1}\le c\le d\le u_p^{n+1}} (F(d,c)-f(c)) \Big) + v_{q,p}^n \Big(\max_{u_q^{n+1}\le c\le d\le u_p^{n+1}} (f(d)-F(c,d)) + \max_{u_q^{n+1}\le c\le d\le u_p^{n+1}} (f(c)-F(c,d)) \Big) \Big] \le \frac{C_v}{\sqrt{h}}. \tag{57}
$$

PROOF

We multiply (12) by ku_p^{n+1}, and sum the result over $p \in \mathcal{T}_R$ and $n \in \mathbb{N}_T$. We can then follow, step by step, the proof of Lemma 2.2, until equation (40), in which the first term of right-hand-side appears with the opposite sign. We can then directly deduce an inequality similar to (44), which suffices to conclude the proof. ∎

3.3 "Time BV" estimate

For the following estimate one uses the fact that $u_0 \in BV(\mathbb{R}^N)$ and that $\mathbf{v}$ does not depend on t.

Lemma 3.2 *Assume (3), (5) and (6). Assume that $u_0 \in BV(\mathbb{R}^N)$ and that $\mathbf{v}$ does not depend on t. Let $\{u_p^n, n \in \mathbb{N}, p \in \mathcal{T}\}$ such that $u_p^n \in [U_m, U_M]$ be the solution of (12), (8) (existence and uniqueness of such a solution is given by Proposition 3.1). Then, there exists C_b, depending only on $\mathbf{v}$, f, F and u_0, such that*

$$
\sum_{p\in\mathcal{T}} \frac{m(p)}{k} |u_p^{n+1} - u_p^n| \le C_b, \ \forall n \in \mathbb{N}. \tag{58}
$$

PROOF

Since $\mathbf{v}$ does not depend on t, one may set $v_{p,q} = v_{p,q}^n$, for $p \in \mathcal{T}$ and $q \in \mathcal{N}(p)$.
For $n \in \mathbb{N}$, one sets

$$
A_n = \sum_{p\in\mathcal{T}} m(p) \frac{|u_p^{n+1} - u_p^n|}{k},
$$

and

$$
B_n = \sum_{p\in\mathcal{T}} \Big| \sum_{q\in\mathcal{N}(p)} v_{p,q} \; F(u_p^n, u_q^n) - v_{q,p} \; F(u_q^n, u_p^n) \Big|.
$$

Since $u_0 \in BV(\mathbb{R}^N)$ (and $\text{div} v = 0$), there exists C_b such that $B_0 \le C_b$. Indeed, C_b depends only on $|u_0|_{BV}$, V, the Lipschitz constant of F on $[U_m, U_M]$ and α (with $|u_0|_{BV} = \sup\{\int u_0(x)\text{div}\varphi(x)dx$, $\varphi \in C_c^\infty(\mathbb{R}^N, \mathbb{R}^N)$; $|\varphi(x)| \le 1$, $\forall x \in \mathbb{R}^N\}$).
From (12), one deduces that $B_{n+1} \le A_n$, for all $n \in \mathbb{N}$. Then, in order to prove Lemma 3.2, one has only to prove that $A_n \le B_n$ for all $n \in \mathbb{N}$ (and to conclude by induction).
Let $n \in \mathbb{N}$, in order to prove that $A_n \le B_n$, recall that the implicit scheme (12) writes

$$
m(p)\frac{u_p^{n+1} - u_p^n}{k} + \sum_{q\in\mathcal{N}(p)} \Big(v_{p,q} \; F(u_p^{n+1}, u_q^{n+1}) - v_{q,p} \; F(u_q^{n+1}, u_p^{n+1}) \Big) = 0. \tag{59}
$$

From (59), one deduces, for all $p \in \mathcal{T}$,

$$
\begin{aligned}
& m(p)\frac{u_p^{n+1}-u_p^n}{k} + \sum_{q\in\mathcal{N}(p)} v_{p,q}\ \left(F(u_p^{n+1},u_q^{n+1}) - F(u_p^n,u_q^{n+1})\right) \\
& + \sum_{q\in\mathcal{N}(p)} v_{p,q}\ \Big(F(u_p^n,u_q^{n+1}) - F(u_p^n,u_q^n)\Big) - \sum_{q\in\mathcal{N}(p)} v_{q,p}\ \Big(F(u_q^{n+1},u_p^{n+1}) - F(u_q^n,u_p^{n+1})\Big) \\
& - \sum_{q\in\mathcal{N}(p)} v_{q,p}\ \Big(F(u_q^n,u_p^{n+1}) - F(u_q^n,u_p^n)\Big) \\
& = - \sum_{q\in\mathcal{N}(p)} v_{p,q} F(u_p^n,u_q^n) + \sum_{q\in\mathcal{N}(p)} v_{q,p}\ F(u_q^n,u_p^n).
\end{aligned}
$$

Using the monotonicity properties of F, one obtains for all $p\in\mathcal{T}$,

$$
\begin{aligned}
& m(p)\frac{|u_p^{n+1}-u_p^n|}{k} + \sum_{q\in\mathcal{N}(p)} v_{p,q}\ |F(u_p^{n+1},u_q^{n+1}) - F(u_p^n,u_q^{n+1})| \\
& + \sum_{q\in\mathcal{N}(p)} v_{q,p}\ |F(u_q^n,u_p^{n+1}) - F(u_q^n,u_p^n)| \\
& \leq |- \sum_{q\in\mathcal{N}(p)} v_{p,q} F(u_p^n,u_q^n) + \sum_{q\in\mathcal{N}(p)} v_{q,p}\ F(u_q^n,u_p^n)| \\
& + \sum_{q\in\mathcal{N}(p)} v_{p,q}\ |F(u_p^n,u_q^{n+1}) - F(u_p^n,u_q^n)| + \sum_{q\in\mathcal{N}(p)} v_{q,p}\ |F(u_q^{n+1},u_p^{n+1}) - F(u_q^n,u_p^{n+1})|.
\end{aligned}
\tag{60}
$$

In order to deal with convergent series, let us proceed as in the proof of proposition 3.1. For $0<\gamma<1$, let $\varphi_\gamma : \mathbb{R}^N \mapsto \mathbb{R}_+^\star$ be defined by $\varphi_\gamma(x) = \exp(-\gamma|x|)$.
For $p\in\mathcal{T}$, let $\varphi_{\gamma,p}$ be the mean value of φ_γ on p. As in Proposition 3.1, since φ_γ is integrable over $\mathbb{R}^N$, $\sum_{p\in\mathcal{T}}\varphi_{\gamma,p}<\infty$. Therefore, multiplying (60) by $\varphi_{\gamma,p}$ (for a fixed γ) and summing over $p\in\mathcal{T}$ yields six convergent series which can be reordered to give

$$
\begin{aligned}
& \sum_{p\in\mathcal{T}} m(p)\frac{|u_p^{n+1}-u_p^n|}{k}\varphi_{\gamma,p} \\
& \leq \sum_{p\in\mathcal{T}} |- \sum_{q\in\mathcal{N}(p)} v_{p,q} F(u_p^n,u_q^n) + \sum_{q\in\mathcal{N}(p)} v_{q,p}\ F(u_q^n,u_p^n)|\varphi_{\gamma,p} \\
& + \sum_{p\in\mathcal{T}}\sum_{q\in\mathcal{N}(p)} v_{p,q}\ |F(u_p^{n+1},u_q^{n+1}) - F(u_p^n,u_q^{n+1})||\varphi_{\gamma,p}-\varphi_{\gamma,q}| \\
& + \sum_{p\in\mathcal{T}}\sum_{q\in\mathcal{N}(p)} v_{q,p}\ |F(u_q^n,u_p^{n+1}) - F(u_q^n,u_p^n)||\varphi_{\gamma,p}-\varphi_{\gamma,q}|.
\end{aligned}
$$

For $p\in\mathcal{T}$, let $x_p\in\overline{p}$ be such that $\varphi_{\gamma,p}=\varphi_\gamma(x_p)$. Let $p\in\mathcal{T}$ and $q\in\mathcal{N}(p)$. Then there exists $s\in(0,1)$ such that $\varphi_{\gamma,q}-\varphi_{\gamma,p} = \nabla\varphi_\gamma(x_p+s(x_q-x_p))\cdot(x_q-x_p)$. Using $|\nabla\varphi_\gamma(x)| = \gamma\exp(-\gamma|x|)$, this yields $|\varphi_{\gamma,q}-\varphi_{\gamma,p}| \leq 2h\gamma\exp(2h\gamma)\varphi_{\gamma,p} \leq 2h\gamma\exp(2h)\varphi_{\gamma,p}$.
Then, using the assumptions (3) and (6), there exists some a only depending on k, V, h, α, F_1 and F_2 such that

$$
\begin{aligned}
& \sum_{p\in\mathcal{T}} m(p)\frac{|u_p^{n+1}-u_p^n|}{k}\varphi_{\gamma,p}(1-\gamma a) \\
& \leq \sum_{p\in\mathcal{T}} |- \sum_{q\in\mathcal{N}(p)} v_{p,q} F(u_p^n,u_q^n) + \sum_{q\in\mathcal{N}(p)} v_{q,p}\ F(u_q^n,u_p^n)|\varphi_{\gamma,p} \leq B_n.
\end{aligned}
$$

Passing to the limit in the latter inequality as $\gamma\to 0$ yields $A_n\leq B_n$. This completes the proof of Lemma 3.2.

■

4 Entropy inequalities for the approximate solution

4.1 Discrete entropy inequalities

In the case of the explicit scheme, following the 1D terminology (see e.g. [15]), the following lemma asserts that the scheme (10) satisfies a discrete entropy condition.

Lemma 4.1 *Assume (3), (5), (6) and condition (9), let $u_{\mathcal{T},k}$ be given by (11), (10), (8); then, for all $\kappa \in \mathbb{R}$, $p \in \mathcal{T}$ and $n \in \mathbb{N}$, the following inequality holds:*

$$m(p)\frac{|u_p^{n+1}-\kappa|-|u_p^n-\kappa|}{k} + \sum_{q\in\mathcal{N}(p)} \Big[\; v_{p,q}^n \Big(F(u_p^n\top\kappa, u_q^n\top\kappa) - F(u_p^n\bot\kappa, u_q^n\bot\kappa)\Big) - v_{q,p}^n \Big(F(u_q^n\top\kappa, u_p^n\top\kappa) - F(u_q^n\bot\kappa, u_p^n\bot\kappa)\Big)\Big] \le 0. \tag{61}$$

Proof
From relation (10), we express u_p^{n+1} as a function of u_p^n and u_q^n, $q \in \mathcal{N}(p)$,

$$u_p^{n+1} = u_p^n + \frac{k}{m(p)} \sum_{q\in\mathcal{N}(p)} (v_{q,p}^n \; F(u_q^n, u_p^n) - v_{p,q}^n \; F(u_p^n, u_q^n)). \tag{62}$$

The right hand side is nondecreasing with respect to u_q^n, $\sigma \in S(p)$. It is also nondecreasing with respect to u_p^n, thanks to the CFL condition (9), and the Lipschitz continuity of F.
Therefore, for all $\kappa \in \mathbb{R}$, using $\text{div}\mathbf{v} = 0$, we have:

$$u_p^{n+1}\top\kappa \le u_p^n\top\kappa + \frac{k}{m(p)} \sum_{q\in\mathcal{N}(p)} (v_{q,p}^n \; F(u_q^n\top\kappa, u_p^n\top\kappa) - v_{p,q}^n \; F(u_p^n\top\kappa, u_q^n\top\kappa)). \tag{63}$$

and

$$u_p^{n+1}\bot\kappa \ge u_p^n\bot\kappa + \frac{k}{m(p)} \sum_{q\in\mathcal{N}(p)} (v_{q,p}^n \; F(u_q^n\bot\kappa, u_p^n\bot\kappa) - v_{p,q}^n \; F(u_p^n\bot\kappa, u_q^n\bot\kappa)). \tag{64}$$

The difference between (63) and (64) leads directly to (61). Note that using $\text{div}\mathbf{v} = 0$ leads to:

$$m(p)\frac{|u_p^{n+1}-\kappa|-|u_p^n-\kappa|}{k} + \sum_{q\in\mathcal{N}(p)} \Big[\; v_{p,q}^n \Big(F(u_p^n\top\kappa, u_q^n\top\kappa) - f(u_p^n\top\kappa) - F(u_p^n\bot\kappa, u_q^n\bot\kappa) + f(u_p^n\bot\kappa)\Big) - v_{q,p}^n \Big(F(u_q^n\top\kappa, u_p^n\top\kappa) - f(u_p^n\top\kappa) - F(u_q^n\bot\kappa, u_p^n\bot\kappa) + f(u_p^n\bot\kappa)\Big)\Big] \le 0. \tag{65}$$

For the implicit scheme, one obtains the same kind of dicrete entropy inequalities.

Lemma 4.2 *Assume (3), (5) and (6). Let $\{u_p^n,\ n \in \mathbb{N},\ p \in \mathcal{T}\} \subset [U_m, U_M]$ be the solution of (12), (8) (existence and uniqueness of such a solution is given by Proposition 3.1). Then, for all $\kappa \in \mathbb{R}$, $p \in \mathcal{T}$ and $n \in \mathbb{N}$, the following inequality holds:*

$$m(p)\frac{|u_p^{n+1}-\kappa|-|u_p^n-\kappa|}{k} + \sum_{q\in\mathcal{N}(p)} \Big[v_{p,q}^n \Big(F(u_p^{n+1}\top\kappa, u_q^{n+1}\top\kappa) - F(u_p^{n+1}\bot\kappa, u_q^{n+1}\bot\kappa)\Big) - v_{q,p}^n \Big(F(u_q^{n+1}\top\kappa, u_p^{n+1}\top\kappa) - F(u_q^{n+1}\bot\kappa, u_p^{n+1}\bot\kappa)\Big)\Big] \le 0. \tag{66}$$

PROOF of lemma 4.2

For all $p \in \mathcal{T}$ and $n \in \mathbb{N}$, Equation (12) gives u_p^{n+1} as an implicit function of u_p^n and u_q^{n+1}, for all $q \in \mathcal{N}(p)$. The monotonicity properties of this implicit function, and the fact that its value is κ, for all $\kappa \in \mathbb{R}$, if $u_p^n = \kappa$ and $u_q^{n+1} = \kappa$ for all $q \in \mathcal{N}(p)$, allows us to write analogous equations to (63) and (64), and therefore to conclude (66). ■

4.2 Continuous entropy estimates for the approximate solution

For $\Omega = \mathbb{R}^N$ or $\mathbb{R}^N \times \mathbb{R}_+$, we denote by $\mathcal{M}(\Omega)$ the set of positive measures on Ω, that is of σ-additive mappings from the Borel σ-algebra of Ω in $\overline{\mathbb{R}}_+$. If $\mu \in \mathcal{M}(\Omega)$ and $g \in C_c(\Omega)$, one sets $\langle \mu, g\rangle = \int g d\mu$. The following theorems give the entropy inequalities which are satisfied by the approximate solutions, $u_{\mathcal{T},k}$, in the case of the explicit scheme (Theorem 4.1) and of the implicit scheme (Theorem 4.2).

Theorem 4.1 *Assume (3), (5), (6) and condition (9), let $u_{\mathcal{T},k}$ be given by (11), (10), (8); then there exist $\mu_{\mathcal{T},k} \in \mathcal{M}(\mathbb{R}^N \times \mathbb{R}_+)$ and $\mu_{\mathcal{T}} \in \mathcal{M}(\mathbb{R}^N)$ such that:*

$$\begin{aligned}
\int_{\mathbb{R}^N\times\mathbb{R}_+} \Big(& |u_{\mathcal{T},k}(x,t)-\kappa|\varphi_t(x,t) + \\
& (f(u_{\mathcal{T},k}(x,t)\top\kappa) - f(u_{\mathcal{T},k}(x,t)\bot\kappa))\mathbf{v}(x,t)\cdot\nabla\varphi(x,t)\Big)dxdt \quad + \\
\int_{\mathbb{R}^N} & |u_0(x)-\kappa|\varphi(x,0)dx \quad \geq \\
-\int_{\mathbb{R}^N\times\mathbb{R}_+} & \Big(|\varphi_t(x,t)| + |\nabla\varphi(x,t)|\Big)d\mu_{\mathcal{T},k}(x,t) - \int_{\mathbb{R}^N}\varphi(x,0)d\mu_{\mathcal{T}}(x),
\end{aligned} \tag{67}$$

$$\forall \kappa \in \mathbb{R}, \quad \forall \varphi \in C_c^\infty(\mathbb{R}^N \times \mathbb{R}_+, \mathbb{R}_+).$$

The measures $\mu_{\mathcal{T},k}$ and $\mu_{\mathcal{T}}$ satisfy the following properties:

1. *For all $R > 0$ and $T > 0$, there exists C depending only on $\mathbf{v}$, F, u_0, α, ξ, R, T such that, for $h \leq R$:*
$$\mu_{\mathcal{T},k}(B(0,R)\times[0,T]) \leq C\sqrt{h}. \tag{68}$$
2. *The measure $\mu_{\mathcal{T}}$ is the measure of density $|u_0(\cdot) - u_{\mathcal{T},0}(\cdot)|$ w.r.t. the Lebesgue measure. If $u_0 \in BV(\mathbb{R}^N)$, then, for all $R > 0$, there exists D depending only on u_0, α and R such that:*
$$\mu_{\mathcal{T}}(B(0,R)) \leq Dh. \tag{69}$$

Remark 4.1 *Let u be the weak entropy solution to (1)-(2). Then (67) is satisfied with u instead of $u_{\mathcal{T},k}$ and $\mu_{\mathcal{T},k} = 0$ and $\mu_{\mathcal{T}} = 0$.*

PROOF of Theorem 4.1

Let $\varphi \in C_c^\infty(\mathbb{R}^N \times \mathbb{R}_+, \mathbb{R}_+)$ and $\kappa \in \mathbb{R}$. Let $T > 0$ and $R > 0$ such that $\varphi(x,t) \neq 0$ implies $|x| \leq R - h$ and $t \leq T$. Let us multiply (65) by $\frac{1}{m(p)}\int_{nk}^{(n+1)k}\int_p \varphi(x,t)dxdt$, and sum the result for all $p \in \mathcal{T}$ and $n \in \mathbb{N}$. One obtains:

$$T_1 + T_2 \leq 0, \tag{70}$$

with ($N_T = \max\{n \in \mathbb{N}, n \leq T/k\}$),

$$T_1 = \sum_{n=0}^{N_T}\sum_{p\in\mathcal{T}_R} \frac{|u_p^{n+1}-\kappa| - |u_p^n - \kappa|}{k}\int_{nk}^{(n+1)k}\int_p \varphi(x,t)dxdt, \tag{71}$$

and

$$T_2 = \sum_{n=0}^{N_T} \sum_{(p,q)\in\mathcal{E}_R^n} \Big[\begin{array}{l} \frac{v_{p,q}^n}{m(p)} \int_{nk}^{(n+1)k} \int_p \varphi(x,t)dxdt \\ \Big(F(u_p^n \top \kappa, u_q^n \top \kappa) - f(u_p^n \top \kappa) - F(u_p^n \bot \kappa, u_q^n \bot \kappa) + f(u_p^n \bot \kappa)\Big) \quad - \\ \frac{v_{p,q}^n}{m(q)} \int_{nk}^{(n+1)k} \int_q \varphi(x,t)dxdt \\ \Big(F(u_p^n \top \kappa, u_q^n \top \kappa) - f(u_q^n \top \kappa) - F(u_p^n \bot \kappa, u_q^n \bot \kappa) + f(u_q^n \bot \kappa)\Big) \quad - \\ \frac{v_{q,p}^n}{m(p)} \int_{nk}^{(n+1)k} \int_p \varphi(x,t)dxdt \\ \Big(F(u_q^n \top \kappa, u_p^n \top \kappa) - f(u_p^n \top \kappa) - F(u_q^n \bot \kappa, u_p^n \bot \kappa) + f(u_p^n \bot \kappa)\Big) \quad + \\ \frac{v_{q,p}^n}{m(q)} \int_{nk}^{(n+1)k} \int_q \varphi(x,t)dxdt \\ \Big(F(u_q^n \top \kappa, u_p^n \top \kappa) - f(u_q^n \top \kappa) - F(u_q^n \bot \kappa, u_p^n \bot \kappa) + f(u_q^n \bot \kappa)\Big)\Big]. \end{array} \tag{72}$$

One has to prove:

$$T_{10} + T_{20} \le \int_{\mathbb{R}^N \times \mathbb{R}_+} \Big(|\varphi_t(x,t)| + |\nabla\varphi(x,t)|\Big) d\mu_{\mathcal{T},k}(x,t) + \int_{\mathbb{R}^N} \varphi(x,0) d\mu_{\mathcal{T}}(x), \tag{73}$$

for some convenient measures $\mu_{\mathcal{T},k}$ and $\mu_{\mathcal{T}}$, where T_{10}, T_{20} are defined as follows:

$$T_{10} = -\int_{\mathbb{R}^N \times \mathbb{R}_+} |u_{\mathcal{T},k}(x,t) - \kappa| \varphi_t(x,t) dxdt - \int_{\mathbb{R}^N} |u_0(x) - \kappa| \varphi(x,0) dx, \tag{74}$$

$$T_{20} = -\int_{\mathbb{R}^N \times \mathbb{R}_+} \Big((f(u_{\mathcal{T},k}(x,t) \top \kappa) - f(u_{\mathcal{T},k}(x,t) \bot \kappa)) \mathbf{v}(x,t) \cdot \nabla\varphi(x,t) \Big) dxdt. \tag{75}$$

In order to prove (73), one compares T_1 and T_{10} (this will give $\mu_{\mathcal{T}}$, and a part of $\mu_{\mathcal{T},k}$) and one compares T_2 and T_{20} (this will give another part of $\mu_{\mathcal{T},k}$).
Estimate (22) (in the comparison of T_1 and T_{10}) and estimate (21) (in the comparison of T_2 and T_{20}) will be used in order to obtain (68).

Comparison of T_1 and T_{10}

We have, using the definition of $u_{\mathcal{T},k}$ and introducing the function $u_{\mathcal{T},0}(x) = u_p^0$, for all $x \in p$:

$$\begin{array}{l} T_{10} = \displaystyle\sum_{n=0}^{N_T} \sum_{p\in\mathcal{T}_R} \frac{|u_p^{n+1} - \kappa| - |u_p^n - \kappa|}{k} \int_{nk}^{(n+1)k} \int_p \varphi(x,(n+1)k) dxdt \ + \\ \displaystyle\int_{\mathbb{R}^N} (|u_{\mathcal{T},0}(x) - \kappa| - |u_0(x) - \kappa|) \varphi(x,0) dx. \end{array} \tag{76}$$

The function $|\cdot - \kappa|$ is Lipschitz continuous with Lipschitz constant equal to 1, we then obtain:

$$\begin{array}{ll} |T_1 - T_{10}| \le & \displaystyle\sum_{n=0}^{N_T} \sum_{p\in\mathcal{T}_R} \frac{|u_p^{n+1} - u_p^n|}{k} \int_{nk}^{(n+1)k} \int_p |\varphi(x,(n+1)k) - \varphi(x,t)| dxdt \ + \\ & \displaystyle\int_{\mathbb{R}^N} |u_0(x) - u_{\mathcal{T},0}(x)| \varphi(x,0) dx, \end{array} \tag{77}$$

which leads to:

$$\begin{array}{ll} |T_1 - T_{10}| \le & \displaystyle\sum_{n=0}^{N_T} \sum_{p\in\mathcal{T}_R} |u_p^{n+1} - u_p^n| \int_{nk}^{(n+1)k} \int_p |\varphi_t(x,t)| dxdt \ + \\ & \displaystyle\int_{\mathbb{R}^N} |u_0(x) - u_{\mathcal{T},0}(x)| \varphi(x,0) dx. \end{array} \tag{78}$$

Inequality (78) gives:

$$|T_1 - T_{10}| \le \int_{\mathbb{R}^N \times \mathbb{R}_+} |\varphi_t(x,t)| d\nu_{\mathcal{T},k}(x,t) + \int_{\mathbb{R}^N} \varphi(x,0) d\mu_{\mathcal{T}}(x), \tag{79}$$

where the measures $\mu_{\mathcal{T}} \in \mathcal{M}(\mathbb{R}^N)$ and $\nu_{\mathcal{T},k} \in \mathcal{M}(\mathbb{R}^N \times \mathbb{R}_+)$ are defined, by their action on $C_c(\mathbb{R}^N)$ and $C_c(\mathbb{R}^N \times \mathbb{R}_+)$, as follows:

$$\langle \mu_{\mathcal{T}}, g\rangle = \int_{\mathbb{R}^N} |u_0(x) - u_{\mathcal{T},0}(x)| g(x) dx, \ \forall g \in C_c(\mathbb{R}^N), \tag{80}$$

$$\begin{aligned} &\langle \nu_{\mathcal{T},k}, g\rangle = \sum_{n\in\mathbb{N}} \sum_{p\in\mathcal{T}} |u_p^{n+1} - u_p^n| \int_{nk}^{(n+1)k} \int_p g(x,t) dx dt, \\ &\forall g \in C_c(\mathbb{R}^N \times \mathbb{R}_+). \end{aligned} \tag{81}$$

The measures $\mu_{\mathcal{T}}$ and $\nu_{\mathcal{T},k}$ are absolutely continuous w.r.t. the Lebesgue measure. Indeed, one has $d\mu_{\mathcal{T}}(x) = |u_0(x) - u_{\mathcal{T},0}(x)| dx$ and $d\nu_{\mathcal{T},k}(x,t) = (\sum_{n\in\mathbb{N}} \sum_{p\in\mathcal{T}} |u_p^{n+1} - u_p^n| 1_{p\times[nk,(n+1)k[}) dx dt$ (where 1_Ω denotes the characteristic function of Ω).

If $u_0 \in BV(\mathbb{R}^N)$, the measure $\mu_{\mathcal{T}}$ satisfies (69). The function D depends on $|u_0|_{BV}$ and α, with, $|u_0|_{BV}$ = $\sup\{\int u_0(x) \mathrm{div}\varphi(x) dx,\ \varphi \in C_c^\infty(\mathbb{R}^N, \mathbb{R}^N);\ |\varphi(x)| \le 1,\ \forall x \in \mathbb{R}^N\}$.

The measure $\nu_{\mathcal{T},k}$ satisfies (68), with $\nu_{\mathcal{T},k}$ instead of $\mu_{\mathcal{T},k}$, thanks to (22) and condition (9) (which gives $k \le C_0 h$, where C_0 depends only on $\mathbf{v}$, F, u_0, α, ξ).

Comparison of T_2 and T_{20}

Using $\mathrm{div}\mathbf{v} = 0$, and gathering (75) by edges, we get:

$$\begin{aligned} T_{20} = -\sum_{n=0}^{N_T} \sum_{(p,q)\in\mathcal{E}_R^n} \Big[& \Big((f(u_p^n \top \kappa) - f(u_p^n \bot \kappa)) - (f(u_q^n \top \kappa) - f(u_q^n \bot \kappa)) \Big) \\ & \int_{\sigma_{p,q}} \int_{nk}^{(n+1)k} \Big(\mathbf{v}(\gamma,t) \cdot \mathbf{n}_{p,q} \varphi(\gamma,t) \Big) d\gamma dt \Big]. \end{aligned} \tag{82}$$

We can now, in (82), write $\mathbf{v}(\gamma,t) \cdot \mathbf{n}_{p,q} = (\mathbf{v}(\gamma,t) \cdot \mathbf{n}_{p,q} \top 0) + (\mathbf{v}(\gamma,t) \cdot \mathbf{n}_{p,q} \bot 0)$. We introduce the differences of the average of φ on p and on $\sigma_{p,q}$:

$$r_{p,q}^{n+} = \Big|\frac{v_{p,q}^n}{k\ m(p)} \int_{nk}^{(n+1)k} \int_p \varphi(x,t) dx dt - \frac{1}{k} \int_{nk}^{(n+1)k} \int_{\sigma_{p,q}} \Big((\mathbf{v}(\gamma,t) \cdot \mathbf{n}_{p,q}) \top 0 \Big) \varphi(\gamma,t) d\gamma dt\Big|, \tag{83}$$

and

$$r_{p,q}^{n-} = \Big|\frac{v_{q,p}^n}{k\ m(p)} \int_{nk}^{(n+1)k} \int_p \varphi(x,t) dx dt + \frac{1}{k} \int_{nk}^{(n+1)k} \int_{\sigma_{p,q}} \Big((\mathbf{v}(\gamma,t) \cdot \mathbf{n}_{p,q}) \bot 0 \Big) \varphi(\gamma,t) d\gamma dt\Big|. \tag{84}$$

From (72) and (82), one gets

$$\begin{aligned} & |T_2 - T_{20}| \le \sum_{n=0}^{N_T} k \sum_{(p,q)\in\mathcal{E}_R^n} \Big[\\ & r_{p,q}^{n+} \Big(F(u_p^n \top \kappa, u_q^n \top \kappa) - f(u_p^n \top \kappa) + F(u_p^n \bot \kappa, u_q^n \bot \kappa) - f(u_p^n \bot \kappa) \Big) \quad + \\ & r_{q,p}^{n-} \Big(F(u_p^n \top \kappa, u_q^n \top \kappa) - f(u_q^n \top \kappa) + F(u_p^n \bot \kappa, u_q^n \bot \kappa) - f(u_q^n \bot \kappa) \Big) \quad + \\ & r_{p,q}^{n-} \Big(f(u_p^n \top \kappa) - F(u_q^n \top \kappa, u_p^n \top \kappa) + f(u_p^n \bot \kappa) - F(u_q^n \bot \kappa, u_p^n \bot \kappa) \Big) \quad + \\ & r_{q,p}^{n+} \Big(f(u_q^n \top \kappa) - F(u_q^n \top \kappa, u_p^n \top \kappa) + f(u_q^n \bot \kappa) - F(u_q^n \bot \kappa, u_p^n \bot \kappa) \Big) \Big]. \end{aligned} \tag{85}$$

For all $\kappa \in \mathbb{R}$, the following inequality holds:

$$0 \leq F(u_p^n \top \kappa, u_q^n \top \kappa) - f(u_p^n \top \kappa) \leq \max_{u_q^n \leq c \leq d \leq u_p^n} (F(d,c) - f(d)), \tag{86}$$

more precisely, one has $F(u_p^n \top \kappa, u_q^n \top \kappa) - f(u_p^n \top \kappa) = 0$, if $\kappa \geq u_p^n$, and one has $F(u_p^n \top \kappa, u_q^n \top \kappa) - f(u_p^n \top \kappa) = F(d,c) - f(d)$ with $c = \kappa$ and $d = u_p^n$ if $\kappa \in [u_q^n, u_p^n]$, and with $c = u_q^n$ and $d = u_p^n$ if $\kappa \leq u_q^n$. In the same way, we can assert that:

$$0 \leq F(u_p^n \bot \kappa, u_q^n \bot \kappa) - f(u_p^n \bot \kappa) \leq \max_{u_q^n \leq c \leq d \leq u_p^n} (F(d,c) - f(d)). \tag{87}$$

The same analysis can be applied to the other six terms of (85).
To conclude the estimate on $|T_2 - T_{20}|$, it remains to estimate the four quantities $r_{\sigma\pm}^{n\pm}$. This will be done with convenient measures applied to $|\nabla\varphi|$ and $|\varphi_t|$. In order to estimate $r_{\sigma+}^{n+}$, for instance, one remarks that:

$$\begin{aligned}
r_{p,q}^{n+} &\leq \frac{1}{k^2 m(p)} \int_{nk}^{(n+1)k} \int_{nk}^{(n+1)k} \int_p \int_{\sigma_{p,q}} |\varphi(x,t) - \varphi(\gamma,s)| \Big((\mathbf{v}(\gamma,s) \cdot \mathbf{n}_{p,q}) \top 0 \Big) d\gamma dx dt ds \\
&= \frac{1}{k^2 m(p)} \int_{nk}^{(n+1)k} \int_{nk}^{(n+1)k} \int_p \int_{\sigma_{p,q}} \int_0^1 |\nabla\varphi(x + \theta(\gamma - x), t + \theta(s-t)) \cdot (\gamma - x) + \\
&\qquad \varphi_t(x + \theta(\gamma - x), t + \theta(s-t))(s-t)| \\
&\qquad \Big((\mathbf{v}(\gamma,s) \cdot \mathbf{n}_{p,q}) \top 0 \Big) d\theta d\gamma dx dt ds \\
&\leq \frac{1}{k^2 m(p)} \int_{nk}^{(n+1)k} \int_{nk}^{(n+1)k} \int_p \int_{\sigma_{p,q}} \int_0^1 \Big(h |\nabla\varphi(x + \theta(\gamma - x), t + \theta(s-t))| + \\
&\qquad k |\varphi_t(x + \theta(\gamma - x), t + \theta(s-t))| \Big) \\
&\qquad \Big((\mathbf{v}(\gamma,s) \cdot \mathbf{n}_{p,q}) \top 0 \Big) d\theta d\gamma dx dt ds.
\end{aligned} \tag{88}$$

This leads to the definition of a measure $\mu_{p,q}^{n+}$, given by its action on $C_c(\mathbb{R}^N \times \mathbb{R}_+)$:

$$\begin{aligned}
\langle \mu_{p,q}^{n+}, g \rangle = \frac{1}{k^2 m(p)} \int_{nk}^{(n+1)k} \int_{nk}^{(n+1)k} \int_p \int_{\sigma_{p,q}} \int_0^1 & \Big((h+k) g(x + \theta(\gamma - x), t + \theta(s-t)) \Big) \\
& \Big((\mathbf{v}(\gamma,s) \cdot \mathbf{n}_{p,q}) \top 0 \Big) d\theta d\gamma dx dt ds,
\end{aligned} \tag{89}$$

for all $g \in C_c(\mathbb{R}^N \times \mathbb{R}_+)$.
We define in the same way $\mu_{p,q}^{n-}$ and we finally define the measure $\mu_{\mathcal{T},k}$:

$$\begin{aligned}
\langle \mu_{\mathcal{T},k}, g \rangle = \langle \nu_{\mathcal{T},k}, g \rangle + \sum_{n=0}^{N_T} k \sum_{(p,q) \in \mathcal{E}_R^n} \Big[& \Big(\max_{u_q^n \leq c \leq d \leq u_p^n} (F(d,c) - f(d)) \Big) \langle \mu_{p,q}^{n+}, g \rangle \quad + \\
& \Big(\max_{u_q^n \leq c \leq d \leq u_p^n} (F(d,c) - f(c)) \Big) \langle \mu_{q,p}^{n-}, g \rangle \quad + \\
& \Big(\max_{u_q^n \leq c \leq d \leq u_p^n} (f(d) - F(c,d)) \Big) \langle \mu_{p,q}^{n-}, g \rangle \quad + \\
& \Big(\max_{u_q^n \leq c \leq d \leq u_p^n} (f(d) - F(c,d)) \Big) \langle \mu_{q,p}^{n+}, g \rangle \quad \Big].
\end{aligned} \tag{90}$$

Note that the measure $\mu_{\mathcal{T},k}$ does not appear to be absolutely continuous with respect to the Lebesgue measure.
Thanks to (21), and convenient estimates on $\mu_{p,q}^{n\pm}$, one gets (68).
Finally, from (79), (85) and the definition of $\mu_{\mathcal{T},k}$ (that is (90)), one deduces (67). ∎

The following theorem investigates the case of the implicit scheme.

Theorem 4.2 *Assume (3), (5) and (6). Let* $\{u_p^n, n \in \mathbb{N}, p \in \mathcal{T}\}$ *such that* $u_p^n \in [U_m, U_M]$ *be the solution of (12), (8) (existence and uniqueness of such a solution are given by Proposition 3.1). Let* $u_{\mathcal{T},k}$ *be given by (13). Assume that* $\mathbf{v}$ *does not depend on* t *and that* $u_0 \in BV(\mathbb{R}^N)$*. Then, there exist* $\mu_{\mathcal{T},k} \in \mathcal{M}(\mathbb{R}^N \times \mathbb{R}_+)$ *and* $\mu_{\mathcal{T}} \in \mathcal{M}(\mathbb{R}^N)$ *such that:*

$$\begin{aligned}
\int_{\mathbb{R}^N\times\mathbb{R}_+}\Big(\quad & |u_{\mathcal{T},k}(x,t)-\kappa|\varphi_t(x,t)+ \\
& (f(u_{\mathcal{T},k}(x,t)\top\kappa)-f(u_{\mathcal{T},k}(x,t)\bot\kappa))\mathbf{v}(x,t)\cdot\nabla\varphi(x,t)\Big)dxdt \quad + \\
\int_{\mathbb{R}^N} \quad & |u_0(x)-\kappa|\varphi(x,0)dx \qquad\qquad\qquad\qquad \geq \\
-\int_{\mathbb{R}^N\times\mathbb{R}_+} \quad & \Big(|\varphi_t(x,t)|+|\nabla\varphi(x,t)|\Big)d\mu_{\mathcal{T},k}(x,t) - \int_{\mathbb{R}^N}\varphi(x,0)d\mu_{\mathcal{T}}(x),
\end{aligned} \tag{91}$$

$$\forall\kappa\in\mathbb{R},\quad \forall\varphi\in C_c^\infty(\mathbb{R}^N\times\mathbb{R}_+,\mathbb{R}_+).$$

The measures $\mu_{\mathcal{T},k}$ *and* $\mu_{\mathcal{T}}$ *satisfy the following properties:*

1. *For all* $R>0$ *and* $T>0$*, there exists* C *depending only on* $\mathbf{v}$*,* F*,* u_0*,* α*,* R*,* T *such that, for* $h \leq R$*:*
$$\mu_{\mathcal{T},k}(B(0,R)\times[0,T]) \leq C(k+\sqrt{h}). \tag{92}$$
2. *The measure* $\mu_{\mathcal{T}}$ *is the measure of density* $|u_0(\cdot)-u_{\mathcal{T},0}(\cdot)|$ *w.r.t. the Lebesgue measure and, for all* $R>0$*, there exists* D *depending only on* u_0*,* α *and* R *such that:*
$$\mu_{\mathcal{T}}(B(0,R)) \leq Dh. \tag{93}$$

PROOF

Similarly to the proof of Theorem 4.1, we introduce a test function $\varphi \in C_c^\infty(\mathbb{R}^N\times\mathbb{R}_+,\mathbb{R}_+)$ and $\kappa\in\mathbb{R}$. We multiply (66) by $\frac{1}{m(p)}\int_{nk}^{(n+1)k}\int_p \varphi(x,t)dxdt$, and sum the result for all $p\in\mathcal{T}$ and $n\in\mathbb{N}$. We then define T_1 and T_2 such that $T_1+T_2\leq 0$ using equations (71) and (72) in which we replace u_p^n by u_p^{n+1} and u_q^n by u_q^{n+1}. Therefore we obtain (79), where the measure $\nu_{\mathcal{T},k}$ is such that, for all $R>0$ and $T>0$, there exists C_ν depending only on $\mathbf{v}$, F, u_0, α, R, T such that:

$$\nu_{\mathcal{T},k}(B(0,R)\times[0,T]) \leq Ck, \tag{94}$$

using Lemma 3.2, which holds if $\mathbf{v}$ does not depend on t. For the same reason, the treatment of T_2 leads to the definition of a measure $\mu_{p,q}^{n+}$, given by its action on $C_c(\mathbb{R}^N\times\mathbb{R}_+)$:

$$\begin{aligned}
\langle\mu_{p,q}^{n+},g\rangle = \frac{1}{km(p)}\int_{nk}^{(n+1)k}\int_p\int_{\sigma_{p,q}}\int_0^1 \quad & \Big(h\ g(x+\theta(\gamma-x),t)\Big) \\
& \Big((\mathbf{v}(\gamma)\cdot\mathbf{n}_{p,q})\top 0\Big)d\theta d\gamma dxdt,
\end{aligned} \tag{95}$$

for all $g\in C_c(\mathbb{R}^N\times\mathbb{R}_+)$. This measure contributes to the final expression of $\mu_{\mathcal{T},k}$, which then satisfies (92) thanks to Lemma 3.1. ∎

Remark 4.2 In the case where $\mathbf{v}$ depends on t, Lemma 3.2 cannot be used, and we get inequality (89) again. Then (92) is replaced by

$$\mu_{\mathcal{T},k}(B(0,R)\times[0,T]) \leq C(\frac{k}{\sqrt{h}}+\sqrt{h}), \tag{96}$$

which leads to the result given in Remark 1.3.

5 Error estimate

5.1 The error estimate theorem

Theorem 5.1 *Assume (3) and* $u_0 \in BV(\mathbb{R}^N)$. *Let* $\tilde{u} \in L^\infty(\mathbb{R}^N \times \mathbb{R}_+)$ *such that* $U_m \le \tilde{u} \le U_M$ *a.e.. Assume that there exist* $\mu \in \mathcal{M}(\mathbb{R}^N \times \mathbb{R}_+)$ *and* $\mu_0 \in \mathcal{M}(\mathbb{R}^N)$ *such that*

$$\begin{array}{lll} \displaystyle\int_{\mathbb{R}^N \times \mathbb{R}_+} \Big(& |\tilde{u}(x,t) - \kappa|\varphi_t(x,t) + & \\ & (f(\tilde{u}(x,t)\top\kappa) - f(\tilde{u}(x,t)\bot\kappa))\mathbf{v}(x,t)\cdot\nabla\varphi(x,t)\Big)dxdt & + \\ \displaystyle\int_{\mathbb{R}^N} & |u_0(x) - \kappa|\varphi(x,0)dx & \ge \\ \displaystyle -\int_{\mathbb{R}^N \times \mathbb{R}_+} & \Big(|\varphi_t(x,t)| + |\nabla\varphi(x,t)|\Big)d\mu(x,t) - \displaystyle\int_{\mathbb{R}^N} |\varphi(x,0)|d\mu_0(x), & \end{array} \tag{97}$$

$$\forall \kappa \in \mathbb{R}, \quad \forall \varphi \in C_c^\infty(\mathbb{R}^N \times \mathbb{R}_+, \mathbb{R}_+)$$

Let u be the unique entropy weak solution of (1), that is:

$$\begin{array}{l} \displaystyle\int_{\mathbb{R}^N \times \mathbb{R}_+} \Big[|u(y,s) - \kappa|\varphi_s(y,s) + \Big(f(u(y,s)\top\kappa) - f(u(y,s)\bot\kappa)\Big)\mathbf{v}(y,s)\cdot\nabla\varphi(y,s)\Big]dyds + \\ \displaystyle\int_{\mathbb{R}^N} |u_0(y) - \kappa|\varphi(y,0)dy \ge 0, \quad \forall \kappa \in \mathbb{R}, \quad \forall \varphi \in C_c^\infty(\mathbb{R}^N \times \mathbb{R}_+, \mathbb{R}_+). \end{array} \tag{98}$$

(Note that (98) is equivalent to (97) with u *instead of* $\tilde{u}$ *and* $\mu = 0$, $\mu_0 = 0$.*)*
Then, for all compact subsets E *of* $\mathbb{R}^N \times \mathbb{R}_+$, *there exist* C_e, R *and* T *depending only on* E, $\mathbf{v}$, f *and* u_0 *such that the following inequality holds :*

$$\int_E |\tilde{u}(x,t) - u(x,t)|dxdt \le C_e(\mu_0(B(0,R)) + (\mu(B(0,R)\times[0,T]))^{\frac{1}{2}} + \mu(B(0,R)\times[0,T])). \tag{99}$$

The proof of this theorem (Theorem 5.1) consists in using (97) and (98), making $\kappa = u(y,s)$ in (97), $\kappa = \tilde{u}(x,t)$ in (98) and introducing mollifiers in order to make y close of x and s close of t. This proof is quite technical and will be developed in the following subsections.
From Theorem 4.1 and Theorem 5.1 one deduces easily Theorem 1.1 (which gives an error estimate for the numerical scheme (10), (8)) and Theorem 1.2 (which gives an error estimate for the numerical scheme (12), (8)).

5.2 A preliminary lemma

To prove Theorem 5.1, the first step is the following lemma.

Lemma 5.1 *Assume (3) and* $u_0 \in BV(\mathbb{R}^N)$. *Let* $\tilde{u} \in L^\infty(\mathbb{R}^N \times \mathbb{R}_+)$ *such that* $U_m \le \tilde{u} \le U_M$ *a.e.. Assume that there exist* $\mu \in \mathcal{M}(\mathbb{R}^N \times \mathbb{R}_+)$ *and* $\mu_0 \in \mathcal{M}(\mathbb{R}^N)$ *satisfying (97). Let* u *be the unique solution to (98).*
Then, for all $\psi \in C_c^\infty(\mathbb{R}^N \times \mathbb{R}_+, \mathbb{R}_+)$, *there exists* C, *depending only on* ψ *(more precisely on* $||\psi||_\infty$, $||\psi_t||_\infty$, $||\nabla\psi||_\infty$, *and of the support of* ψ*),* $\mathbf{v}$, f, *and* u_0, *such that:*

$$\left\{ \begin{array}{ll} \displaystyle\int_{\mathbb{R}^N \times \mathbb{R}_+} & \Big[|\tilde{u}(x,t) - u(x,t)|\psi_t(x,t) + \\ & \Big(f(\tilde{u}(x,t)\top u(x,t)) - f(\tilde{u}(x,t)\bot u(x,t))\Big)\ (\mathbf{v}(x,t)\cdot\nabla\psi(x,t))\Big]\ dxdt \ \ge \\ & -C(\mu_0(\{\psi(\cdot,0)\neq 0\}) + (\mu(\{\psi\neq 0\}))^{\frac{1}{2}} + \mu(\{\psi \neq 0\})), \end{array} \right. \tag{100}$$

PROOF of Lemma 5.1

For $p=1$ and $p=N$, one defines $\rho_p \in C_c^\infty(\mathbb{R}^p,\mathbb{R})$ satisfying the following properties:

$$\operatorname{supp}(\rho_p) = \{x\in\mathbb{R}^p;\, \rho_p(x)\neq 0\} \subset \{x\in\mathbb{R}^p; |x|\leq 1\}, \tag{101}$$
$$\rho_p(x)\geq 0,\ \forall x\in\mathbb{R}^p, \tag{102}$$
$$\int_{\mathbb{R}^p}\rho_p(x)dx = 1, \tag{103}$$

and furthermore, for $p=1$:

$$\rho_1(x) = 0,\ \forall x\in\mathbb{R}_+. \tag{104}$$

For $r\in\mathbb{R}$, $r>0$, one defines $\rho_{p,r}(x) = r^p\rho_p(rx)$, for all $x\in\mathbb{R}^p$.
Let $\psi\in C_c^\infty(\mathbb{R}^N\times\mathbb{R}_+,\mathbb{R}_+)$, one sets:

$$\varphi(x,t,y,s) = \psi(x,t)\rho_{N,r}(x-y)\rho_{1,r}(t-s). \tag{105}$$

Note that, for any $(y,s)\in\mathbb{R}^N\times\mathbb{R}_+$, one has $\varphi(\cdot,\cdot,y,s)\in C_c^\infty(\mathbb{R}^N\times\mathbb{R}_+,\mathbb{R}_+)$ and, for any $(x,t)\in\mathbb{R}^N\times\mathbb{R}_+$, one has $\varphi(x,t,\cdot,\cdot)\in C_c^\infty(\mathbb{R}^N\times\mathbb{R}_+,\mathbb{R}_+)$.
Let us take $\varphi(\cdot,\cdot,y,s)$ for the test function φ in (97) and $\varphi(x,t,\cdot,\cdot)$ for the test function φ in (98). We take, in (97), $\kappa = u(y,s)$ and we take, in (98), $\kappa=\tilde u(x,t)$. We then integrate (97) for $(y,s)\in\mathbb{R}^N\times\mathbb{R}_+$, and (98) for $(x,t)\in\mathbb{R}^N\times\mathbb{R}_+$. Adding both inequations yields:

$$E_{11}+E_{12}+E_{13}+E_{14}\geq -E_2, \tag{106}$$

where:

$$E_{11} = \int_{(\mathbb{R}^N\times\mathbb{R}_+)^2}\Big[|\tilde u(x,t)-u(y,s)|\psi_t(x,t)\rho_{N,r}(x-y)\rho_{1,r}(t-s)\Big]\ dxdtdyds, \tag{107}$$

$$\begin{aligned}E_{12} = \int_{(\mathbb{R}^N\times\mathbb{R}_+)^2}\Big[\ &\Big(f(\tilde u(x,t)\top u(y,s)) - f(\tilde u(x,t)\bot u(y,s))\Big)\\ &(\mathbf{v}(x,t)\cdot\nabla\psi(x,t))\ \rho_{N,r}(x-y)\rho_{1,r}(t-s)\Big]dxdtdyds,\end{aligned} \tag{108}$$

$$\begin{aligned}E_{13} = -\int_{(\mathbb{R}^N\times\mathbb{R}_+)^2}\ &\Big(f(\tilde u(x,t)\top u(y,s)) - f(\tilde u(x,t)\bot u(y,s))\Big)\psi(x,t)\\ &((\mathbf{v}(y,s)-\mathbf{v}(x,t)).\nabla\rho_{N,r}(x-y))\rho_{1,r}(t-s)\ dxdtdyds,\end{aligned} \tag{109}$$

$$E_{14} = \int_{\mathbb{R}^N\times\mathbb{R}_+\times\mathbb{R}^N}|u_0(x)-u(y,s)|\psi(x,0)\rho_{N,r}(x-y)\rho_{1,r}(-s)dydsdx \tag{110}$$

and

$$\begin{aligned}E_2 = \ &\int_{\mathbb{R}^N\times\mathbb{R}_+}\int_{\mathbb{R}^N\times\mathbb{R}_+}\Big(\ |\rho_{N,r}(x-y)(\psi_t(x,t)\rho_{1,r}(t-s)+\psi(x,t)\rho'_{1,r}(t-s))|+\\ &\qquad |\ \rho_{1,r}(t-s)(\nabla\psi(x,t)\rho_{N,r}(x-y)+\psi(x,t)\nabla\rho_{N,r}(x-y))|\Big)\\ &\qquad d\mu(x,t)dyds\\ &+\int_{\mathbb{R}^N\times\mathbb{R}_+}\int_{\mathbb{R}^N}\ |\ \psi(x,0)\rho_{N,r}(x-y)\rho_{1,r}(-s)|d\mu_0(x)dyds.\end{aligned} \tag{111}$$

Note that, in order to obtain (106), one does not make use of the fact that the entropy weak solution u of (1) satisfies the initial condition of (1). Indeed, this initial condition appears only in the third term of the left hand side of (98) and, for all $(x,t)\in\mathbb{R}^N\times\mathbb{R}_+$, one has $\varphi(x,t,\cdot,0)=0$. Then, the third term

of the left hand side of (98) is zero when one takes $\varphi(x,t,\cdot,\cdot)$ as test function in (98). The fact that u satisfies the initial condition of (1) will be used in order to get a bound on E_{14}.
One has to study, now, the four terms of (106). In the following, one denotes by C_i $(i \in \mathbb{N})$ various real functions depending only on $\|\psi\|_\infty$, $\|\psi_t\|_\infty$, $\|\nabla\psi\|_\infty$, $\mathbf{v}$, f, and u_0. One sets $K = \{(x,t) \in \mathbb{R}^N \times \mathbb{R}_+;\ \psi(x,t) \neq 0\}$ and $K_0 = \{x \in \mathbb{R}^N;\ \psi(x,0) \neq 0\}$.
Equality (111) leads to:

$$E_2 \leq (r+1)C_1\mu(K) + C_2\mu_0(K_0). \tag{112}$$

Let us handle the term E_{11}. For all $x \in \mathbb{R}^N$ and for all $t \in \mathbb{R}_+$, one has (using (104)):

$$\int_{\mathbb{R}^N\times\mathbb{R}_+} \rho_{N,r}(x-y)\rho_{1,r}(t-s)dsdy = 1. \tag{113}$$

Then,

$$\begin{gathered}|E_{11} - \int_{\mathbb{R}^N\times\mathbb{R}_+} \Big[|\tilde{u}(x,t) - u(x,t)|\psi_t(x,t)\Big]\ dxdt| \leq \\ \int_{(\mathbb{R}^N\times\mathbb{R}_+)^2} \Big[|u(x,t) - u(y,s)||\psi_t(x,t)|\rho_{N,r}(x-y)\rho_{1,r}(t-s)\Big]\ dxdtdyds \leq |\psi_t|_\infty \varepsilon(r,K),\end{gathered} \tag{114}$$

with

$$\varepsilon(r,K) = \sup\{\int_K |u(x,t) - u(x+\eta, t+\tau)|dxdt;\ |\eta| \leq \frac{1}{r},\ 0 \leq \tau \leq \frac{1}{r}\}. \tag{115}$$

Since $u_0 \in BV(\mathbb{R}^N)$, one has $u \in BV(\mathbb{R}^N \times [0,T])$ (for all T), and then

$$\varepsilon(r,K) \leq \frac{C_3}{r}. \tag{116}$$

This gives:

$$|E_{11} - \int_{\mathbb{R}^N\times\mathbb{R}_+} \Big[|\tilde{u}(x,t) - u(x,t)|\psi_t(x,t)\Big]\ dxdt| \leq \frac{C_4}{r}. \tag{117}$$

In the same way, one obtains:

$$\begin{gathered}|E_{12} - \int_{\mathbb{R}^N\times\mathbb{R}_+} \Big(f(\tilde{u}(x,t)\top u(x,t)) - f(\tilde{u}(x,t)\bot u(x,t))\Big) \\ (\mathbf{v}(x,t)\cdot\nabla\psi(x,t))dxdt| \leq C_5\varepsilon(r,K) \leq \tfrac{C_6}{r}.\end{gathered} \tag{118}$$

Let us now turn to E_{13}. We compare this term with:

$$\begin{gathered}E_{13b} = -\int_{(\mathbb{R}^N\times\mathbb{R}_+)^2} \Big(f(\tilde{u}(x,t)\top u(x,t)) - f(\tilde{u}(x,t)\bot u(x,t))\Big)\psi(x,t) \\ (\mathbf{v}(y,s) - \mathbf{v}(x,t))\cdot\nabla\rho_{N,r}(x-y)\rho_{1,r}(t-s)\ dxdtdyds.\end{gathered} \tag{119}$$

Since $\mathrm{div}(\mathbf{v}(\cdot,s) - \mathbf{v}(x,t)) = 0$ (on $\mathbb{R}^N$) for all $x \in \mathbb{R}^N$, $t \in \mathbb{R}_+$ and $s \in \mathbb{R}_+$, one has $E_{13b} = 0$. Therefore, substracting E_{13b} from E_{13} yields:

$$\begin{gathered}E_{13} \leq C_7 \int_{(\mathbb{R}^N\times\mathbb{R}_+)^2} |u(x,t) - u(y,s)|\psi(x,t) \\ |(\mathbf{v}(y,s) - \mathbf{v}(x,t))\cdot\nabla\rho_{N,r}(x-y)|\rho_{1,r}(t-s)\ dxdtdyds.\end{gathered} \tag{120}$$

The right hand side of (120) is then smaller than $C_8\varepsilon(r,K)$, since $|(\mathbf{v}(y,s) - \mathbf{v}(x,t))\cdot\nabla\rho_{N,r}(x-y)|$ is bounded by $C_9 r^N$. Then, with (116), one has:

$$E_{13} \leq \frac{C_{10}}{r}. \tag{121}$$

In order to study E_{14}, let us take in (98), for $x \in \mathbb{R}^N$ fixed, $\varphi(x,y,s) = \psi(x,0)\rho_{N,r}(x-y)\int_s^\infty \rho_{1,r}(-\tau)d\tau$, and $\kappa = u_0(x)$. This choice for φ leads to a function $\varphi(x,\cdot,\cdot)$ of $C^\infty(\mathbb{R}^N \times \mathbb{R}, \mathbb{R}_+)$, with a compact support in $\mathbb{R}^N \times \mathbb{R}_+$. We then integrate the resulting inequality with respect to $x \in \mathbb{R}^N$. We get:

$$-E_{14} + E_{15} + E_{16} \geq 0, \tag{122}$$

with:

$$E_{15} = -\int_{\mathbb{R}^N\times\mathbb{R}_+\times\mathbb{R}^N}\int_s^\infty \begin{array}{l}(f(u(y,s)\top u_0(x)) - f(u(y,s)\bot u_0(x)))\\ \mathbf{v}(y,s)\cdot(\psi(x,0)\nabla\rho_{N,r}(x-y))\rho_{1,r}(-\tau)d\tau dy dx ds,\end{array} \tag{123}$$

$$E_{16} = \int_{\mathbb{R}^N\times\mathbb{R}^N}\int_0^\infty \psi(x,0)\rho_{N,r}(x-y)\rho_{1,r}(-\tau)|u_0(x)-u_0(y)|d\tau dy dx. \tag{124}$$

In order to obtain a bound on E_{15}, one introduces E_{15b} defined as:

$$E_{15b} = \int_{\mathbb{R}^N\times\mathbb{R}_+\times\mathbb{R}^N}\int_s^\infty \begin{array}{l}(f(u(y,s)\top u_0(y)) - f(u(y,s)\bot u_0(y)))\\ (\mathbf{v}(y,s)\cdot\nabla\rho_{N,r}(x-y))\psi(x,0)\rho_{1,r}(-\tau)d\tau dy dx ds,\end{array} \tag{125}$$

Integrating by parts for the x variable yields:

$$E_{15b} = -\int_{\mathbb{R}^N\times\mathbb{R}_+\times\mathbb{R}^N}\int_s^\infty \begin{array}{l}(f(u(y,s)\top u_0(y)) - f(u(y,s)\bot u_0(y)))\\ (\mathbf{v}(y,s)\cdot\nabla\psi(x,0))\rho_{N,r}(x-y)\rho_{1,r}(-\tau)d\tau dy dx ds,\end{array} \tag{126}$$

Then, noting that the time support of this integration is reduced to $[0, 1/r]$, one has:

$$E_{15b} \leq \frac{C_{11}}{r}. \tag{127}$$

Furthermore, one has:

$$|E_{15} + E_{15b}| \leq C_{12}\int_{\mathbb{R}^N\times\mathbb{R}_+\times\mathbb{R}^N}\int_s^\infty |u_0(x)-u_0(y)||\mathbf{v}(y,s)\cdot\nabla\rho_{N,r}(x-y)|\psi(x,0)\rho_{1,r}(-\tau)d\tau dy dx ds,$$

which is bounded by $C_{13}\varepsilon_0(r, K_0)$ (since the time support of the integration is reduced to $[0, 1/r]$) where $\varepsilon_0(r, K_0)$ is defined by:

$$\varepsilon_0(r, K_0) = \sup\{\int_{K_0}|u_0(x) - u_0(x+\eta)|dx;\ |\eta| \leq \frac{1}{r}\}. \tag{128}$$

Since $u_0 \in BV(\mathbb{R}^N)$, one has $\varepsilon_0(r, K_0) \leq \frac{C_{14}}{r}$ and therefore, with (127), $E_{15} \leq \frac{C_{15}}{r}$.
Thanks to the fact that $u_0 \in BV(\mathbb{R}^N)$, it easily seen that the term E_{16} is again bounded by C_{16}/r.
Hence, since $E_{14} \leq E_{15} + E_{16}$,

$$E_{14} \leq \frac{C_{17}}{r}. \tag{129}$$

Using (106), (112), (117), (118),(121), (129), one obtains:

$$\left\{\begin{array}{l} \displaystyle\int_{\mathbb{R}^N\times\mathbb{R}_+} \Big[|\tilde{u}(x,t)-u(x,t)|\psi_t(x,t) + \\ \Big(f(\tilde{u}(x,t)\top u(x,t)) - f(\tilde{u}(x,t)\bot u(x,t))\Big)\ (\mathbf{v}(x,t)\cdot\nabla\psi(x,t))\Big]\ dxdt\ \geq \\ -C_1(r+1)\mu(K) - C_2\mu_0(K_0) - \frac{C_{18}}{r}, \end{array}\right. \tag{130}$$

which, taking $r = \frac{1}{\sqrt{\mu(K)}}$ (or $r\to\infty$ if $\mu(K)=0$), gives (100).
This concludes the proof of the Lemma 5.1. ∎

5.3 Conclusion of the proof of Theorem 5.1

Let K be a compact subset of $\mathbb{R}^N\times\mathbb{R}_+$. One sets $\omega = VM$, where V is given in (3) and M is the Lipschitz constant of f in $[U_m,U_M]$ (indeed, since $f\in C^1(\mathbb{R},\mathbb{R})$, one has $M=\sup\{|f'(s)|; s\in[U_m,U_M]\}$). Let $R>0$, $T\in]0,\frac{R}{\omega}[$ such that $K\subset\cup_{0\leq t\leq T}(B(0,R-\omega t)\times\{t\})$.
Let $\rho\in C^1_c(\mathbb{R}_+,[0,1])$ be a function such that $\rho(r)=1$ if $r\in[0,R]$, $\rho(r)=0$ if $r\in[R+1,\infty[$ and $\rho'(r)\leq 0$, for all $r\in\mathbb{R}_+$. One takes, in (100), ψ defined by $\psi(x,t)=\rho(|x|+\omega t)\frac{T-t}{T}$, if $x\in\mathbb{R}^N$ and $t\in[0,T]$, $\psi(x,t)=0$ if $x\in\mathbb{R}^N$ and $t\geq T$. This function is not in $C^\infty_c(\mathbb{R}^N\times\mathbb{R}_+,\mathbb{R}_+)$, but, using a classical regularisation technique, one proves that one can take such a function in (100).
Then, inequality (100) leads to:

$$\left\{\begin{array}{l} \displaystyle\int_{\mathbb{R}^N\times[0,T]} \Big[|\tilde{u}(x,t)-u(x,t)|\Big(\frac{T-t}{T}\omega\rho'(|x|+\omega t) - \frac{1}{T}\rho(|x|+\omega t)\Big) + \\ \Big(f(\tilde{u}(x,t)\top u(x,t)) - f(\tilde{u}(x,t)\bot u(x,t))\Big)\ \frac{T-t}{T}\rho'(|x|+\omega t)(\mathbf{v}(x,t)\cdot\frac{x}{|x|}))\Big]\ dxdt\ \geq \\ -C(\mu_0(B(0,R+1)) + (\mu(B(0,R+1)\times[0,T]))^{\frac{1}{2}} + \mu(B(0,R+1)\times[0,T])), \end{array}\right. \tag{131}$$

where C, R, T, depend only on K, $\mathbf{v}$, f and u_0.
Since $\omega=VM$ and $\rho'\leq 0$, one has:

$$\begin{array}{l} \Big(f(\tilde{u}(x,t)\top u(x,t)) - f(\tilde{u}(x,t)\bot u(x,t))\Big)\ \frac{T-t}{T}\rho'(|x|+\omega t)(\mathbf{v}(x,t)\cdot\frac{x}{|x|}))\Big) \leq \\ |\tilde{u}(x,t)-u(x,t)|\frac{T-t}{T}\omega(-\rho'(|x|+\omega t)), \end{array} \tag{132}$$

and therefore,

$$\int_K |\tilde{u}(x,t)-u(x,t)|dxdt \leq CT(\mu_0(B(0,R+1)) + (\mu(B(0,R+1)\times[0,T]))^{\frac{1}{2}} + \mu(B(0,R+1)\times[0,T])). \tag{133}$$

This completes the proof of Theorem 5.1. ∎

Recall that from the entropy inequality given in Theorem 4.1 and the error estimate given in Theorem 5.1 one deduces easily Theorem 1.1, which gives an error estimate for the explicit numerical scheme (10), (8), and Theorem 1.2, which gives an error estimate for the implicit numerical scheme (12), (8).

6 Conclusion

Theorem 1.1 gives an error estimate of order $h^{\frac{1}{4}}$ for the approximate solution of a nonlinear hyperbolic equation of the form $u_t + div\mathbf{v}(x,t)f(u)=0$, with initial data in $L^\infty\cap BV$ by the explicit finite volume scheme (10), (8), under a usual CFL condition $k\leq Ch$ (see (9), note that there is no "inverse" CFL condition required here). Note that, in fact, the same estimate holds if u_0 is only locally BV. More generally, if the inital data u_0 is only in L^∞, then one still obtains an error estimate in terms of the quantities $\varepsilon(r,K) = \sup\{\int_K|u(x,t)-u(x+\eta,t+\tau)|dxdt;\ |\eta|\leq\frac{1}{r}, 0\leq\tau\leq\frac{1}{r}\}$ and $\varepsilon_0(r,K_0) = \sup\{\int_{K_0}|u_0(x)-u_0(x+\eta)|dx;\ |\eta|\leq\frac{1}{r}\}$ (see (115) and (128)). This is again an obvious consequence of Theorem 4.1 and Theorem 5.1.

A crucial ingredient of the proof is the "BV weak estimate", namely (21), which is in fact "three times weak" for the following reasons:
(i) the estimate is of order $\frac{1}{\sqrt{h}}$, and not of order 1.
(ii) In the left hand side of (21), the quantity which is associated to the (p,q) interface is zero if f is constant on the interval $[u_p^n - u_q^n]$ thus preventing the appearance of $|u_p^n - u_q^n|$ in the estimate.
(iii) The left hand side of (21) involves $|\mathbf{v} \cdot \mathbf{n}|$ wich depends on the mesh $\mathcal{T}$ and is not uniformly bounded by a positive constant.
Note that a "twice weak BV" estimate in the sense (ii) and (iii), but of order 1, would yield a sharp error estimate, i.e. of order $h^{\frac{1}{2}}$.

In this paper, we also considered the implicit schemes, which seem to be much more widely used in industrial codes in order to ensure their robustness. The implicit case required additional work in order
(i) to prove the existence of the solution to the finite volume scheme,
(ii) to obtain the strong "time BV" estimate (58) in the case where $\mathbf{v}$ does not depend on t.
For $\mathbf{v}$ depending on t, Theorem 1.2 yields an estimate of order $h^{\frac{1}{4}}$ if k behaves as h; however, in the case where $\mathbf{v}$ does not depend on t, then an estimate of order $h^{\frac{1}{4}}$ is obtained for a behaviour of k as $\sqrt{h}$; Indeed, recent numerical experiments have shown that taking k of the order of $\sqrt{h}$ yields results of the same precision than taking k of the order of h, with an obvious reduction of the computational cost.
Note the method described here may also be extended to higher order schemes for the same equation, [1]; other methods have been used for error estimates for higher order schemes with a nonlinearity of the form $F(u)$ [6], [21]. However, it is still an open problem, to our knowledge, to improve the order or the error estimate in the case of higher order schemes.

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