$$+|x(t)|^2 - \|x_t\|^2 + \int_{t-h}^{t} \frac{d}{dt}\|x^2(\cdot)\|^{[s,t]}\, ds$$
$$\le (-2a(t) + |b(t)| + \alpha(t) + 1)|x(t)|^2 + 2|x(t)||f(t,x_t)| - \|x_t\|^2$$
$$- \frac{1}{h}(1 - |b(t)|h^2)\int_{t-h}^{t} \alpha(s)|x(s)|^2\, ds + \int_{t-h}^{t} \frac{d}{dt}\|x^2(\cdot)\|^{[s,t]}\, ds.$$

For each fixed s, if $\|x^2(\cdot)\|^{[s,t]} = |x(\theta)|^2$ with $s \le \theta < t$ and $|x(\tau)| < |x(\theta)|$ for all $\theta < \tau \le t$, then $\frac{d}{dt}\|x^2(\cdot)\|^{[s,t]} = 0$ (see Hale [4], p. 127). Now suppose that $\|x^2(\cdot)\|^{[s,t]} = |x(t)|^2$. Then $|x(t)| \ge |x(\tau)|$ for $\tau \in [s,t]$ and

$$\frac{d}{dt}\|x^2(\cdot)\|^{[s,t]} \le -2a(t)|x(t)|^2 + |b(t)|\,|x(t)|^2$$
$$+ |b(t)|h\int_{t-h}^{t} |C(s,x(s))|^2\, ds + 2|x(t)|\,|f(t,x_t)|$$
$$\le |b(t)|h\int_{t-h}^{t} |C(s,x(s))|^2\, ds + 2|x(t)|\,|f(t,x_t)|.$$

Thus,

$$\begin{aligned} V'_{(E)}(t,x_t) &\le (-2a(t) + |b(t)| + \alpha(t) + 1)|x(t)|^2 + 2(1+h)|x(t)|\,|f(t,x_t)| \\ &- \|x_t\|^2 - \frac{1}{h}[1 - |b(t)|(1+h)h^2]\int_{t-h}^{t} \alpha(s)|x(s)|^2\, ds \\ &\le (-2a(t) + |b(t)| + \alpha(t) + 1)|x(t)|^2 + 2(1+h)p\|x_t\|^2 - \|x_t\|^2 \\ &\le -\beta(t)|x(t)|^2. \end{aligned} \tag{11}$$

It follows from (10) that

$$(1+h)|\phi(0)|^2 \le V(t,\phi) \le 2(1+L)[|\phi(0)| + |\phi|_2]^2 + h\|\phi\|^2. \tag{12}$$

Let $W(r) = r^2$, $W_2(r) = 2(1+L)r^2$, $W_1(r) = (1+h)r^2$, and $W_3(r) = hr^2$. Then all conditions of Corollary 7 are satisfied and the zero solution of (E) is UAS.

REFERENCES

[1] T.A. Burton, *Uniform asymptotic stability in functional differential equations*, Proc. Amer. Math. Soc., 68 (1978), 195–199.

[2] T.A. Burton, "Stability and Periodic Solutions of Ordinary and Functional Differential Equations," Academic Press, Orlando, FL 1985.

[3] K.L. Cooke, *Stability of nonautonomous delay differential equations by Liapunov functionals*, Infinite-Dimensional Systems (F. Kappel and W. Schappacher, eds), Lecture Notes in Math., 1076, Springer, Berlin, 1984.

[4] J.K. Hale, "Theory of Functional Differential Equations," Springer, New York, 1977.

[5] R.H. Hering, "Boundedness and Stability in Functional Differential Equations," Ph.D. Dissertation, Southern Illinois University, 1988.

[6] R.H. Hering, *Boundedness and periodic solutions in infinite delay systems*, J. Math. Anal. Appl., 163 (1992), 521–535.

[7] R.H. Hering, *Uniform asymptotic stability in infinite delay systems*, J. Math. Anal. Appl., to appear.

[8] J. Kato, *A conjecture in Liapunov method for functional differential equations*, Preprint.

[9] N.N. Krasovskii, "Stability of Motion," Moscow 1959, Translation, Stanford Univ. Press, 1963.

[10] T. Yoshizawa, "Stability Theory by Liapunov's Second Method," Math. Soc., Japan, 1966.

Differential and Integral Equations Volume 9, Number 1, January 1996, pp. 209–212

ANISOTROPIC EQUATIONS IN L^1

L. BOCCARDO

Dipartimento di Matematica, Università di Roam I.

T. GALLOUET

Departement de Mathematiques, Universié de Savoie

P. MARCELLINI

Dipartimento di Matematica, Università di Firenze

1. Introduction. Existence results for weak solutions of equations involving the p-Laplacian with right hand side measure have been recently obtained in [1]. G. Stampacchia [5] first considered the problem in the linear case; he obtained an existence result through a *duality method*: if $T \in W^{-1,r}(\Omega)$, $r > N$, then the solution of the Dirichlet problem

$$w \in H_0^1(\Omega) : -\mathrm{div}(A(x)Dw) = T \tag{1.1}$$

lies in $C^0(\Omega)$ and the mapping $T \to w$ is linear and continuous from $W^{-1,r}(\Omega)$ to $C^0(\Omega)$. Therefore the adjoint operator maps $M_b(\Omega)$ into $W_0^{1,q}(\Omega)$, for any $q < \frac{N}{N-1}$.

In this paper we consider anisotropic equations with right hand side measures. We confine ourselves to the model case

$$\begin{cases} -\mathrm{div}(j(Du)) = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where $j(\xi)$ is the vector field whose components are $|\xi_i|^{p_i-2}\xi_i$ $(i = 1, \dots, N;\ p_i > 1)$. We shall prove the existence of solutions of (1.2). More precisely we obtain the existence of a solution in the Sobolev space

$$W_0^{1,q_i}(\Omega)\big\{v \in W_0^{1,1}(\Omega) : \frac{\partial v}{\partial x_i} \in L^{q_i}(\Omega), \quad \forall i = 1, \dots, N\big\},$$

where q_i (for $i = 1, \dots, N$) is any real number greater than 1 and such that

$$1 < q_i < \frac{N(\bar{p}-1)}{\bar{p}(N-1)}\, p_i,$$

and

$$\frac{1}{\bar{p}} = \frac{1}{N}\sum_{i=1}^{N} \frac{1}{p_i}, \qquad p_i > 1. \tag{1.3}$$

Received March 1993.

AMS Subject Classification: 35A35.

Our existence result can be seen as a "nonlinear version" of the Stampacchia method, since it is known that the Dirichlet problem

$$\begin{cases} -\operatorname{div}(j(Dw)) = -\operatorname{div}(j(F)) \\ u \in W_0^{1,p_i}(\Omega) \end{cases} \tag{1.4}$$

has an L^∞ solution if each component F_i of the vector field F belongs to $L^{r_i}(\Omega)$, $r_i > N\frac{p_i}{\bar{p}}$ (see [6], [3], [4]).

2. Existence theorems. Let Ω be an open bounded set of $\mathbb{R}^N$ $(N \geq 1)$, $p_i > 1$ $(i = 1, \dots, N)$ and $\bar{p} < N$. The aim of this paper is to obtain weak solutions of the (model) problem

$$\begin{cases} -\sum_{i=1}^N \frac{\partial}{\partial x_i}(|\frac{\partial u}{\partial x_i}|^{p_i-2}\frac{\partial u}{\partial x_i}) = \mu & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{2.1}$$

in the sense of distributions, when μ is a bounded Randon measure on Ω (i.e., $\mu \in M_b(\Omega)$).

We shall prove the following existence theorem.

Theorem 1. *Let $\mu \in M_b(\Omega)$. Then there exists $u \in W_0^{1,1}(\Omega)$ (such that $\frac{\partial u}{\partial x_i} \in L^{q_i}(\Omega)$, $q_i < \frac{N(\bar{p}-1)}{\bar{p}(N-1)}p_i$) solution of (2.1)*

Proof. As in [1] we consider a sequence of smooth functions f_k converging to μ in $M_b(\Omega)$ weak, such that $\|f_k\|_{L^1(\Omega)} \leq \|\mu\|_{M_b(\Omega)}$, and the solutions u_k of the equations

$$u_k \in W_0^{1,p_i}(\Omega) : -\sum_{i=1}^N \frac{\partial}{\partial x_i}(|\frac{\partial u_k}{\partial x_i}|^{p_i-2}\frac{\partial u}{\partial x_i}) = f_k. \tag{2.2}$$

We define

$$\varphi(s) = \begin{cases} 0 & 0 \leq s \leq n \\ s - n & n < s < n+1 \\ 1 & s \geq n+1 \\ -\varphi(-s) & s < 0 \end{cases}$$

and we use $\varphi(u_k)$ as test function in (2.2). Let $B_{n,k} = \{x \in \Omega : n \leq |u_k(x)| < n+1\}$. Then

$$\int_{B_{n,k}} \sum_{i=1}^N |\frac{\partial u_k}{\partial x_i}|^{p_i} \leq c_1,$$

which implies

$$\int_{B_{n,k}} |\frac{\partial u_k}{\partial x_i}|^{p_i} \leq c_1, \quad i = 1, \dots, N. \tag{2.3}$$

Let $q_i = \theta p_i$, $\lambda = \frac{(1-\theta)\bar{q}^*}{\theta}$ and $\theta \in (0, \frac{N(\bar{p}-1)}{\bar{p}(N-1)})$. Then, using (2.3), $\lambda > 1$ and Holder inequality,

$$\begin{aligned} \int_\Omega |\frac{\partial u_k}{\partial x_i}|^{q_i} &\leq \Big[\int_\Omega |\frac{\partial u_k}{\partial x_i}|^{p_i}(1+|u_k|)^{-\lambda}\Big]^{\frac{q_i}{p_i}} \Big[\int_\Omega (1+|u_k|)^{\frac{\lambda q_i}{p_i-q_i}}\Big]^{1-\frac{q_i}{p_i}} \\ &\leq \Big[\sum_{n=0}^\infty \frac{1}{(1+n)^\lambda}\int_{B_{n,k}} |\frac{\partial u_k}{\partial x_i}|^{p_i}\Big]^{\frac{q_i}{p_i}} \Big[\int_\Omega (1+|u_k|)^{\frac{\lambda q_i}{p_i-q_i}}\Big]^{1-\frac{q_i}{p_i}}. \end{aligned} \tag{2.4}$$

The use of the nonisotropic Sobolev inequality (see [7]) leads to

$$\begin{aligned} \|u_k\|_{L^{\bar{q}^*}} &\leq c_3 \prod_{i=1}^N \|\frac{\partial u_k}{\partial x_i}\|_{L^{q_i}}^{1/N} \leq c_4 \prod_{i=1}^N \Big[\int_\Omega (1+|u_k|)^{\frac{\lambda q_i}{p_i-q_i}}\Big]^{(\frac{1}{q_i}-\frac{1}{p_i})\frac{1}{N}} \\ &= c_4 \prod_{i=1}^N \Big[\int_\Omega (1+|u_k|)^{\frac{\lambda\theta}{1-\theta}}\Big]^{(\frac{1}{\theta}-1)\frac{1}{Np_i}} = c_4\Big[\int_\Omega (1+|u_k|)^{\bar{q}^*}\Big]^{\frac{1-\theta}{\theta}\frac{1}{\bar{p}}}. \end{aligned} \tag{2.5}$$

By estimate (2.4), $\frac{\partial u_k}{\partial x_i}$ is bounded in $L^{q_i}(\Omega)$. Then we can assume (for some u and for some subsequence still denoted u_k) that

$$\frac{\partial u_k}{\partial x_i} \to \frac{\partial u}{\partial x_i} \quad \text{weakly in } L^{q_i}(\Omega), \tag{2.6}$$

$$u_k \to u \quad \text{strongly in } L^{\bar{q}}(\Omega). \tag{2.7}$$

This is not sufficient to pass to the limit, but the monotonicity properties of the differential operator allow us to use the techniques of [1] in order to deduce that we have also

$$\frac{\partial u_k}{\partial x_i} \to \frac{\partial u}{\partial x_i} \quad \text{in } L^{r_i}(\Omega), \quad r_i < q_i. \tag{2.8}$$

Indeed, by (2.2), we have, for any $\eta > 0$,

$$\sum_{i=1}^N \int_\Omega \Big[|\frac{\partial u_k}{\partial x_i}|^{p_i-2}\frac{\partial u_k}{\partial x_i} - |\frac{\partial u_h}{\partial x_i}|^{p_i-2}\frac{\partial u_h}{\partial x_i}\Big]\frac{\partial}{\partial x_i}T_\eta(u_k - u_h) = \int (f_k - f_h)T_\eta(u_k - u_h),$$

where T_η is the truncation at levels $\pm\eta$ $(\eta > 0)$. Then, for $i = 1, \dots, N$,

$$\int_{|u_k-u_h|\leq\eta} \Big[|\frac{\partial u_k}{\partial x_i}|^{p_i-2}\frac{\partial u_k}{\partial x_i} - |\frac{\partial u_h}{\partial x_i}|^{p_i-2}\frac{\partial u_h}{\partial x_i}\Big]\frac{\partial}{\partial x_i}(u_k - u_h) \leq 2\eta\|\mu\|.$$

If, for a fixed i, p_i is greater than or equal to 2, we deduce that

$$\int_{|u_k-u_h|\leq\eta} |\frac{\partial(u_k - u_h)}{\partial x_i}|^{p_i} \leq c_5\eta$$

and then

$$\begin{aligned} \int_\Omega |\frac{\partial(u_k - u_h)}{\partial x_i}|^{p_i} &\leq c_6\Big[\int_{|u_k-u_h|\leq\eta} |\frac{\partial(u_k - u_h)}{\partial x_i}|^{p_i}\Big]^{\frac{r_i}{p_i}} \\ &\quad + c_7 \operatorname{mis}\{x \in \Omega : |u_k(x) - u_h(x)| > \eta\}^{1-\frac{r_i}{q_i}} \\ &\leq c_8\eta^{\frac{r_i}{p_i}} + c_7 \operatorname{mis}\{x \in \Omega : |u_k(x) - u_h(x)| > \eta\}^{1-\frac{r_i}{q_i}}. \end{aligned}$$

Recall that u_k is a Cauchy sequence is measure. The arbitrariness of $\eta > 0$ leads to the conclusion that $\frac{\partial u_k}{\partial x_i}$ is a Cauchy sequence in $L^{r_i}(\Omega)$. A slight modification is needed to obtain (2.8) for $p_i < 2$. Then by (2.8), we have

$$|\frac{\partial u_k}{\partial x_i}|^{p_i-2}\frac{\partial u_k}{\partial x_i} \to |\frac{\partial u}{\partial x_i}|^{p_i-2}\frac{\partial u}{\partial x_i} \quad \text{in } L^1(\Omega)$$

and therefore u is a solution of equation (2.1).

Theorem 1 allows us to find a solution when, for instance, the data are L^1-functions. Anyway, to obtain $q_i = \frac{N(\bar{p}-1)}{\bar{p}(N-1)}p_i$, we have to make stronger assumptions on the right hand side. This is stated in the following.

Theorem 2. *Let f be a measurable function such that*

$$\int_\Omega |f|\log(1+|f|) < \infty.$$

Then there exists a solution u of the Dirichlet problem

$$u \in W_0^{1,q_i}(\Omega) : -\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\Big(\Big|\frac{\partial u}{\partial x_i}\Big|^{p_i-2}\frac{\partial u}{\partial x_i}\Big) = f,$$

where $q_i = \frac{N(\bar{p}-1)}{\bar{p}(N-1)}p_i$.

Proof. We modify the previous proof with the help of techniques used in [2]. Using $\log(1+|u_k|)\mathrm{sgn}(u_k)$ as test function in (2.2), the inequality (2.3) becomes

$$\int_\Omega \Big|\frac{\partial u_k}{\partial x_i}\Big|^{p_i} \cdot \frac{1}{1+|u|} \le \int_\Omega f\log(1+|u_k|)$$
$$\le \int_\Omega |f|\log(1+|f|) + \int (1+|u_k|) \le c_1 + \int_\Omega (1+|u_k|).$$

This implies the change in inequality (2.5),

$$\|u_k\|_{\bar{q}^*} \le c_2 + c_2\Big[\int_\Omega (1+|u_k|)^{\frac{\theta}{1-\theta}}\Big]^{\frac{1-\theta}{\bar{q}}},$$

and so the sequence u_k is bounded in $W_0^{1,q_i}(\Omega)$. The a priori estimate and the strong L^1 convergence of $\frac{\partial u_k}{\partial x_i}$ allow us to pass to the limit.

Acknowledgments. Part of this work originated during a visit of the second and third author in Roma on November 12th, 1990.

REFERENCES

[1] L. Boccardo and T. Gallouet, *Nonlinear elliptic and parabolic equations involving measure data*, J. Funct. Anal., 87 (1989), 149–169.
[2] L. Boccardo and T. Gallouet, *Nonlinear elliptic equations with right hand side measure*, Comm. P.D.E., 17 (1992), 641–655.
[3] L. Boccardo, P. Marcellini and C. Sbordone, *L^∞-regularity for variational problems with sharp non-standard growth conditions*, Boll. U.M.I., 4A (1990), 219–225.
[4] P. Marcellini, *Regularity of minimizers of integrals of calculus of variations with nonstandard growth conditions*, Arch. Rat. Mech. Anal., 105 (1989), 267–284.
[5] G. Stampacchia, *Le problème de Dirichlet pour les équation elliptiques du second ordre à coefficients discontinus*, Ann. Inst. Fourier. (Grenoble), 15 (1965), 189–258.
[6] B. Stroffolini, *Some remarks on the regularity of anisotropic variational problems*, Boll. U.M.I., 5A (1991), 345–352.
[7] M. Troisi, *Teoremi di inlcusione per spazi di Sobolev non isotropi*, Ricerche Mat., 18 (1969), 3–24.

Differential and Integral Equations Volume 9, Number 1, January 1996, pp. 213 – 237

A FINITE RANGE OPERATOR WITH A QUASI-PERIODIC POTENTIAL

STEVE SURACE
Drew University, Department of Mathematics, Madison, NJ 07940

(Submitted by: Roger Nussbaum)

Abstract. We consider the operator $H = \varepsilon h + \cos(\alpha \cdot j + \vartheta)$ for $j \in \mathbb{Z}^\nu$, where h is self-adjoint, translation invariant and finite range. The vector $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_\nu)$ is assumed to have the diophantine property $|j \cdot \alpha \bmod 2\pi| \ge C/|j|^2$, where $j \neq 0$ and C is some constant. For ε sufficiently small we prove that H has pure point spectrum for almost every ϑ. Moreover, every polynomially bounded eigenfunction of H decays exponentially fast. Finally, we will show how this operator comes up in the study of electrons in a transverse magnetic field subject to a two dimensional periodic potential.

1. Introduction. We consider the equation

$$H\psi \equiv (\varepsilon h + V)\psi = E\psi$$

on the lattice $\mathbb{Z}^\nu$. We assume that h is self-adjoint, translation invariant and finite range; i.e., the matrix elements of h satisfy, $h_{ij} = 0$ when $|i-j| \ge R$, where $i, j \in \mathbb{Z}^\nu$ and R is some positive constant. The potential V is given by matrix elements

$$V_{ij}(\vartheta) = \delta_{ij} v_j(\vartheta) = \delta_{ij}\cos(\alpha \cdot j + \vartheta)$$

and the vector $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_\nu)$ is assumed to have the Diophantine property

$$|j \cdot \alpha \bmod 2\pi| \ge C/|j|^2, \qquad (1.1)$$

where $j \neq 0$ and C is some constant.

We state our main result:

Theorem. *Consider the operator $H = \varepsilon h + \cos(\alpha \cdot j + \vartheta)$ ($j \in \mathbb{Z}^\nu$) subject to the conditions listed above. For ε sufficiently small H has pure point spectrum for almost every ϑ. Moreover, every polynomially bounded eigenfunction of H decays exponentially fast.*

Our motivation for considering operators of this type comes from the study of electrons in a transverse magnetic field subject to a two dimensional periodic potential, [26], [27]. In Section 4, we will examine the Schrödinger equation in a Magnetic field. We will make a series expansion of the solution to the differential equation and show that the recursion formula for the coefficients in the expansion leads us to the finite difference equation treated in this paper. In recent years many problems of this type have been studied. We now mention a few of the latest results.

Received for publication February 1993.