A naïve Riemann solver to compute a non-conservative hyperbolic system

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Abstract. An approximate Riemann solver is presented in this contribution, which enables to compute shock waves in compressible flows using one or two-equation turbulence model.

1. Introduction

Among the numerous turbulence models, the (one- or two-equation) $K-\epsilon$ closed models, are widely used, in spite of their drawbacks, for industrial applications, mainly in the case of incompressible flows. When focusing on compressible turbulence, they have not been studied very much (except in [5], [18], [11]). We focus here on the numerical approximation of the simplest one, which only contains one (non conservative) equation ; this one governs the motion of the turbulent kinetic energy $K =< \rho\, u_i'' \, u_i'' >$ using Favre's averaging formalism.
The numerical approach is based on a new approximate Riemann solver proposed recently in [12] in order to solve complex hyperbolic systems in conservative form. This approach was investigated in detail in [17], and a slightly modified version of the scheme was suggested in [3], and used in [4] and [7] in order to compute some single-phase or two-phase flows. One may wonder now whether this approach applies to hyperbolic systems in non conservative form, and this is the main objective of the present contribution. We examine the capability of the proposed scheme to provide as stable results as those obtained using a Godunov-type solver ([11]), and compare efficiencies.

2. Governing equations

Before we introduce the approximate Riemann solver which is used to compute turbulent compressible flows with shocks, we first recall some basic results connected with a simple one equation closure. The set of equations (see [1]) is obtained using Favre's averaging process ([10]):

$$
\begin{cases}
\rho_{,t} + (\rho\, U_j)_{,j} = 0 \\
(\rho\, U_i)_{,t} + (\rho\, U_i U_j)_{,j} + (P\, \delta_{ij} + \Sigma^v_{ij})_{,j} + (R_{ij})_{,j} = 0 \\
E_{,t} + (U_j\, (E+P))_{,j} + (U_k\, (R_{kj} + \Sigma^v_{kj}))_{,j} = (\sigma_K (\frac{K}{\rho})_{,j})_{,j} + (\sigma_E (\frac{P}{\rho})_{,j})_{,j} \\
K_{,t} + (K\, U_j)_{,j} - (\sigma_K (\frac{K}{\rho})_{,j})_{,j} = -R_{jk}\, U_{j,k} - \epsilon
\end{cases}
\tag{1}
$$

with:

$$
\begin{aligned}
R_{ij} &= \tfrac{2}{3} K\, \delta_{ij} - \mu_t (U_{i,j} + U_{j,i} - \tfrac{2}{3}\, U_{l,l}\, \delta_{ij}) \\
\Sigma^v_{ij} &= -\mu (U_{i,j} + U_{j,i} - \tfrac{2}{3}\, U_{l,l}\, \delta_{ij})
\end{aligned}
\tag{2}
$$

Moreover, it is assumed that the perfect gas state law holds ; hence:

$$
P = (\gamma - 1)(E - \frac{\rho\, U_j\, U_j}{2} - K) \tag{3}
$$

Admissible states are such as:

$$
\rho(x,t) \geq 0\,; \quad P(x,t) \geq 0\,; \quad K(x,t) \geq 0 \tag{4}
$$

Here, ρ stands for the mean density, $\mathbf{U}$ is the mean velocity, E denotes the mean total energy and P is the mean pressure. The Reynolds stress tensor is $\mathbf{R}$, and K is the turbulent kinetic energy ($K = trace(\mathbf{R})/2$) ; γ is the constant ratio of specific heats ; μ is the molecular viscosity. It is assumed that σ_E, σ_K and μ_t are positive bounded functions. The mechanical dissipation ϵ is given through an algebraic relation (see [1]).

We now consider the following entropy-consistent fractional step method which requires solving successively:
Step 1:

$$
\begin{cases}
\rho_{,t} + (\rho\, U_j)_{,j} = 0 \\
(\rho\, U_i)_{,t} + (\rho\, U_i U_j)_{,j} + P_{,i} + \frac{2}{3}\, K_{,i} = -(\Sigma^v_{ij})_{,j} \\
E_{,t} + (U_j\, (E + P + \frac{2}{3}\, K))_{,j} = -(U_k\, \Sigma^v_{kj})_{,j} \\
K_{,t} + (K\, U_j)_{,j} + \frac{2}{3}\, K\, U_{j,j} = 0
\end{cases}
\tag{5}
$$

Step 2:

$$
\begin{cases}
\rho_{,t} = 0 \\
(\rho\, U_i)_{,t} = -(\Sigma^t_{ij})_{,j} \\
E_{,t} = -(U_k\, \Sigma^t_{kj})_{,j} + (\sigma_K (K/\rho)_{,j})_{,j} + (\sigma_E (P/\rho)_{,j})_{,j} \\
K_{,t} = -\epsilon + (\sigma_K (\frac{K}{\rho})_{,j})_{,j} - \frac{1}{2}\, \Sigma^t_{kj}\, (U_{k,j} + U_{j,k})
\end{cases}
\tag{6}
$$

noting: $\Sigma^t_{kj} = -\mu_t (U_{k,j} + U_{j,k} - \frac{2}{3}\, U_{l,l}\, \delta_{kj})$.

3. Recalling some basic results

In a 2D framework, we introduce the "conservative" state variable:

$$\mathbf{Z}^t = (\rho, \rho U, \rho V, E, K).$$

Given some vector $\mathbf{n}$ in $\mathbb{R}^2$ with unit norm, we also introduce:

$$U_n = \mathbf{U}.\mathbf{n}$$

We know ([11], [16]) that we may derive an entropy inequality for viscous solutions of (1)(2)(3), using the following entropy-entropy flux pair:

$$\eta = -\rho \, \ln(P\rho^{-\gamma})$$
$$\mathbf{F}_\eta^{nv}(\mathbf{Z}) = \mathbf{U}\,\eta.$$

If $\mathbf{n}$ is normal to the shock which speed is σ, the entropy inequality:

$$-\sigma\,[\eta] + [\mathbf{F}_\eta^{nv}.\mathbf{n}] \leq 0, \tag{7}$$

where [.] represents the jump through the discontinuity, enables to connect states through genuinely non linear fields in a physically meaningful way, when neglecting viscous effects and investigating the one-dimensional Riemann problem. Actually ([11]), the non viscous non conservative first-order differential system (namely the left hand side of step 1):

$$\mathbf{Z}_{,t} + \sum_{i=1}^{2}(\mathbf{F}_i(\mathbf{Z}))_{,i} + \sum_{i=1}^{2}\mathbf{A}_\mathbf{i}^\mathbf{nc}(\mathbf{Z})\,\mathbf{Z}_{,i} = 0 \tag{8}$$

is a non strictly hyperbolic system, whose eigenvalues are:

$$\lambda_1 = U_n - c_1\,; \quad \lambda_2 = \lambda_3 = \lambda_4 = U_n\,; \quad \lambda_5 = U_n + c_1,$$

setting:

$$c_1 = \left(\frac{\gamma\,P}{\rho} + \frac{10}{9}\frac{K}{\rho}\right)^{1/2}.$$

The 1-wave and the 5-wave are genuinely non linear. If we restrict to sufficiently weak shocks and connect states $(1/\rho, U, V, P, K)$ across some discontinuity travelling in the x-direction with a linear path (see [6], [8], [14], [15]), we get the following approximate jump conditions:

$$\left\{\begin{array}{l} -\sigma\,[\rho] + [\rho\,U] = 0 \\ -\sigma\,[\rho\,U] + [\rho\,U^2 + 2\,K/3 + P] = 0 \\ -\sigma\,[\rho\,V] + [\rho\,U\,V] = 0 \\ -\sigma\,[E] + [U(E + 2\,K/3 + P)] = 0 \\ -\sigma\,[K] + [U\,K] + 2/3\,\bar{K}\,[U] = 0 \end{array}\right. \tag{9}$$

where σ stands for the speed of the travelling discontinuity, and if L and R subscripts refer to the left and right hand side of the discontinuity, we have noted:

$$[\phi] = \phi_R - \phi_L, \quad \bar{\phi} = (\phi_L + \phi_R)/2.$$

Obviously, when the turbulence is set to zero, (9) identifies with exact jump conditions for Gas Dynamics. It may then be proved ([11]) that the 1D-Riemann problem associated with (8) and approximate jump conditions (9) has a unique entropy-consistent solution, provided that the following condition on the data on each side of the initial discontinuity holds

$$U_R - U_L < X_L + X_R$$

setting

$$X_i = \int_0^{\rho_i} \frac{1}{\rho} \left(\frac{\gamma P_i}{\rho_i} (\frac{\rho}{\rho_i})^{\gamma-1} + \frac{10}{9} \frac{K_i}{\rho_i} (\frac{\rho}{\rho_i})^{2/3} \right)^{1/2} d\rho, \quad i = L \; or \; R.$$

In addition, the approximate solution is in agreement with (4). Based on the latter results, a Godunov-type scheme was previously proposed, which unfortunately requires solving two non linear scalar equations with two unknowns, and the whole results in a rather expensive solver. The main objective here is to derive an approximate Godunov scheme which does not have this drawback.

4. An approximate Riemann solver

Restricting to the two dimensional frame, we present below an extension ([3], [4]) of the approximate Godunov solver which was recently proposed in [12] and [17]. We introduce $\mathbf{B_1}(\mathbf{Z})$ and $\mathbf{B_2}(\mathbf{Z})$:

$$\mathbf{B_1}(\mathbf{Z}) = \frac{d\mathbf{F}_1}{d\mathbf{Z}}(\mathbf{Z}) + \mathbf{A_1^{nc}}(\mathbf{Z})\,; \quad \mathbf{B_2}(\mathbf{Z}) = \frac{d\mathbf{F}_2}{d\mathbf{Z}}(\mathbf{Z}) + \mathbf{A_2^{nc}}(\mathbf{Z})$$

noting:

$$\mathbf{F}_1(\mathbf{Z}) = \begin{bmatrix} \rho U \\ \rho U^2 + 2K/3 + P \\ \rho U V \\ U(E + 2K/3 + P) \\ U K \end{bmatrix}, \quad \mathbf{F}_2(\mathbf{Z}) = \begin{bmatrix} \rho V \\ \rho U V \\ \rho V^2 + 2K/3 + P \\ V(E + 2K/3 + P) \\ V K \end{bmatrix} \tag{10}$$

We introduce the (explicit) scheme:

$$|\Omega_i|(\mathbf{Z}_i^{n+1} - \mathbf{Z}_i^n) + \Delta t \sum_{j \in V(i)} \Gamma_{ij} \left(\sum_{k=1}^{2} n_k \, \mathbf{F}_k(\mathbf{Z}(\mathbf{Y}^*)) \right)_{ij} + \Delta t \int_{\Omega_i} \mathbf{S}(\mathbf{Z}(\mathbf{Y}^*))\, d\Omega = 0$$

where:

$$\int_{\Omega_i} \mathbf{S}^t(\mathbf{Z}(\mathbf{Y}^*))\, d\Omega = (0,\, 0,\, 0,\, 0,\, \frac{2}{3} \hat{K}_i \sum_{j \in V(i)} \Gamma_{ij} (\mathbf{U}^*.\mathbf{n})_{ij})$$

$$\hat{K}_i = \sum_{j \in V(i)} K_{ij}^* \,/\, card(V(i))$$

Δt is a constant time step, $V(i)$ refers to the neighbouring cells of Ω_i, and $\mathbf{n}$ stands for the outward (from Ω_i to Ω_j) normal vector of the interface (which length is

Γ_{ij}) between cells Ω_i and Ω_j. Approximate starred values of states at interface are obtained as follows. Owing to the invariance under frame rotation, and neglecting as usual transverse variations, we may examine the Riemann problem associated with the one-dimensional hyperbolic system in the x-direction in the following fully non conservative form:

$$\mathbf{Z}_{,t} + \mathbf{B_1}(\mathbf{Z})\ \mathbf{Z}_{,x} = 0$$

Restricting to regular solutions, we introduce the following change of variables $\mathbf{Z} \mapsto \mathbf{Y}(\mathbf{Z})$:

$$\mathbf{Y}^t = (\tau, U, V, P, K)$$

(see also [4], [9], [20]), where $\tau = 1/\rho$. Thus comes:

$$\mathbf{Y}_{,t} + (\mathbf{Z_Y}(\mathbf{Y}))^{-1}\,\mathbf{B_1}(\mathbf{Z}(\mathbf{Y}))\,\mathbf{Z_Y}(\mathbf{Y})\ \mathbf{Y}_{,x} = 0$$

or:

$$\mathbf{Y}_{,t} + \mathbf{C_1}(\mathbf{Y})\,\mathbf{Y}_{,x} = 0 \tag{11}$$

with:

$$\mathbf{Y} = \begin{pmatrix} \tau \\ U \\ V \\ P \\ K \end{pmatrix} \quad \mathbf{C_1}(\mathbf{Y}) = \begin{pmatrix} U & -\tau & 0 & 0 & 0 \\ 0 & U & 0 & \tau & 2\tau/3 \\ 0 & 0 & U & 0 & 0 \\ 0 & \gamma P & 0 & U & 0 \\ 0 & 5K/3 & 0 & 0 & U \end{pmatrix} \tag{12}$$

The Riemann problem associated with the one-dimensional non linear hyperbolic system (11)(12) and initial conditions:

$$\mathbf{Y}(x, t=0) = \begin{cases} \mathbf{Y}_L = \mathbf{Y}(\mathbf{Z}_L) & if\ x < 0 \\ \mathbf{Y}_R = \mathbf{Y}(\mathbf{Z}_R) & if\ x > 0 \end{cases} \tag{13}$$

is linearized as follows:

$$\mathbf{Y}_{,t} + \mathbf{C_1}(\bar{\mathbf{Y}})\,\mathbf{Y}_{,x} = 0 \tag{14}$$

where:

$$\bar{\mathbf{Y}} = (\mathbf{Y}_R + \mathbf{Y}_L)/2.$$

System (14)(12) contains three distinct linearly degenerated fields, and its solution only requires computing five real coefficients noted $\bar{\alpha}_i$ (for: i=1 to 5), setting:

$$\mathbf{Y}_R - \mathbf{Y}_L = \sum_{i=1}^{5} \bar{\alpha}_i\ \bar{\mathbf{r}}_i \tag{15}$$

where $\{\bar{\mathbf{r}}_i\}$ represent the basis of right eigenvectors of the matrix $\mathbf{C_1}(\bar{\mathbf{Y}})$. Intermediate states are then uniquely defined, and hence the state $\mathbf{Y}^*$ at the initial location of the data discontinuity. The above mentionned approximate Riemann solver has the following properties:

Remark 4.1. *Numerical intermediate states (indexed 1 and 4) are in agreement with the solution of the Riemann problem since:*

$$\begin{aligned} &U_1 = U_4 \\ &P_1 + \tfrac{2}{3}\,K_1 = P_4 + \tfrac{2}{3}\,K_4 \end{aligned} \tag{16}$$

Proof. The right eigenvectors of $\mathbf{C_1}(\bar{\mathbf{Y}})$ associated with the eigenvalue $\lambda_2 = \lambda_3 = \lambda_4 = \bar{U}$ are:

$$\bar{\mathbf{r}}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \bar{\mathbf{r}}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \bar{\mathbf{r}}_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -2/3 \\ 1 \end{bmatrix}$$

Hence, the jump between the two intermediate states is given by:

$$\mathbf{Y}_4 - \mathbf{Y}_1 = \begin{bmatrix} \bar{\alpha}_2 \\ 0 \\ \bar{\alpha}_3 \\ -2\bar{\alpha}_4/3 \\ \bar{\alpha}_4 \end{bmatrix}$$

We obtain (16) by considering on one hand the second equation, and on the other hand, the two last equations. □

The expression of vectors or coefficients which occur in the numerical scheme for solving linearized problem (14)(12)(13) is:

$$\bar{\mathbf{r}}_1 = \begin{bmatrix} \bar{\tau} \\ \tilde{c_1} \\ 0 \\ -\gamma\bar{P} \\ -5\bar{K}/3 \end{bmatrix}, \quad \bar{\mathbf{r}}_5 = \begin{bmatrix} \bar{\tau} \\ -\tilde{c_1} \\ 0 \\ -\gamma\bar{P} \\ -5\bar{K}/3 \end{bmatrix}$$

$$\bar{\alpha}_1 = \frac{1}{2\,\tilde{c_1}}\left([U] - \frac{\bar{\tau}}{\tilde{c_1}}\,[P + \frac{2}{3}\,K]\right), \quad \bar{\alpha}_5 = \frac{1}{2\,\tilde{c_1}}\left(-[U] - \frac{\bar{\tau}}{\tilde{c_1}}\,[P + \frac{2}{3}\,K]\right)$$

where: $\tilde{c_1}^2 = \bar{\tau}\,(\gamma\bar{P} + \frac{10}{9}\,\bar{K})$.

Remark 4.2. *Consider one travelling (single) discontinuity. Then, approximate jump conditions (9), associated with the non linear system are equivalent to those, associated with the linear hyperbolic system (14), i.e.:*

$$-\sigma\,[\mathbf{Y}] + \mathbf{C_1}(\bar{\mathbf{Y}})\,[\mathbf{Y}] = 0$$

The proof is similar to the one presented in [4] in the case of Euler system.
Let's note that even for zero turbulence, this is not consistency with the integral form of the conservation law.

Remark 4.3. *This scheme obviously requires an entropy correction at sonic points in rarefaction waves.*

Remark 4.4. *When focusing on the Riemann problem associated with:*

$$[U] = [P + 2\,K/3] = 0, \tag{17}$$

present scheme identifies with Godunov scheme.

Proof. Under condition (17), we have $\bar{\alpha}_1 = \bar{\alpha}_5 = 0$,
hence $\mathbf{Y}_1 = \mathbf{Y}_L$ and $\mathbf{Y}_4 = \mathbf{Y}_R$.
Assuming that $U = U_L = U_R > 0$ provides $\mathbf{Y}^* = \mathbf{Y}_L$. We deduce:

$$\mathbf{F}(\mathbf{Z}(\mathbf{Y}^*)) = \mathbf{F}(\mathbf{Z}_L).$$

In particular, we have also $U^* = U$, and so:

$$\int_{\Omega_i} \mathbf{S}^t(\mathbf{Z}(\mathbf{Y}^*))\, d\Omega = (0,0,0,0,0).$$

□

5. Numerical results

When focusing on conservative hyperbolic systems, it has been underlined in [3], [7], [12], [17] that the behaviour of Godunov scheme and scheme proposed in [12] are quite similar. In particular, restricting to isentropic or non isentropic Euler equations, the measured rates of convergence for all variables are similar for both schemes.

The present numerical scheme has been applicated on several test cases, in particular on the computation of Riemann problems with: two rarefaction waves, two shock waves, one rarefaction wave and one shock wave. Only this last configuration is presented here for which initial conditions are given by:
test 1 :

$$(\rho_L,\ u_L,\ P_L,\ K_L) = (1,\ 0,\ 5.10^5,\ 5000)$$
$$(\rho_R,\ u_R,\ P_R,\ K_R) = (0.125,\ 0,\ 10^4,\ 100)$$

test 2 :

$$(\rho_L,\ u_L,\ P_L,\ K_L) = (1,\ 0,\ 10^5,\ 100)$$
$$(\rho_R,\ u_R,\ P_R,\ K_R) = (0.125,\ 0,\ 10^4,\ 1000)$$

and γ is fixed to 1.4. The first turbulent shock tube experiment presents a supersonic zone, whereas the second remains everywhere subsonic. In both cases, the initial turbulent Mach numbers (namely the square root of the ratio K/P) are rather small.
Regular one dimensional meshes have been used. Several profiles corresponding to test 1 (resp. test 2) are plotted in figure 1 (resp. figure 3) in the case of a mesh containing 20000 points. It is emphasized that the preservation of Riemann invariants U and $P + 2K/3$ through the contact discontinuity is good, even of much coarser grids. Also, Riemann invariants of the 1-wave, namely $P\,\rho^{-\gamma}$ and $K\,\rho^{-5/3}$, are rather well predicted (see profiles on figure 2 in the case of test 1). Let's note on figure 3 that the values of P are different on both sides of the contact discontinuity.

6. Conclusion

Present scheme is based on an initial proposition to compute conservative systems (Euler equations with real gas state law). Its stability actually enables to perform computations of strong rarefaction waves, contact discontinuities and shocks, even when the ratio K/P is not negligible, which is much useful for industrial purposes. It is obviously much cheaper than the basic Godunov scheme introduced in [11]. Obviously, derivation of meaningful approximate jump conditions still requires more works (see [2]).

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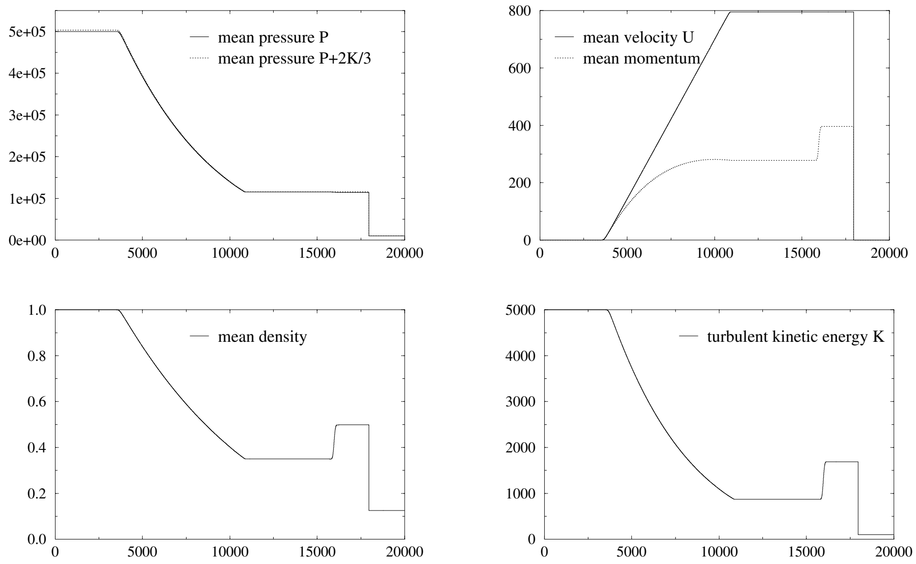

FIGURE 1. Profiles for a supersonic turbulent shock tube test.

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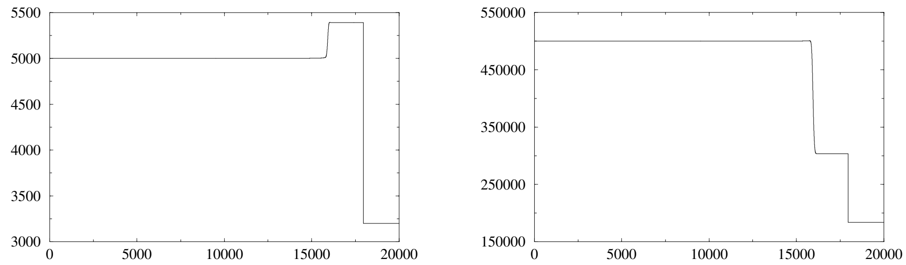

(a) (b)

FIGURE 2. Profiles of 1-Riemann invariants for a supersonic turbulent shock tube test: (a) $P\,\rho^{-\gamma}$, (b) $K\,\rho^{-5/3}$

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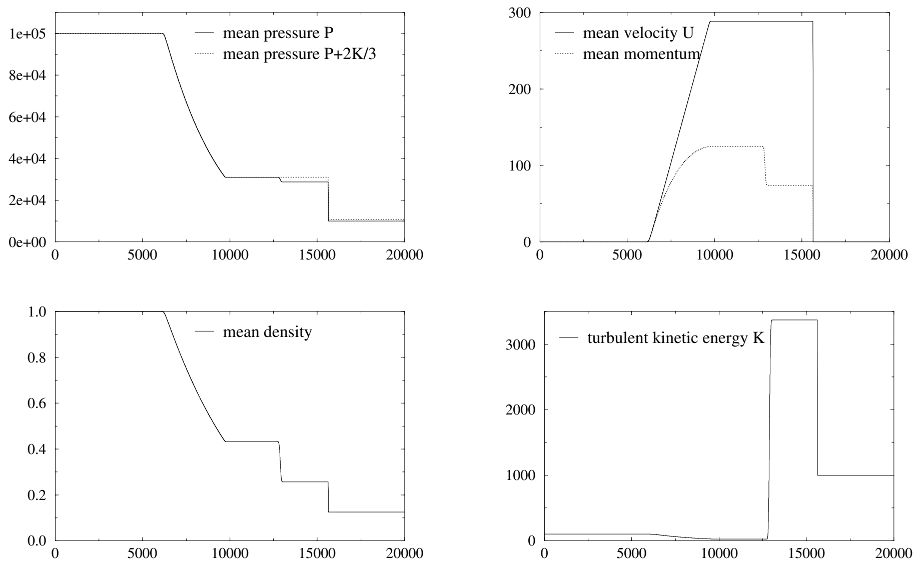

FIGURE 3. Profiles for a subsonic turbulent shock tube test.