Finite volume approximation of elliptic problems with irregular data

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abstract We prove here the convergence of a cell-centered finite volume scheme for the discretization on a non-structured grid of the Laplace equation with irregular data towards the weak solution of the equation.

Keys words *Finite volumes scheme, non-structured mesh, diffusion equation, irregular data.*

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1. Introduction

We are interested here in proving the convergence of the finite volume method in the case of the following model equation:

$$-\Delta u = \mu, \quad \text{in } \Omega, \tag{1}$$

with Dirichlet boundary condition:

$$u = 0, \quad \text{in } \partial\Omega, \tag{2}$$

where

Assumption 1
1. Ω *is an open bounded polygonal subset of* $\mathbb{R}^d$, $d = 2$ *or* 3,
2. $\mu \in L^p(\Omega)$ *for* $p \in [1, +\infty]$ *or* μ *is a signed bounded measure.*

Such problems arise for instance when modelling heat transfers in the presence of electric current in which case the heat term due to ohmic loss writes

$\mu = \sigma\nabla\Phi\nabla\Phi$ where $\sigma \in L^\infty(\Omega)$ is the electric conductivity and $\Phi \in H^1(\Omega)$ is the electric potential; hence $\mu \in L^1(\Omega)$ (see e.g. [FH 94]). Another field where such a problem arises is in oil reservoir simulation, where the dimension of the well is often small enough with respect to the size or the domain of simulation so that it is modelled by a Dirac measure in the two-dimensional case ($d = 2$).

The purpose of the proposed presentation is to show that the finite volume method is well adapted to this type of problem; we can show in particular that the analysis tools recently developped by Boccardo, Gallouët [BG 89] for the study of nonlinear partial differential equations with measure data can be adapted to show the strong convergence as the size of the mesh tends to 0 of the approximate finite volume solution in $W_0^{1,p}$ for any $p \in [d, \frac{d}{d-1}[$ towards a weak solution of (1)-(2) which is a function u from Ω to $\mathbb{R}$ satisfying:

$$\left\{ \begin{array}{l} u \in \cap_{1\leq p<\frac{d}{d-1}} W_0^{1,p}(\Omega), \\ \displaystyle\int_\Omega \nabla u(x)\nabla\varphi(x)dx = \int_\Omega v(x)d\mu(x),\ \forall v \in \cup_{q>d} W_0^{1,q}(\Omega). \end{array} \right. \tag{3}$$

Remark 1

The Laplace operator is considered here for the sake of simplicity, but more general elliptic operators are possible to handle, for instance operators of the form $-div(a(u)\nabla u)$ with adequate assumptions on a.

A by-product of the convergence analysis which is presented here is the existence of a solution of (3).

2. The finite volume scheme

The finite volume scheme is found by integrating equation (1) on a given control volume of a discretization mesh and finding an approximation of the fluxes on the control volume boundary in terms of the discrete unknowns. Let us first give the assumptions which are needed on the mesh.

Definition 1 (Admissible meshes) *Let Ω be an open bounded polygonal subset of $\mathbb{R}^d$. An admissible finite volume mesh of Ω, denoted by $\mathcal{T}$, is given by a family of "control volumes", which are polygonal convex subsets of Ω (with positive measure), a family of subsets of $\overline{\Omega}$ contained in hyperplanes of $\mathbb{R}^d$, denoted by $\mathcal{E}$ (these are the edges of the control volumes), with strictly positive $(d-1)$-dimensional measure, and a family of points of Ω denoted by $\mathcal{P}$ satisfying the following properties (in fact, we shall denote, somewhat incorrectly, by $\mathcal{T}$ the family of control volumes):*

(i) the set of all control volumes is a partition of Ω;

(ii) *For any $K \in \mathcal{T}$, there exists a subset $\mathcal{E}_K$ of $\mathcal{E}$ such that $\partial K = \overline{K} \setminus K = \cup_{\sigma \in \mathcal{E}_K} \overline{\sigma}$. Let $\mathcal{E} = \cup_{K \in \mathcal{T}} \mathcal{E}_K$.*

(iii) *For any $(K, L) \in \mathcal{T}^2$ with $K \neq L$, either the $(d-1)$-dimensional Lebesgue measure of $\overline{K} \cap \overline{L}$ is 0 or $\overline{K} \cap \overline{L} = \overline{\sigma}$ for some $\sigma \in \mathcal{E}$, which will then be denoted by $K|L$.*

(iv) *The family $\mathcal{P} = (x_K)_{K \in \mathcal{T}}$ is such that $x_K \in \overline{K}$ (for all $K \in \mathcal{T}$) and, if $\sigma = K|L$, it is assumed that $x_K \neq x_L$, and that the straight line $\mathcal{D}_{K,L}$ going through x_K and x_L is orthogonal to $K|L$.*

In the sequel, the following notations are used. The mesh size is defined by: $\mathrm{size}(\mathcal{T}) = \sup\{\mathrm{diam}(K),\ K \in \mathcal{T}\}$. For any $K \in \mathcal{T}$ and $\sigma \in \mathcal{E}$, $\mathrm{m}(K)$ is the d-dimensional measure of K and $\mathrm{m}(\sigma)$ the $(d-1)$-dimensional measure of σ. The set of interior (resp. boundary) edges is denoted by $\mathcal{E}_{\mathrm{int}}$ (resp. $\mathcal{E}_{\mathrm{ext}}$), that is $\mathcal{E}_{\mathrm{int}} = \{\sigma \in \mathcal{E};\ \sigma \not\subset \partial\Omega\}$ (resp. $\mathcal{E}_{\mathrm{ext}} = \{\sigma \in \mathcal{E};\ \sigma \subset \partial\Omega\}$). The set of neighbours of K is denoted by $\mathcal{N}(K)$, that is $\mathcal{N}(K) = \{L \in \mathcal{T};\ \exists \sigma \in \mathcal{E}_K,\ \overline{\sigma} = \overline{K} \cap \overline{L}\}$. For any $K \in \mathcal{T}$ and $\sigma \in \mathcal{E}_K$ we denote by $d_{K,\sigma}$ the Euclidean distance between x_K and σ. For any $\sigma \in \mathcal{E}$, we define $d_\sigma = d_{K,\sigma} + d_{L,\sigma}$ if $\sigma = K|L \in \mathcal{E}_{\mathrm{int}}$ (in which case d_σ is the Euclidean distance between x_K and x_L) and $d_\sigma = d_{K,\sigma}$ if $\sigma \in \mathcal{E}_{\mathrm{ext}} \cap \mathcal{E}_K$.

For any $\sigma \in \mathcal{E}$; the "transmissibility" through σ is defined by $\tau_\sigma = \mathrm{m}(\sigma)/d_\sigma$ if $d_\sigma \neq 0$ and $\tau_\sigma = 0$ if $d_\sigma = 0$. In some results and proofs given below, there are summations over $\sigma \in \mathcal{E}_0$, with $\mathcal{E}_0 = \{\sigma \in \mathcal{E};\ d_\sigma \neq 0\}$. For simplicity, (in these results and proofs) $\mathcal{E} = \mathcal{E}_0$ is assumed.

We may now introduce the space of piecewise constant functions associated with an admissible mesh and some "discrete $W_0^{1,p}$" norm for this space. This discrete norm will be used to obtain some estimates on the approximate solution given by a finite volume scheme.

Definition 2 (Discrete norm) *Let Ω be an open bounded polygonal subset of $\mathbb{R}^d$, $d = 2$ or 3, and let $\mathcal{T}$ be an admissible mesh. Define $X(\mathcal{T})$ as the set of functions from Ω to $\mathbb{R}$ which are constant over each control volume of the mesh.*

For $u \in X(\mathcal{T})$, and $p \in [1, +\infty)$, define the discrete $W_0^{1,p}$ norm by

$$||u||_{1,p,\mathcal{T}} = \Big(\sum_{\sigma \in \mathcal{E}} \mathrm{m}(\sigma) d_\sigma (\frac{D_\sigma u}{d_\sigma})^p\Big)^{\frac{1}{p}} \tag{4}$$

where, for any $\sigma \in \mathcal{T}$,

$D_\sigma u = |u_K - u_L|$ if $\sigma \in \mathcal{E}_{\mathrm{int}}$, $\sigma = K|L$,

$D_\sigma u = |u_K|$ if $\sigma \in \mathcal{E}_{\mathrm{ext}} \cap \mathcal{E}_K$,

where u_K denotes the value taken by u on the control volume K and the sets $\mathcal{E}$, $\mathcal{E}_{\mathrm{int}}$, $\mathcal{E}_{\mathrm{ext}}$ and $\mathcal{E}_K$ are defined in Definition 1.

Let $\mathcal{T}$ be an admissible mesh. Let us now define a finite volume scheme to discretize (1)-(2).

Let $(u_K)_{K\in\mathcal{T}}$ denote the discrete unknowns associated with the control volumes $K \in \mathcal{T}$. In order to describe the scheme in the most general way, one introduces some auxiliary unknowns namely the fluxes $F_{K,\sigma}$, for all $K \in \mathcal{T}$ and $\sigma \in \mathcal{E}_K$, and some (expected) approximation of u on an edge σ, denoted by u_σ, for all $\sigma \in \mathcal{E}$. For $K \in \mathcal{T}$ and $\sigma \in \mathcal{E}_K$, let $\mathbf{n}_{K,\sigma}$ denote the normal unit vector to σ outward to K and $v_{K,\sigma} = \int_\sigma \mathbf{v}(x)\cdot\mathbf{n}_{K,\sigma} d\gamma(x)$. Note that $d\gamma$ is the integration symbol for the $(d-1)$-dimensional Lebesgue measure on the considered hyperplane.

We may now write the finite volume scheme for the discretization of Problem (1)-(2) under Assumption 1 as the following set of equations:

$$\sum_{\sigma\in\mathcal{E}_K} F_{K,\sigma} = \mu(K),\ \forall K \in \mathcal{T}, \tag{5}$$

where $F_{K,\sigma}$ is defined by

$$F_{K,\sigma} = -F_{L,\sigma}, \forall \sigma \in \mathcal{E}_{\rm int}, \text{ if } \sigma = K|L, \tag{6}$$

$$F_{K,\sigma} d_{K,\sigma} = -\mathrm{m}(\sigma)(u_\sigma - u_K), \forall \sigma \in \mathcal{E}_K, \forall K \in \mathcal{T}, \tag{7}$$

and

$$u_\sigma = 0,\ \forall \sigma \in \mathcal{E}_{\rm ext}. \tag{8}$$

Note that the values u_σ for $\sigma \in \mathcal{E}_{\rm int}$ are auxiliary values which may be eliminated so that (5)-(8) leads to a linear system of N equations with N unknowns, namely the $(u_K)_{K\in\mathcal{T}}$, with $N = \mathrm{card}(\mathcal{T})$.

3. Existence and estimates for the approximate solution

Let us first prove the existence of the approximate solution and an estimate on this solution. This estimate will yield convergence thanks to a compactness theorem which we recall below.

Lemma 1 (Existence and estimate) *Under Assumptions 1, let $\mathcal{T}$ be an admissible mesh in the sense of Definition 1, and let:*

$$\zeta = \min_{K\in\mathcal{T}} \min_{\sigma\in\mathcal{E}_K} \frac{d_{K,\sigma}}{d_\sigma}, \tag{9}$$

then there exists a solution $(u_K)_{K\in\mathcal{T}}$ to the system of equations (5)-(8).

Furthermore, let $p \in [1, \frac{d}{d-1})$, and let $u_\mathcal{T} \in X(\mathcal{T})$ be defined by $u_\mathcal{T}(x) = u_K$ for a.e. $x \in K$, and for any $K \in \mathcal{T}$; there exists $C \in \mathbb{R}$, only depending on Ω, ζ, p and μ, such that

$$\|u_\mathcal{T}\|_{1,p,\mathcal{T}} \le C \text{ and } \|u_\mathcal{T}\|_{L^{p^*}(\Omega)} \le C. \tag{10}$$

Proof of Lemma 1

The existence and uniqueness to the solution of the scheme was proved in e.g. [H 95]. Let us now turn to the estimate. For $\theta \in (1, +\infty)$, let φ be the bounded function from $\mathbb{R}$ to $\mathbb{R}$ defined by $\varphi(s) = \int_0^s \frac{dt}{1+|t|^\theta}$. Multiplying (5) by $\varphi(u_K)$ and summing over $K \in \mathcal{T}$ yields

$$\sum_{K\in\mathcal{T}}\sum_{\sigma\in\mathcal{E}_K} F_{K,\sigma}\varphi(u_K) = \sum_{K\in\mathcal{T}} \mu(K)\varphi(u_K).$$

By a discrete integration by part and from the fact that

$$\varphi(u_K) - \varphi(u_L) = (u_K - u_L)\int_0^1 \varphi'(u_K + t(u_L - u_K))dt,$$

one obtains:

$$\sum_{\sigma\in\mathcal{E}} \frac{\mathrm{m}(\sigma)}{d_\sigma} a_\sigma (D_\sigma u)^2 \le ||\varphi||_\infty \mu(\Omega) \tag{11}$$

where $a_\sigma = \int_0^1 \varphi'(u_K + t(u_L - u_K))dt$ if $\sigma = K|L \in \mathcal{E}_{\text{int}}$, and $a_\sigma = \int_0^1 \varphi'((1-t)u_K)dt$ if $\sigma \in \mathcal{E}_{\text{ext}} \cap \mathcal{E}_K$.

Now for $1 \le p < 2$, by Hölder's inequality and from (11)

$$||u_\mathcal{T}||^p_{1,p,\mathcal{T}} \le (||\varphi||_\infty \mu(\Omega))^{\frac{p}{2}} \Big(\sum_{\sigma\in\mathcal{E}} d_\sigma \mathrm{m}(\sigma) a_\sigma^{-\frac{p}{2-p}}\Big)^{\frac{2-p}{2}}.$$

Let us reorder the summation over the control volumes in the right-hand-side and remark that by definition of ζ, one has $d_\sigma \le \frac{d_{K,\sigma}}{\zeta}$. This yields the existence of $C_1 \in \mathbb{R}_+$ depending only on μ, Ω, θ, p and ζ, such that

$$||u_\mathcal{T}||^p_{1,p,\mathcal{T}} \le C_1\Big(1 + \sum_{K\in\mathcal{T}} \mathrm{m}(K)|u_K|^{\frac{\theta p}{2-p}}\Big)^{\frac{2-p}{2}}. \tag{12}$$

Using a discrete Sobolev inequality (the proof of which is similar to the one proved in [EGH 97] or [CGH 98]), there exists $C_2 \in \mathbb{R}_+$ depending only on p such that:

$$||u||_{p^*} \le C_2 ||u_\mathcal{T}||_{1,p,\mathcal{T}}. \tag{13}$$

From (12) and (13), there exists $C_3 \in \mathbb{R}_+$ depending only on θ, p and ζ such that

$$||u||^p_{p^*} \le C_3(1 + ||u_\mathcal{T}||^{\frac{\theta}{2}p}_{\frac{\theta p}{2-p}}). \tag{14}$$

Hence, for $p < \frac{d}{d-1}$, one has $\frac{p}{2-p} < p^*$, so that one may choose $\theta \in (1,2)$ such that $\frac{\theta p}{2-p} \le p^*$. Since $p > \frac{\theta}{2}p$, from (14) and (12), there exists C depending only on μ, Ω, p and ζ, such that:

$$||u||_{p^*} \le C \text{ and } ||u_\mathcal{T}||_{1,p,\mathcal{T}} \le C.$$

4. Convergence

Let us now show the convergence of approximate solutions obtained by the above finite volume scheme when the size of the mesh tends to 0. One uses Lemma 1 together with the Kolmogorov compactness theorem given at the end of this chapter to prove the convergence result. In order to use the Kolmogorov compactness theorem, one needs the following lemma.

Lemma 2 (Estimate on the space translates) *Let Ω be an open bounded set of $\mathbb{R}^d$, $d = 2$ or 3. Let $\mathcal{T}$ be an admissible mesh and $u \in X(\mathcal{T})$. One defines $\tilde{u}$ by $\tilde{u} = u$ a.e. on Ω, and $\tilde{u} = 0$ a.e. on $\mathbb{R}^d \setminus \Omega$. Then there exists $C > 0$, only depending on Ω, such that*

$$\|\tilde{u}(\cdot + \eta) - \tilde{u}\|^p_{L^p(\mathbb{R}^d)} \leq \|u\|^p_{1,p,\mathcal{T}} |\eta| \Big(|\eta| + C \operatorname{size}(\mathcal{T}) \Big)^{p-1}, \forall \eta \in \mathbb{R}^d. \tag{15}$$

The proof of this lemma is an easy adaptation of the proof which is available in [EGH 97] of [EGH 99].

We are now able to state the convergence theorem.

Theorem 1 (Convergence) *Under Assumption 1, let $\mathcal{T}$ be an admissible mesh. Let $(u_K)_{K\in\mathcal{T}}$ be the solution of the system given by equations (5)-(8). Define $u_\mathcal{T} \in X(\mathcal{T})$ by $u_\mathcal{T}(x) = u_K$ for a.e. $x \in K$, and for any $K \in \mathcal{T}$. Let $(\mathcal{T}_n)_{n\in\mathbb{N}}$ be a sequence of admissible meshes such that $\operatorname{size}(\mathcal{T}_n) \to 0$ as $n \to +\infty$ and such that*

$$\zeta = \inf_{n\in\mathbb{N}} \min_{K\in\mathcal{T}_n} \min_{\sigma\in\mathcal{E}_K} \frac{d_{K,\sigma}}{d_\sigma} > 0, \tag{16}$$

then there exists a subsequence of $(u_{\mathcal{T}_n})_{n\in\mathbb{N}}$, still denoted $(u_{\mathcal{T}_n})_{n\in\mathbb{N}}$, which converges in $L^p(\Omega)$ for $p < \frac{d}{d-2}$ to a weak solution $u \in \cap_{1\leq q<\frac{d}{d-1}} W_0^{1,q}(\Omega)$ of Problem (3) as $\operatorname{size}(\mathcal{T}) \to 0$.

Remark 2 *In the case of the uniqueness of a solution to (3), for instance if $d = 2$, or if $d = 3$ and Ω is convex (see e.g. [G 97]), then the whole sequence $(u_{\mathcal{T}_n})_{n\in\mathbb{N}}$ tends to the solution of (3), and therefore $u_\mathcal{T} \to u$ in $L^p(\Omega)$ as $\operatorname{size}(\mathcal{T}) \to 0$ under the condition that there exists zeta > 0 such that $\zeta_\mathcal{T} = \min_{K\in\mathcal{T}} \min_{\sigma\in\mathcal{E}_K} \frac{d_{K,\sigma}}{d_\sigma} \geq \zeta$ for all $\mathcal{T}$.*

Proof of Theorem 1

Let Y be the set of approximate solutions, that is the set of functions $u_\mathcal{T}$ as defined in Theorem 1 where $\mathcal{T}$ is an admissible mesh which satisfies (16). Thanks to Lemma 1, for any $p < \frac{d}{d-2}$, there exists $C_1 \in \mathbb{R}$, only depending on Ω, μ, ζ_0 and p, such that $\|u_\mathcal{T}\|_{L^p(\Omega)} \leq C_1$ for all $u_\mathcal{T} \in Y$. Then, thanks to Lemma 2 and to the Kolmogorov compactness result (see e.g. [EGH 99] or [EGH 97] for the case $p = 2$), the set Y is relatively compact in $L^p(\Omega)$. Now by Lemma 1, we know that for any $q \in [1, \frac{d}{d-1})$, there exists $C_2 \in \mathbb{R}$, only depending on Ω, μ, ζ_0 and q, such that $\|u_\mathcal{T}\|_{1,q,\mathcal{T}} \leq C_2$. Hence, adapting a result of [EGH 97], one may show that any possible limit (in $L^p(\Omega)$) of a sequence $(u_{\mathcal{T}_n})_{n\in\mathbb{N}} \subset Y$ (such that size$(\mathcal{T}_n) \to 0$) belongs to $W_0^{1,q}(\Omega)$. Therefore, there remains to prove that if $(u_{\mathcal{T}_n})_{n\in\mathbb{N}} \subset Y$ converges towards some $u \in W_0^{1,q}(\Omega)$ in $L^p(\Omega)$ and size$(\mathcal{T}_n) \to 0$ (as $n \to \infty$), then u is a solution to (3). We prove this result below, omiting the index n, that is assuming $u_\mathcal{T} \to u$ in $L^p(\Omega)$ as size$(\mathcal{T}) \to 0$.

Let $\psi \in C_c^\infty(\Omega)$ and let size$(\mathcal{T})$ be small enough so that $\psi(x) = 0$ if $x \in K$ and $K \in \mathcal{T}$ is such that $\partial K \cap \partial\Omega \neq \emptyset$. Multiplying (5) by $\psi(x_K)$, and summing the result over $K \in \mathcal{T}$ yields

$$\sum_{K\in\mathcal{T}} \sum_{L\in\mathcal{N}(K)} \tau_{K|L}(u_K - u_L)\psi(x_K) = \sum_{K\in\mathcal{T}} \mu(K)\psi(x_K) \tag{17}$$

Since the set $\{K, K \in \mathcal{T}\}$ is a partition of Ω,

$$\sum_{K\in\mathcal{T}} \mu(K)\psi(x_K) = \int_\Omega \psi_\mathcal{T}(x)d\mu(x),$$

where $\psi_\mathcal{T}$ is defined from Ω to $\mathbb{R}$ by $\psi_\mathcal{T}(x) = \psi(x_K)$ for $x \in K$; since $\psi \in C_c^\infty(\Omega)$, by the Lebesgue dominated convergence theorem, one has:

$$\int_\Omega \psi_\mathcal{T}(x)d\mu(x) \to \int_\Omega \psi(x)d\mu(x) \text{ as size}(\mathcal{T}) \to 0.$$

Now, using the same technique as in the variational framework (see [EGH 97] or [EGH 99]), one has:

$$\sum_{K\in\mathcal{T}} \sum_{L\in\mathcal{N}(K)} \tau_{K|L}(u_K - u_L)\psi(x_K) \to -\int u\Delta\psi \text{ as size}(\mathcal{T}) \to 0.$$

Hence, letting size$(\mathcal{T}) \to 0$ in (17) yields that $u \in \cap_{p\in[1,\frac{d}{d-1})} W_0^{1,p}(\Omega)$ satisfies

$$-\int_\Omega u(x)\Delta\psi(x)dx = \int_\Omega \psi(x)d\mu(x),\ \forall\psi \in C_c^\infty(\Omega),$$

which, in turn, yields (3) thanks to the fact that $u \in W_0^{1,p}(\Omega)$, and to the density of $C_c^\infty(\Omega)$ in $W_0^{1,q}(\Omega)$.

This proves that $u_\mathcal{T} \to u$ in $L^p(\Omega)$ as size$(\mathcal{T}) \to 0$, where u is a solution (in $\cap_{p\in[1,\frac{d}{d-1})} W_0^{1,p}(\Omega)$) to (3) and concludes the proof of Theorem 1.

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