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# Summability of the Solutions of Nonlinear Elliptic Equations With Right Hand Side Measures

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## 1. Introduction

A classical theorem by Guido Stampacchia states that, if $A(x) = (a_{ij}(x))$ is an elliptic bounded measurable matrix, there exists a solution (obtained by *duality*) $u \in W_0^{1,q}(\Omega)$, for every $q < \frac{N}{N-1}$, of the Dirichlet problem

$$\begin{cases} -\mathrm{div}\,(A(x)\,\nabla u) = \mu & \text{in } \Omega, \\ \qquad u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\mu$ is a bounded measure, and $\Omega$ is a bounded open subset of $\mathbb{R}^N$.
Of course, the Stampacchia method is restricted to a linear setting. For example, consider the Dirichlet problem for the so-called $p$-laplacian $(2 - \frac{1}{N} < p < N)$

$$\begin{cases} -\mathrm{div}\,(|\nabla u|^{p-2}\,\nabla u) = \mu & \text{in } \Omega, \\ \qquad u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

Then the duality method cannot be used. Moreover the solutions of (1.2) are not, in general, critical points of the convex functional

$$\int_\Omega |\nabla v|^p\ dx - \int_\Omega v\,d\mu.$$

For instance, the minimum can be $-\infty$. We proved by approximation the existence of solutions $u \in W_0^{1,q}(\Omega), q < \frac{N(p-1)}{N-1}$ of nonlinear Dirichlet problems when the right hand side is a bounded measure.

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We begin stating the hypotheses that will hold throughout the paper.
Let $\Omega$ be a bounded, open subset of $\mathbb{R}^N$, $N \geq 2$. Let $p$ be a real number such that $1 < p \leq N$.
Let $a : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ be a Carathéodory function such that the following holds:

$$a(x,\xi) \cdot \xi \geq \rho_1 \, |\xi|^p \,, \tag{1.3}$$

for almost every $x \in \Omega$, for every $\xi \in \mathbb{R}^N$, where $\rho_1$ is a positive constant;

$$|a(x,\xi)| \leq \lambda(x) + \rho_2 \, |\xi|^{p-1} \,, \tag{1.4}$$

for almost every $x \in \Omega$, for every $\xi \in \mathbb{R}^N$, where $\rho_2$ is a positive constant, and $\lambda$ belongs to $L^{p'}(\Omega)$;

$$(a(x,\xi) - a(x,\eta)) \cdot (\xi - \eta) > 0 \,, \tag{1.5}$$

for almost every $x \in \Omega$, for every $\xi$ and $\eta$ in $\mathbb{R}^N$, with $\xi \neq \eta$.
Let us define the differential operator

$$A(u) = -\mathrm{div}\,(a(x,\nabla u)) \,.$$

Thanks to (1.3), (1.4) and (1.5), $A$ is a monotone and coercive differential operator acting between $W_0^{1,p}(\Omega)$ and $W^{-1,p'}(\Omega)$; hence, it is surjective.
We look for weak solutions, i.e., for functions $u$ such that

$$\begin{cases} u \in W_0^{1,1}(\Omega), \\ \displaystyle\int_\Omega a(x,\nabla u) \cdot \nabla v \, dx = \int_\Omega v \, \mu, \ \forall v \in C_0^\infty(\Omega) \,, \end{cases} \tag{1.6}$$

We recall the following results (see [1] or [2]).

**Theorem 1.1.**
*There exists a solution $u$ of (1.6), and $u \in W_0^{1,q}(\Omega)$, for every $q < \frac{N(p-1)}{N-1} = q_0$.*

**Remark 1.2.** If $p > N$, then $\mu \in W^{-1,p'}(\Omega)$, and the existence theorem is a consequence of the classical theorems on the surjectivity of monotone operators.

**Remark 1.3.** We observe that $q_0 > 1$ if and only if $p > 2 - \frac{1}{N}$.

**Remark 1.4.** If $\mu$ is the Dirac mass, and $\Omega$ is a ball, we can check that the result of Theorem 1.1 is optimal.

If $\mu$ has a density $f$ with respect to the Lebesgue measure, and $f \in L^1(\Omega)$, we recall the following results.

**Theorem 1.5.** *If $f \in L^m(\Omega)$, $1 < m < \frac{Np}{Np-p+N} = (p^*)'$, then $u \in W_0^{1,(p-1)m^*}(\Omega)$.*

**Remark 1.6.** If $m = (p^*)'$, then $L^m(\Omega) \subset W^{-1,p'}(\Omega)$, and $W_0^{1,(p-1)m^*}(\Omega) = W_0^{1,p}(\Omega)$.

**Remark 1.7.** If $m$ tends to 1, we cannot deduce Theorem 1.1 from Theorem 1.5 (see Remark 1.4, and the counterexample in Section 2 of [3]). In order to obtain a solution in $W_0^{1,q_0}(\Omega)$, a sufficient condition is given by the following theorem.

**Theorem 1.8.** *If $\int_\Omega |f| \log(1+|f|)\, dx < \infty$, then there exists a solution $u \in W_0^{1,q_0}(\Omega)$. That is there exists a function $u$ such that*

$$\begin{cases} u \in W_0^{1,1}(\Omega), \\ \displaystyle\int_\Omega a(x,\nabla u)\cdot\nabla v\, dx = \int_\Omega f\, v\, dx, \ \forall v \in C_0^\infty(\Omega)\,, \end{cases} \tag{1.7}$$

We point out that if $m \in (1,(p^*)')$ then, roughly speaking, the function "$m \to$ summability of $(u, |\nabla u|)$" is increasing. On the other hand, if $m > (p^*)'$, only the function "$m \to$ summability of $u$" is increasing .

Other results concerning existence, uniqueness, regularity and generalizations can be found in the papers quoted in the references.

## 2. Regularity results

In this section we present an improvement of the summability of the solution $u$ of the Dirichlet problem (1.6).

**Theorem 2.1.** *The solution $u$ of* (1.6) *given by* Theorem 1.1 *is such that, for every $\beta > \frac{1}{p-1}$,*

$$\frac{u}{[\log(2+|u|)]^\beta} \in L^{q_0^*}(\Omega)\,, \tag{2.1}$$

$$\frac{|\nabla u|}{[\log(2+|u|)]^\beta} \in L^{q_0}(\Omega)\,. \tag{2.2}$$

**Proof.** Let $\{f_k\}$ be a sequence of smooth functions converging to $\mu$ in the sense of the measures. Consider the approximate problems

$$\begin{cases} -\mathrm{div}\,(a(x,\nabla u_k)) = f_k & \text{in } \Omega, \\ u_k = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.3}$$

that have a unique solution $u_k \in W_0^{1,p}(\Omega)$. Define, for $\beta > \frac{1}{p-1}$ and for $s \in \mathbb{R}$,

$$\psi(s) = \frac{s}{\log(2+|s|)^\beta}\,,$$

and, for $n \in \mathbb{N}$,

$$\varphi_n(s) = \begin{cases} 0 & \text{if } 0 \le s \le n, \\ s-n & \text{if } n < s < n+1, \\ 1 & \text{if } s > n+1, \\ -\varphi_n(-s) & \text{if } s < 0. \end{cases}$$

The use of $\varphi_n(u_k)$ as test function in (2.3), and the boundedness of $\{f_k\}$ in $L^1(\Omega)$, imply that

$$\int_{B_n} |\nabla u_k|^p \, dx \le c_1 \,, \tag{2.4}$$

where

$$B_n = \{x \in \Omega : n \le |u_k| < n+1\}$$

(observe that $B_n$ is the set in $\Omega$ where the derivative of $\varphi_n(u_k)$ is different from zero). Then, if $\alpha > 1$,

$$\begin{aligned} \int_\Omega \frac{|\nabla u_k|^p}{(2+|u_k|)\,[\log(2+|u_k|)]^\alpha}\, dx &= \sum_{n=0}^{\infty} \int_{B_n} \frac{|\nabla u_k|^p}{(2+|u_k|)\,[\log(2+|u_k|)]^\alpha}\, dx \\ &\le c_1 \sum_{n=0}^{\infty} \frac{1}{(2+n)\,[\log(2+n)]^\alpha} \le c_2 \,. \end{aligned}$$

Sobolev and Hölder inequalities yield, setting $g(u_k) = (2+|u_k|)\,[\log(2+|u_k|)]^\alpha$ for the sake of simplicity,

$$\begin{aligned} c_3 \left( \int_\Omega |\psi(u_k)|^{q_0^*} \, dx \right)^{\frac{q_0}{q_0^*}} &\le \int_\Omega |\nabla \psi(u_k)|^{q_0} \, dx = \int_\Omega |\nabla u_k|^{q_0} \, |\psi'(u_k)|^{q_0} \, dx \\ &= \int_\Omega \frac{|\nabla u_k|^{q_0}}{g(u_k)^{\frac{q_0}{p}}} \, |\psi'(u_k)|^{q_0} \, g(u_k)^{\frac{q_0}{p}} \, dx \\ &\le \left( \int_\Omega \frac{|\nabla u_k|^p}{g(u_k)} \, dx \right)^{\frac{q_0}{p}} \left( \int_\Omega |\psi'(u_k)|^{\frac{pq_0}{p-q_0}} \, g(u_k)^{\frac{q_0}{p-q_0}} \, dx \right)^{1-\frac{q_0}{p}} \\ &\le c_2^{\frac{q_0}{p}} \left( \int_\Omega |\psi'(u_k)|^{\frac{pq_0}{p-q_0}} \, (2+|u_k|)^{q_0^*} \, [\log(2+|u_k|)]^{\frac{\alpha q_0}{p-q_0}} \, dx \right)^{1-\frac{q_0}{p}} . \end{aligned}$$

Now we point out that

$$\begin{aligned} |\psi'(s)|^{\frac{pq_0}{p-q_0}} \, (2+|s|)^{q_0^*} \, [\log(2+|s|)]^{\frac{\alpha q_0}{p-q_0}} &\le [\log(2+|s|)]^{\alpha q_0^* - \beta p q_0^*} \, (2+|s|)^{q_0^*} \\ &\le c_4 \, (1 + |\psi(s)|^{q_0^*}) \,, \end{aligned}$$

if $\alpha q_0^* - \beta p q_0^* \le -\beta q_0^*$; the previous inequality is verified if $\alpha \le \beta(p-1)$, and this explains the bound on $\beta$ given in the statement. Thus

$$\int_\Omega |\psi(u_k)|^{q_0^*} \, dx \le c_5 \,, \qquad \int_\Omega |\nabla \psi(u_k)|^{q_0} \, dx \le c_6 \,.$$

Then the methods of [1], [2] allow us to pass to the limit in the approximate equations, and we can prove that there exists a solution $u$ of (1.6) such that both (2.1) and (2.2) hold true. □

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## References

[1] L. Boccardo, T. Gallouet: Nonlinear elliptic and parabolic equations involving measure data, J. Funct. Anal. 87 (1989) 149–169,

[2] L. Boccardo, T. Gallouet: Nonlinear elliptic equations with right hand side measures, Comm. Partial Differential Equations 17 (1992) 641–655,

[3] L. Orsina: Solvability of linear and semilinear eigenvalue problems with $L^1$ data, Rend. Sem. Mat. Univ. Padova 90 (1993) 207–238

[4] P. Bènilan, L. Boccardo, T. Gallouet, R. Gariepy, M. Pierre and J.L. Vazquez: An $L^1$ theory of existence and uniqueness of solutions of nonlinear elliptic equations, Ann. Scuola Norm. Sup. Pisa, to appear.

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**Leere Seite**
**366**