$W_0^{1,1}$ SOLUTIONS IN SOME BORDERLINE CASES OF CALDERON-ZYGMUND THEORY

LUCIO BOCCARDO - THIERRY GALLOUET

ABSTRACT. In this paper we study the existence of $W_0^{1,1}(\Omega)$ distributional solutions of Dirichlet problems whose simplest example is

$$\begin{cases} -\mathrm{div}\big(|\nabla u|^{p-2}\nabla u\big) = f(x), & \text{in } \Omega; \\ u = 0, & \text{on } \partial\Omega. \end{cases}$$

1. Introduction

Let Ω be a bounded open set in $I\!R^N$, $N \geq 2$. The simplest example of nonlinear (and variational) boundary value problem is the Dirichlet problem for the p–Laplace operator

$$\begin{cases} -\mathrm{div}\big(|\nabla u|^{p-2}\nabla u\big) = f(x), & \text{in } \Omega; \\ u = 0, & \text{on } \partial\Omega; \end{cases} \tag{1.1}$$

where

$$1 < p < N, \tag{1.2}$$

so that the growth of the differential operator is $p-1$. The classical theory of nonlinear elliptic equations states that $W_0^{1,p}(\Omega)$ is the natural functional spaces framework to find weak solutions of (1.1), if the function f belongs to the dual space of $W_0^{1,p}(\Omega)$ (see [13], [17], [25]).

This approach fails if $p = 1$ or if we consider the problem of non-parametric minimal surfaces (where $f(x) = 0$, but the boundary datum is not zero, see [23]) because of the lack of compactness of bounded sequences (non reflexivity of $W_0^{1,1}(\Omega)$), so that it is only possible to find solutions in the "larger" space $BV(\Omega)$. We recall that, thanks to a purely geometric argument ([12], [22]) or a duality argument ([29]), existence of "generalized" solutions was obtained. More recently, for this kind of problems, some existence results in $W^{1,1}(\Omega)$ have been proved in [2].

On the other hand, if $p > 1$, for the model problem (1.1), the existence of $W_0^{1,p}(\Omega)$ solutions also fails if the right hand side is a function $f \in L^m(\Omega)$ ($m \geq 1$) which does not belong to the dual space of $W_0^{1,p}(\Omega)$: it is possible to find distributional solutions in function spaces "larger" than $W_0^{1,p}(\Omega)$, but contained in $W_0^{1,1}(\Omega)$ (see [7], [8]). In this paper

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we will prove, for general boundary value problems of the type (1.1) and for some values of p and m, the existence of solutions belonging to $W_0^{1,1}(\Omega)$ and not belonging to $W_0^{1,q}(\Omega)$, $1 < q < p$: see also Remark 2.5.

To be more precise, in this paper, we study some existence results of $W_0^{1,1}(\Omega)$ distributional solutions (not so usual in elliptic problems) for nonlinear elliptic boundary value problems of the type

$$\begin{cases} A(u) = f(x), & \text{in } \Omega; \\ u = 0, & \text{on } \partial\Omega; \end{cases} \tag{1.3}$$

where

$$f \in L^m(\Omega),\ m \geq 1, \tag{1.4}$$

and A is the operator, acting on $W_0^{1,p}(\Omega)$, defined by

$$A(v) = -\operatorname{div}\left(a(x, v, \nabla v)\right). \tag{1.5}$$

We assume the standard hypotheses on $a : \Omega \times I\!R \times I\!R^N \to I\!R^N$, that is a is a Carathéodory function such that the following holds for almost every $x \in \Omega$, for every $s \in I\!R$, for every $\xi \neq \eta \in I\!R^N$:

$$\begin{cases} a(x, s, \xi)\xi \geq \alpha\, |\xi|^p\,, \\ |a(x, s, \xi)| \leq \beta |\xi|^{p-1}\,, \\ [a(x, s, \xi) - a(x, s, \eta)](\xi - \eta) > 0\,, \end{cases} \tag{1.6}$$

where α, β are positive constants.

Thus A is a pseudomonotone and coercive differential operator and it is surjective (see [25], [13], [17]). The simplest example is given by the differential operator $A(v) = -\operatorname{div}(|\nabla v|^{p-2}\nabla v)$, appearing in (1.1).

The existence of $W_0^{1,1}(\Omega)$ solutions, instead of $W_0^{1,p}(\Omega)$ or $W_0^{1,q}(\Omega)$ (with $1 < q < p$) solutions of the boundary value problem (1.3) is a consequence of the poor summability of the right hand side, even if the "growth" of the operator A is not zero, but $p - 1 > 0$.

Existence of solutions for problem (1.3) with nonregular right hand side has been obtained by G. Stampacchia in [28] (if A is a linear elliptic operator), by H. Brezis and W. Strauss in [16] and [15] (for semilinear problems; see also [20]) and in [7], [8], [10], [1], for general nonlinear problems; in particular, we recall the following results contained in [7], [8].

THEOREM 1.1. *Let $m = 1$ and*

$$2 - \frac{1}{N} < p < N. \tag{1.7}$$

Then there exists a distributional solution $u \in W_0^{1,q}(\Omega)$, $q < \frac{N(p-1)}{N-1}$, of (1.3); that is

$$\int_\Omega a(x, u, \nabla u)\nabla v = \int_\Omega f\, v, \quad \forall v \in W_0^{1,\infty}(\Omega)\,.$$

Observe that $\frac{N(p-1)}{N-1} > 1$ if and only if $p > 2 - \frac{1}{N}$.

THEOREM 1.2. *Let* $2 - \frac{1}{N} < p < N$. *If*

$$\int_\Omega |f| \log(1 + |f|) < \infty, \tag{1.8}$$

then there exists a distributional solution $u \in W_0^{1,\frac{N(p-1)}{N-1}}(\Omega)$ *of (1.3).*

THEOREM 1.3 (Calderon-Zygmund theory for infinite energy solutions).
If $f \in L^m(\Omega)$, $\frac{N}{N(p-1)+1} < m < \frac{Np}{pN+p-N} = (p^\star)'$, $p > 1 + \frac{1}{m} - \frac{1}{N}$, *then there exists a distributional solution* $u \in W_0^{1,(p-1)m^\star}(\Omega)$ *of (1.3).*

Moreover, if f belongs to $L^1(\Omega)$ (see also [10], where the datum is sum of an element in $W^{-1,p'}(\Omega)$ and of a function in $L^1(\Omega)$), we recall that in [1] have been introduced notions of gradient and of solution for (1.3), with the purpose of proving its uniqueness (if the function $a(x, s, \xi)$ does not depend on s) and of proving its existence if p does not satisfy (1.7).

In this paper we study the existence of $W_0^{1,1}(\Omega)$ distributional solutions (without the functional framework of [1]) as consequence of the fact that we improve the existence results of Theorem 1.2 and Theorem 1.3 in some borderline cases. Another elliptic problem with $W_0^{1,1}(\Omega)$ solutions is studied in [5].

2. Existence

We recall the definition of $T_k(s)$, for s and k in $I\!R$, with $k \geq 0$: $T_k(s) = \max(-k, \min(k, s))$ and that, in the existence proof, we started in [7], [8], [10], [1] with the Dirichlet problems

$$u_n \in W_0^{1,p}(\Omega) : \; A(u_n) = f_n, \tag{2.1}$$

with $f_n = T_n(f)$. Thus every u_n is a bounded function (see [28]). Moreover in [1] is proved that the use of $T_k(u_n)$ as test function yields (see also [9], [4])

$$\alpha \int_\Omega |\nabla T_k(u_n)|^p \leq \, k \int_\Omega |f| \, . \tag{2.2}$$

Furthermore we have the following estimate.

LEMMA 2.1. *Let* $f \in L^1(\Omega)$, $p > 1$. *The sequence* $\{\log(1+|u_n|)\text{sign}(u_n)\}$ *is bounded in* $W_0^{1,p}(\Omega)$.

PROOF. The use of $[1 - (1 + |u_n|)^{1-p}]\text{sign}(u_n)$ as test function yields

$$\begin{aligned} &\alpha \int_\Omega |\nabla \log(1 + |u_n|)|^p \leq \alpha \int_\Omega \frac{|\nabla u_n|^p}{(1 + |u_n|)^p} \\ &\leq \int_\Omega |f_n|[1 - (1 + |u_n|)^{1-p}] \leq \int_\Omega |f|, \end{aligned} \tag{2.3}$$

which implies the result □

As a consequence of the previous lemma, there exists a subsequence (not relabelled) such that

$$\log(1+|u_n|)\mathrm{sign}(u_n) \text{ converges weakly in } W_0^{1,p}(\Omega) \text{ and a.e.} \tag{2.4}$$

Then $u_n(x)$ converges a.e. to a measurable function $u(x)$ such that $\log(1+|u|)\mathrm{sign}(u) \in W_0^{1,p}(\Omega)$.

THEOREM 2.2. *Let $f \in L^m(\Omega)$, $m = \frac{N}{N(p-1)+1}$, $1 < p < 2 - \frac{1}{N}$. Then there exists a distributional solution $u \in W_0^{1,1}(\Omega)$ of (1.3).*

PROOF. STEP 1 - Note that $m = \frac{N}{N(p-1)+1}$ implies $m < \frac{N}{p}$. The first part of the proof follows the approach of [8]. Let $\theta = \frac{(p-1)m'}{pm'-p^\star}$. Note that $pm' - p^\star > 0$, since $m < \frac{N}{p}$, and that $\theta < 1$, since $m < \frac{pN}{pN+p-N}$. Let ϵ be a strictly positive real number. The function $v_\epsilon = [(\epsilon+|u_n|)^{1-p(1-\theta)} - \epsilon^{1-p(1-\theta)}]\mathrm{sign}(u_n)$ is bounded since $1-p(1-\theta) > 0$ (which is equivalent to $p > 1$). Thus we can use v_ϵ as test function in (2.1) and we have

$$\left|\begin{array}{l} C_{2,p}\left[\int_\Omega \left\{(\epsilon+|u_n|)^\theta - \epsilon^\theta\right\}^{p^\star}\right]^{\frac{p}{p^\star}} \le C_{1,p}\int_\Omega \frac{|\nabla u_n|^p}{(\epsilon+|u_n|)^{p(1-\theta)}} \\ \le \left[\int_\Omega |f|^m\right]^{\frac{1}{m}} \left[\int_\Omega \left\{(\epsilon+|u_n|)^{1-p(1-\theta)} - \epsilon^{1-p(1-\theta)}\right\}^{m'}\right]^{\frac{1}{m'}}, \end{array}\right. \tag{2.5}$$

where $C_{i,p}$ denotes a strictly positive constant. The limit as ϵ tends to zero yields, thanks to the Fatou Lemma,

$$C_{2,p}\left[\int_\Omega |u_n|^{\theta\, p^\star}\right]^{\frac{p}{p^\star}} \le \alpha \int_\Omega \frac{|\nabla u_n|^p}{|u_n|^{p(1-\theta)}} \le \left[\int_\Omega |f|^m\right]^{\frac{1}{m}} \left[\int_\Omega |u_n|^{[1-p(1-\theta)]m'}\right]^{\frac{1}{m'}}.$$

Note that $\frac{p}{p^\star} > \frac{1}{m'}$ since $m < \frac{N}{p}$. Moreover the choice of θ implies $\theta\, p^\star = [1-p(1-\theta)]m' = \frac{(mp)^\star}{p'} = \frac{N}{N-1}$. Thus we proved that

$$C_{2,p}\left[\int_\Omega |u_n|^{\frac{N}{N-1}}\right]^{\frac{1}{m}-\frac{p}{N}} \le \left[\int_\Omega |f|^m\right]^{\frac{1}{m}}. \tag{2.6}$$

This estimate also implies (see the previous inequality) the boundedness, with respect to n, of

$$\int_\Omega \frac{|\nabla u_n|^p}{|u_n|^{p(1-\theta)}}.$$

and the following estimate

$$\mathrm{meas}\{k \le |u_n|\} \le \frac{C_{3,p}}{k^{\frac{N}{N-1}}}, \tag{2.7}$$

so that, if we fix $\epsilon > 0$, there exists k_ϵ such that, for $k \ge k_\epsilon$, we have

$$\mathrm{meas}\{k \le |u_n|\} \le \epsilon, \quad \text{uniformly with respect to } n. \tag{2.8}$$

Now we can estimate $\int_\Omega |\nabla u_n|$. Indeed we have

$$\int_\Omega |\nabla u_n| = \int_\Omega \frac{|\nabla u_n|}{|u_n|^{(1-\theta)}}|u_n|^{(1-\theta)} \le \left[\int_\Omega \frac{|\nabla u_n|^p}{|u_n|^{p(1-\theta)}}\right]^{\frac{1}{p}} \left[\int_\Omega |u_n|^{p'(1-\theta)}\right]^{\frac{1}{p'}}.$$

Note that $p'(1-\theta) = \frac{N}{N-1}$, so the right hand side is bounded; then the sequence $\{u_n\}$ is bounded in $W_0^{1,1}(\Omega)$, subsequently there exists $R > 0$ such that

$$\|u_n\|_{W_0^{1,1}(\Omega)} \le R. \tag{2.9}$$

Thus there exists a subsequence (not relabelled) $\{u_n\}$ converging to u in $L^r(\Omega)$, $1 \le r < \frac{N}{N-1}$, and almost everywhere. Moreover (2.2) implies that $\nabla T_k(u_n)$ converges weakly to $\nabla T_k(u)$ in $W_0^{1,p}(\Omega)$.

Step 2 - Now we need an estimate not only of $\int_\Omega |\nabla u_n|$, but also of $\int_{\{k\le|u_n|\}} |\nabla u_n|$. We adapt the method of Step 1. Thus we use $[|u_n|^{1-p(1-\theta)} - k^{1-p(1-\theta)}]^+\mathrm{sign}(u_n)$ as test function in (2.1), with θ as before, and we have, thanks to (2.6),

(2.10)

$$\begin{aligned} & C_{4,p} \int_{\{k\le|u_n|\}} \frac{|\nabla u_n|^p}{|u_n|^{p(1-\theta)}} \\ & \le \left[\int_{\{k\le|u_n|\}} |f|^m\right]^{\frac{1}{m}} \left[\int_{\{k\le|u_n|\}} \left\{|u_n|^{1-p(1-\theta)} - k^{1-p(1-\theta)}\right\}^{m'}\right]^{\frac{1}{m'}} \\ & \le \left[\int_{\{k\le|u_n|\}} |f|^m\right]^{\frac{1}{m}} \left[\int_{\{k\le|u_n|\}} |u_n|^{[1-p(1-\theta)]m'}\right]^{\frac{1}{m'}} \le C_{5,p}\left[\int_{\{k\le|u_n|\}} |f|^m\right]^{\frac{1}{m}}. \end{aligned}$$

By Hölder inequality we have (using again that $p'(1-\theta) = \frac{N}{N-1}$)

(2.11)

$$\begin{aligned} & \int_{\{k\le|u_n|\}} |\nabla u_n| = \int_{\{k\le|u_n|\}} \frac{|\nabla u_n|}{|u_n|^{(1-\theta)}}|u_n|^{(1-\theta)} \\ & \le \left[\int_{\{k\le|u_n|\}} \frac{|\nabla u_n|^p}{|u_n|^{p(1-\theta)}}\right]^{\frac{1}{p}} \left[\int_\Omega |u_n|^{p'(1-\theta)}\right]^{\frac{1}{p'}} \le C_{6,p}\left[\int_{\{k\le|u_n|\}} |f|^m\right]^{\frac{1}{m}}. \end{aligned}$$

Thus, for every measurable subset E, thanks to (2.2) and (2.11), we have

$$\begin{aligned} & \int_E \left|\frac{\partial u_n}{\partial x_i}\right| \le \int_E |\nabla u_n| \le \int_E |\nabla T_k(u_n)| + \int_{\{k\le|u_n|\}} |\nabla u_n| \\ & \le \mathrm{meas}(E)^{\frac{1}{p'}} \left[\frac{k}{\alpha}\|f\|_{L^1(\Omega)}\right]^{\frac{1}{p}} + C_{6,p}\left[\int_{\{k\le|u_n|\}} |f|^m\right]^{\frac{1}{m}} \end{aligned} \tag{2.12}$$

Now we want to prove that

$$u_n \text{ weakly converges to } u \text{ in } W_0^{1,1}(\Omega) \tag{2.13}$$

and we follow [5]. The estimate (2.12) implies that the sequence $\{\frac{\partial u_n}{\partial x_i}\}$ is equiintegrable, thanks to (2.8) and the absolute continuity of the integral. Thus, by Dunford-Pettis theorem, and up to subsequences, there exists Y_i in $L^1(\Omega)$ such that $\frac{\partial u_n}{\partial x_i}$ weakly converges to Y_i in $L^1(\Omega)$. Since $\frac{\partial u_n}{\partial x_i}$ is the distributional partial derivative of u_n, we have, for every n in $I\!N$,

$$\int_\Omega \frac{\partial u_n}{\partial x_i}\,\varphi = -\int_\Omega u_n\,\frac{\partial \varphi}{\partial x_i}\,, \quad \forall \varphi \in C_0^\infty(\Omega)\,.$$

We now pass to the limit in the above identities, using that $\partial_i u_n$ weakly converges to Y_i in $L^1(\Omega)$, and that u_n strongly converges to u in $L^1(\Omega)$: we obtain

$$\int_\Omega Y_i\,\varphi = -\int_\Omega u\,\frac{\partial \varphi}{\partial x_i}\,, \quad \forall \varphi \in C_0^\infty(\Omega)\,.$$

This implies that $Y_i = \frac{\partial u}{\partial x_i}$, and this result is true for every i. Since Y_i belongs to $L^1(\Omega)$ for every i, u belongs to $W_0^{1,1}(\Omega)$, as desired.

The almost everywhere convergence of ∇u_n to ∇u, proved in Lemma 5.1 in the Appendix, and (2.13) allow us to use the Vitali Theorem. Thus

$$\nabla u_n \to \nabla u \quad \text{in } (L^1(\Omega))^N. \tag{2.14}$$

STEP 3 - The inequality

$$|a(x,u_n,\nabla u_n)| \le \beta\,|\nabla u_n|^{p-1}$$

and (again) the Vitali Theorem imply that $a(x,u_n,\nabla u_n)$ converges to $a(x,u,\nabla u)$ in $(L^{\frac{1}{p-1}}(\Omega))^N$. Note that $\frac{1}{p-1} > 1$. Then it is possible to pass to the limit in (2.1). Thus we proved that $u \in W_0^{1,1}(\Omega)$ is a distributional solution of (1.3). □

THEOREM 2.3. *Assume (1.8) and* $p = 2 - \frac{1}{N}$. *Then there exists a distributional solution* $u \in W_0^{1,1}(\Omega)$ *of (1.3).*

PROOF. STEP 1 - Let $1 < \lambda < p$. Taking $[1-(1+|u_n|)^{1-\lambda}]\text{sign}(u_n)$ as test function in the weak formulation of (1.3), we obtain

$$\alpha\int_\Omega \frac{|\nabla u_n|^p}{(1+|u_n|)^\lambda} \le \frac{1}{\lambda-1}\|f\|_{L^1(\Omega)}.$$

Then, using the Sobolev embedding Theorem we have

$$\left[\int_\Omega \{(1+|u_n|)^{1-\frac{\lambda}{p}}-1\}^{p^\star}\right]^{\frac{p}{p^\star}} \le C_{p,\lambda}\,\|f\|_{L^1(\Omega)}. \tag{2.15}$$

Noting that $\lambda > 1$ implies $(1-\frac{\lambda}{p})p^\star < \frac{N}{N-1}$, we prove that the sequence $\{u_n\}$ is bounded in $L^r(\Omega)$, $1 \le r < \frac{N}{N-1}$: $\|u_n\|_{L^r(\Omega)} \le C_r$.

STEP 2 - We will use the inequality

$$\begin{cases} \text{there exists } a_r > 0, \text{ only depending on } r, \text{ such that} \\ t\log(1+s) \le t\log(1+t) + s^r + a_r \text{ for all } s,t \in \mathbb{R}_+. \end{cases}$$

Taking $[\log(1+|u_n|)]\mathrm{sign}(u_n)$ as test function in the weak formulation of (1.3), we obtain

$$(2.16)\qquad \left| \begin{array}{l} \alpha \displaystyle\int_\Omega \frac{|\nabla u_n|^p}{1+|u_n|} \le \int_\Omega |f|\log(1+|u_n|) \\ \le \|f\log(1+|f|)\|_{L^1(\Omega)} + \displaystyle\int_\Omega |u_n|^r + a_r \mathrm{meas}(\Omega). \end{array} \right.$$

Then the use of Hölder and Sobolev inequalities yield, since $\frac{p'}{p} = \frac{N}{N-1}$ and $1 - \frac{1}{p} = \frac{N-1}{2N-1}$,

(2.17)

$$\left| \begin{array}{l} S_1 \alpha^{\frac{1}{p}} \left[\displaystyle\int_\Omega |u_n|^{\frac{N}{N-1}} \right]^{\frac{N-1}{N}} \\ \le \alpha^{\frac{1}{p}} \displaystyle\int_\Omega |\nabla u_n| \le \left[\alpha \int_\Omega \frac{|\nabla u_n|^p}{1+|u_n|} \right]^{\frac{1}{p}} \left[\int_\Omega (1+|u_n|)^{\frac{p'}{p}} \right]^{\frac{1}{p'}} \\ \le \left[\|f\log(1+|f|)\|_{L^1(\Omega)} + C_r^r + a_r \mathrm{meas}(\Omega) \right]^{\frac{1}{p}} \left[\displaystyle\int_\Omega (1+|u_n|)^{\frac{N}{N-1}} \right]^{\frac{N-1}{2N-1}}. \end{array} \right.$$

Since $\frac{N-1}{2N-1} < \frac{N-1}{N}$, we proved that the sequence $\{u_n\}$ is bounded in $W_0^{1,1}(\Omega)$ and so it is compact in $L^r(\Omega)$, $1 \le r < \frac{N}{N-1}$.

Thus there exists $L^r(\Omega)$, $1 \le r < \frac{N}{N-1}$ and a subsequence (not relabelled) $\{u_n\}$ such that u_n converges to u in $L^r(\Omega)$ and almost everywhere.

STEP 3 - Taking $[\log(1+|u_n|) - \log(1+k)]\mathrm{sign}(u_n)$ as test function in the weak formulation of (1.3), we obtain

$$\left| \begin{array}{l} \alpha \displaystyle\int_{\{k\le|u_n|\}} \frac{|\nabla u_n|^p}{1+|u_n|} \le \int_{\{k\le|u_n|\}} |f|\log(1+|u_n|) \\ \le \displaystyle\int_{\{k\le|u_n|\}} |f|\log(1+|f|) + \int_{\{k\le|u_n|\}} |u_n|^r + a_r \mathrm{meas}\{k \le |u_n|\}, \end{array} \right.$$

which implies (following (2.17))

(2.18)

$$\left| \begin{array}{l} \displaystyle\int_{\{k\le|u_n|\}} |\nabla u_n| \le \\ \le C_1 \left[\displaystyle\int_{\{k\le|u_n|\}} |f|\log(1+|f|) + \int_{\{k\le|u_n|\}} |u_n|^r + a_r \mathrm{meas}\{k \le |u_n|\} \right]^{\frac{1}{p}}. \end{array} \right.$$

Thus, for every measurable subset E, thanks to (2.2), we can follow (2.12) and we obtain

(2.19)

$$\left| \begin{aligned} &\int_E \left|\frac{\partial u_n}{\partial x_i}\right| \le \int_E |\nabla u_n| \le \int_E |\nabla T_k(u_n)| + \int_{\{k\le |u_n|\}} |\nabla u_n| \\ &\le \operatorname{meas}(E)^{\frac{1}{p'}} \left[\frac{k}{\alpha}\|f\|_{L^1(\Omega)}\right]^{\frac{1}{p}} \\ &+C_1\left[\int_{\{k\le |u_n|\}} |f|\log(1+|f|) + \int_{\{k\le |u_n|\}} |u_n|^r + a_r \operatorname{meas}\{k\le |u_n|\}\right]^{\frac{1}{p}} \end{aligned}\right.$$

Thus we proved again the convergence (2.13) and we can repeat the last part of the proof of the previous theorem (mainly the convergence (2.14)) and then we can prove that $u \in W_0^{1,1}(\Omega)$ is a distributional solution of (1.3). □

Remark 2.4. *Note that*

$$\lim_{p\to 1} \frac{N}{N(p-1)+1} = N, \quad \lim_{p\to 2-\frac{1}{N}} \frac{N}{N(p-1)+1} = 1$$

Remark 2.5. Let $1 < p \le 2 - \frac{1}{N}$ and $\Omega = B(0, \frac{1}{2})$. Consider the boundary value problem

$$\begin{cases} -\Delta_p(u) = f(x) = \dfrac{1}{|x|^\alpha(-\log|x|)^\beta}, & \text{in } \Omega; \\ u = 0, & \text{on } \partial\Omega; \end{cases} \tag{2.20}$$

with α, $\beta > 0$. We look for radial solutions $u(x) = u(r)$, $r = |x|$, so that we have

$$-\frac{1}{r^{N-1}}\left(r^{N-1}|u'|^{p-2}u'\right)' = \frac{1}{r^\alpha(-\log r)^\beta}$$

and

$$|u'(s)| = \sqrt[p-1]{\frac{1}{s^{N-1}}\int_0^s \frac{t^{N-1-\alpha}}{(-\log t)^\beta}dt}\,. \tag{2.21}$$

Let now $\alpha = \frac{N}{m}$ and $\beta > \frac{N(p-1)+1}{N}$; thus $\alpha < N$ if $m > 1$. Then it risults

(2.22)

$$\int_{B(0,\frac{1}{2})} |\nabla u| = \int_0^{\frac{1}{2}} |u'(s)|s^{N-1}ds = \int_0^{\frac{1}{2}} s^{\frac{(N-1)(p-2)}{p-1}}\left(\int_0^s \frac{t^{N-1-\alpha}}{(-\log t)^\beta}dt\right)^{\frac{1}{p-1}}ds.$$

Now note that (using the de l'Hôpital rule)

$$\lim_{t\to 0}\frac{\displaystyle\int_0^s \frac{t^{N-1-\alpha}}{(-\log t)^\beta}dt}{\dfrac{t^{N-\alpha}}{(-\log t)^\beta}}=\frac{1}{N-\alpha}.$$

Thus, in (2.22), ∇u belongs to $(L^1(B(0,\frac12)))^N$ if

$$\begin{cases}\dfrac{\beta}{p-1}>1,\ \text{that is}\ \beta>p-1,\\ \dfrac{(N-1)(p-2)+N-\alpha}{p-1}=-1,\ \text{that is}\ \alpha=N(p-1)+1.\end{cases}$$

Note that $f\in L^m(\Omega)$, if $\alpha=\frac{N}{m}$ and $\beta>\frac{1}{m}$, which means now $m=\frac{N}{N(p-1)+1}$ and $\beta>\frac{N(p-1)+1}{N}$ (which is greater than $p-1$). Thus the example shows that the statement of Theorem 2.2 is optimal in the sense that u belongs to $W_0^{1,1}(\Omega)$ and u does not belong to $W_0^{1,q}(\Omega)$, $q>1$.

Let now $\alpha=N$, $\beta>2$. Then (2.21) is

$$|u'(s)|=\sqrt[p-1]{\frac{1}{s^{N-1}}\int_0^s\frac{(-\log t)^{-\beta}}{t}dt}=\sqrt[p-1]{\frac{1}{(\beta-1)s^{N-1}(-\log s)^{\beta-1}}}$$

$\frac{\beta-1}{p-1}>1$ Then ∇u belongs to $(L^1(B(0,\frac12)))^N$ if

$$\int_{B(0,\frac12)}|\nabla u|=\int_0^{\frac12}|u'(s)|s^{N-1}ds=C_\beta\int_0^{\frac12}\frac{1}{s(-\log s)^{\frac{\beta-1}{p-1}}}ds$$

is finite; that is if $\frac{\beta-1}{p-1}>1$. If $p=2-\frac1N$, the last inequality is $\beta>2-\frac1N$ and note that $N+1>2-\frac1N$. Moreover $\int_\Omega|f|\,\log(1+|f|)<\infty$ means that

$$\int_{B(0,\frac12)}\frac{1}{|x|^N(-\log|x|)^\beta}\ \log\Big(1+\frac{1}{|x|^N(-\log|x|)^\beta}\Big)<\infty,$$

which is true as consequence of $\int_{B(0,\frac12)}\frac{1}{|x|^N(-\log|x|)^{\beta-1}}<\infty$ (since $\beta>2$). Thus the example shows that the statement of Theorem 2.3 is optimal in the sense that u belongs to $W_0^{1,1}(\Omega)$ and u does not belong to $W_0^{1,q}(\Omega)$, $q>1$.

However, we recall that, as consequence of the convergence (2.14) and of a Theorem by De La Vallée Poussin, we can state that there exists a positive, continuous, even and convex real function, with the property

$$\lim_{t\to\infty}\frac{Q(t)}{t}=\infty,$$

such that

$$\sup_n\int_\Omega Q(|\nabla u_n|)<\infty.$$

Then the Fatou Lemma implies that $Q(|\nabla u|) \in L^1(\Omega)$.

3. Uniqueness

The uniqueness of infinite energy distributional solutions, in general, is not true: see [27].

However, in [19] it is observed that, if $p > 2 - \frac{1}{N}$, it is possible to select a solution: the only solution which is found by means of approximations. The author calls it the solution obtained as limit of approximations (SOLA). Here we follow this approach. A different point of view can be found in [1], [26].

In this section the differential operator does not depend on v, that is $A(v) = -\text{div}(a(x, \nabla v))$, and we study the uniqueness of the solution found by of approximation.

To be more precise, we assume (1.6) and the standard assumption

$$1 < p < 2, \quad [a(x,\xi) - a(x,\eta)][\xi - \eta] \geq \alpha \frac{|\xi - \eta|^2}{(1 + |\xi| + |\eta|)^{2-p}}. \tag{3.1}$$

Lemma 3.1. *Let $f \in L^m(\Omega)$, $m = \frac{N}{N(p-1)+1}$, $1 < p < 2 - \frac{1}{N}$. Consider the sequences $\{u_n\}$ and $\{f_n\}$ of Theorem 2.2, a sequence $\{g_n\}$ converging to f in $L^m(\Omega)$ and the solutions w_n of the Dirichlet problems*

$$w_n \in W_0^{1,p}(\Omega) : \; A(w_n) = g_n. \tag{3.2}$$

Then there exists a positive constant $Q = Q(\alpha, p, N, m)$ such that

$$\left| \begin{array}{l} S_1 \Big[\displaystyle\int_\Omega |\log(1 + |u_n - w_n|)|^{\frac{N}{N-1}} \Big]^{\frac{N-1}{N}} \leq \displaystyle\int_\Omega \frac{|\nabla(u_n - w_n)|}{1 + |u_n - w_n|} \leq \\ Q \Big[\displaystyle\int_\Omega |f_n - g_n| \Big]^{\frac{1}{2}}, \end{array} \right. \tag{3.3}$$

where S_1 is the Sobolev constant.

Proof. Define

$$g(t) = \frac{t}{1 + |t|}$$

and use $g(u_n - w_n)$ as test function in (2.1) and (3.2). Then we have

$$\int_\Omega [a(x, \nabla u_n) - a(x, \nabla w_n)] \nabla(u_n - w_n)\, g'(u_n - w_n) \leq \int_\Omega (f_n - g_n) g(u_n - w_n).$$

The assumption (3.1) gets

$$\int_\Omega \frac{|\nabla(u_n - w_n)|^2}{(1 + |\nabla u_n| + |\nabla w_n|)^{2-p}}\, g'(u_n - w_n) \leq \frac{1}{\alpha} \int_\Omega (f_n - g_n) g(u_n - w_n).$$

Then

$$\int_\Omega \frac{|\nabla(u_n - w_n)|}{1 + |u_n - w_n|} =$$

$$= \int_\Omega \frac{|\nabla(u_n - w_n)|\sqrt{g'(u_n - w_n)}}{(1+|\nabla u_n| + |\nabla w_n|)^{1-\frac{p}{2}}} \frac{(1+|\nabla u_n| + |\nabla w_n|)^{1-\frac{p}{2}}}{(1+|u_n - w_n|)\sqrt{g'(u_n - w_n)}} =$$

$$\leq \left[\int_\Omega \frac{|\nabla(u_n - w_n)|^2 g'(u_n - w_n)}{(1+|\nabla u_n| + |\nabla w_n|)^{2-p}} \right]^{\frac{1}{2}} \left[\int_\Omega \frac{(1+|\nabla u_n| + |\nabla w_n|)^{2-p}}{(1+|u_n - w_n|)^2 g'(u_n - w_n)} \right]^{\frac{1}{2}}$$

which implies that

$$\int_\Omega \frac{|\nabla(u_n - w_n)|}{1+|u_n - w_n|} \leq \left[\frac{1}{\alpha} \int_\Omega |f_n - g_n| \right]^{\frac{1}{2}} \left[\int_\Omega (1+|\nabla u_n| + |\nabla w_n|)^{2-p} \right]^{\frac{1}{2}}$$

From the assumption $m = \frac{N}{N(p-1)+1}$ and the a priori estimates (2.9) of Theorem 2.2 it follows that the last term is bounded, since $2-p \leq 1$.

□

THEOREM 3.2. *The solution u obtained in Theorem 2.2 is unique.*

PROOF. Consider the sequences $\{u_n\}$ and $\{f_n\}$ of Theorem 2.2, a sequence $\{g_n\}$ converging to f in $L^m(\Omega)$ and the solutions w_n of the Dirichlet problems 3.2. In the proof of Theorem 2.2 is proved that (up to a subsequence) u_n converges to u in $W_0^{1,1}(\Omega)$. The same proof says that (up to a subsequence) w_n converges in $W_0^{1,1}(\Omega)$ to a function w, distributional solution of (1.3). Now we pass to the limit in (3.3) and we obtain

$$S_1 \left[\int_\Omega |\log(1+|u-w|)|^{\frac{N}{N-1}} \right]^{\frac{N-1}{N}} \leq 0,$$

that is $u = w$. □

With the same proof it is possible to prove the following Theorem.

THEOREM 3.3. *The solution u obtained in Theorem 2.3 is unique.*

4. REGULARIZING EFFECT OF A LOWER ORDER TERM

Here we study the existence of $W_0^{1,1}(\Omega)$ solutions of the following "semilinear" problem

$$\begin{cases} A(u) + g(u) = f(x), & \text{in } \Omega; \\ u = 0, & \text{on } \partial\Omega; \end{cases} \tag{4.1}$$

where $g(t)$ is a lipschitz continuous, increasing real function such that

$$tg(t) \geq 0. \tag{4.2}$$

We assume that, for some $T^* \geq 0$,

$$b(t) = \begin{cases} 0, & t \in [0, T^*]; \\ \displaystyle\int_{T^*}^t \frac{ds}{g(s)^{m(p-1)}}, & t > T^*; \\ -b(-t), & t < 0; \end{cases} \tag{4.3}$$

is a bounded function: $|b(t)| \leq B$.

We refer to [14], [21], [11], [24] and [18] for the existence of infinite energy distributional solutions of the "semilinear" problems like (4.1), if the right hand side belongs to $L^m(\Omega)$ (with $m \geq 1$), $p > 2 - \frac{1}{N}$, $g(t)$ has a polynomial growth of order strictly greater than $p-1$.

THEOREM 4.1. *Let* $f \in L^m(\Omega)$, $1 \leq m < \frac{N}{N(p-1)+1}$, $1 < p < 2 - \frac{1}{N}$. *Assume (4.2) and (4.3). Then there exists a distributional solution* u *belonging to* $W_0^{1,1}(\Omega)$ *of the boundary value problem (4.1).*

PROOF. Consider now

$$u_n \in W_0^{1,p}(\Omega): \ A(u_n) + g(u_n) = f_n, \tag{4.4}$$

with $f_n = T_n(f)$. Recall that, for every $n \in I\!N$, u_n is a bounded function and that (see [7])

$$\int_{\{k \leq |u_n|\}} |g(u_n)|^m \leq \int_{\{k \leq |u_n|\}} |f_n|^m \leq \int_{\{k \leq |u_n|\}} |f|^m. \tag{4.5}$$

Moreover the use of $b(u_n)$ as test function in (4.4) yields, dropping a positive term,

$$\alpha \int_\Omega \frac{|\nabla u_n|^p}{|g(u_n)|^{m(p-1)}} \leq B \|f\|_{L^1(\Omega)} .$$

Thus we have

$$\left| \begin{array}{l} \alpha \displaystyle\int_\Omega |\nabla u_n| = \alpha \int_\Omega \frac{|\nabla u_n|}{|g(u_n)|^{m(1-\frac{1}{p})}} |g(u_n)|^{m(1-\frac{1}{p})} \\ \leq B^{\frac{1}{p}} \|f\|_{L^1(\Omega)}^{\frac{1}{p}} \left[\displaystyle\int_\Omega |g(u_n)|^m \right]^{\frac{1}{p'}} \leq B^{\frac{1}{p}} \|f\|_{L^1(\Omega)}^{\frac{1}{p}} \|f\|_{L^m(\Omega)}^{\frac{1}{p'}} , \end{array} \right.$$

which implies that the sequence $\{u_n\}$ is bounded in $W_0^{1,1}(\Omega)$ and it is compact in $L^r(\Omega)$, $1 \leq r < \frac{N}{N-1}$. Thus there exists $L^r(\Omega)$, $1 \leq r < \frac{N}{N-1}$ and a subsequence (not relabelled) $\{u_n\}$ such that u_n converges to u in $L^r(\Omega)$ and almost everywhere. Moreover the inequality (4.5) yields, for every measurable subset E,

$$\int_E |g(u_n)|^m \leq [\sup_{|t| \leq k} |g(t)|^m] \text{meas}(E) + \int_{\{k \leq |u_n|\}} |f|^m,$$

so that the Vitali Theorem implies

$$\text{the convergence in } L^m(\Omega) \text{ of } g(u_n) \text{ to } g(u). \tag{4.6}$$

Moreover, thanks again to (4.5),

$$\left| \begin{array}{l} \alpha \displaystyle\int_{\{k \leq |u_n|\}} |\nabla u_n| \leq \left(B \|f\|_{L^1(\Omega)}\right)^{\frac{1}{p}} \Big[\int_{\{k \leq |u_n|\}} |g(u_n)^m| \Big]^{\frac{1}{p'}} \\ \leq \left(B \|f\|_{L^1(\Omega)}\right)^{\frac{1}{p}} \Big[\displaystyle\int_{\{k \leq |u_n|\}} |f|^m \Big]^{\frac{1}{p'}}, \end{array} \right.$$

so that, for every measurable subset E, we have, thanks to (2.2),

$$\begin{aligned}\alpha\int_E |\nabla u_n| &\le \alpha \int_{\{k\le|u_n|\}} |\nabla u_n| + \alpha\int_E |\nabla T_k(u_n)| \\ &\le \left(B\,\|f\|_{L^1(\Omega)}\right)^{\frac1p}\left[\int_{\{k\le|u_n|\}} |f|^m\right]^{\frac{1}{p'}} + \alpha\left[k\,\frac{\|f\|_{L^1(\Omega)}}{\alpha}\right]^{\frac1p}\operatorname{meas}(E)^{\frac{1}{p'}}\end{aligned}$$

which implies (2.13).

The almost everywhere convergence of ∇u_n to ∇u, proved in Lemma 5.1, and (2.13) allow us to use the Vitali Theorem. Thus we proved again the convergence (2.14).

The third step is equal to the third step of Theorem 2.2. Thus we proved that $u \in W_0^{1,1}(\Omega)$ is a distributional solution of (4.1). □

5. Appendix

In order to have a selfcontained paper, we prove here the following lemma, which is almost the same of the main lemma of [3] and [6].

Lemma 5.1. *Let $\{u_n\}$ be the sequence defined in (2.1). Assume (1.2), (1.4), (1.6) and that*

$$\begin{cases} \|u_n\|_{W_0^{1,1}(\Omega)} \le M, \\ u_n \text{ converges to } u \text{ almost everywhere}, \\ \nabla T_k(u_n) \text{ converges weakly to } \nabla T_k(u) \text{ in } W_0^{1,p}(\Omega). \end{cases} \tag{5.1}$$

Then ∇u_n converges (up to a subsequence) a.e. to ∇u.

Proof. Let $0<\theta<\frac1p$ and $k>0$. Consider

$$I_{\Omega,n} = \int_\Omega \{[a(x,u_n,\nabla u_n) - a(x,u_n,\nabla u)]\nabla(u_n-u)\}^\theta$$

We shall prove that the previous integral converges to zero. Indeed, it is equal to

$$\begin{aligned}&\int_{C_k} \{[a(x,u_n,\nabla u_n) - a(x,u_n,\nabla u)]\nabla(u_n-u)\}^\theta \\ &+\int_{A_k} \{[a(x,u_n,\nabla u_n) - a(x,u_n,\nabla u)]\nabla(u_n-u)\}^\theta \\ &\qquad = I_{C_k,n} + I_{A_k,n},\end{aligned}$$

where

$$C_k = \{x\in\Omega : |u(x)|\le k\}, \qquad A_k = \{x\in\Omega : |u(x)|>k\}.$$

We can write $I_{C_k,n}$ as

$$\int_{C_k} \{[a(x,u_n,\nabla u_n) - a(x,u_n,\nabla T_k(u))]\nabla(u_n - T_k(u))\}^\theta,$$

which is smaller than

$$\int_\Omega \{[a(x,u_n,\nabla u_n)-a(x,u_n,\nabla T_k(u))]\nabla(u_n-T_k(u))\}^\theta \;=\; J_{\Omega,n},$$

since the integrand is positive. Then the use of Hölder inequality (with exponents $\frac{1}{p\theta}$ and $\frac{1}{1-p\theta}$) and (1.6) in $I_{A_k,n}$ imply that

$$I_{C_k,n}+I_{A_k,n} \le J_{\Omega,n}+C_1 \left[\int_{A_k}(|\nabla u_n|+|u_n|+|\nabla u|+|u|)\right]^{p\theta} meas(A_k)^{1-p\theta}.$$

By means of the estimate $\|u_n\|_{W_0^{1,1}(\Omega)} \le M$, we get

$$I_{C_k,n} \;+\; I_{A_k,n} \;\le J_{\Omega,n} \;+\; C_2\; meas(A_k)^{1-p\theta} \;= J_{\Omega,n} \;+\; \omega_1(k).$$

Where denote by by $\omega_i(k)$ quantities such that $\lim_{k\to\infty}\omega_i(k)=0$. Now we study the behaviour of $J_{\Omega,n}$; it can be splitted as ($j \in I\!N$)

$$\int_{\{|u_n(x)-T_k(u)|\le j\}} \{[a(x,u_n,\nabla u_n)-a(x,u_n,\nabla T_k(u))]\nabla[u_n-T_k(u)]\}^\theta$$

$$+\int_{\{|u_n(x)-T_k(u)|>j\}} \{[a(x,u_n,\nabla u_n)-a(x,u_n,\nabla T_k(u))]\nabla[u_n-T_k(u)]\}^\theta.$$

The first integral can be written as

$$\int_\Omega \{[a(x,u_n,\nabla u_n)-a(x,u_n,\nabla T_k(u))]\nabla T_j[u_n-T_k(u)]\}^\theta.$$

Then we use twice the Hölder inequality (with exponents $\frac{1}{\theta}$ and $\frac{1}{1-\theta}$ and with exponents $\frac{1}{p\theta}$ and $\frac{1}{1-p\theta}$) and the estimate $\|u_n\|_{W_0^{1,1}(\Omega)} \le M$ yield

$$\left(\int_\Omega [a(x,u_n,\nabla u_n)-a(x,u_n,\nabla T_k(u))]\nabla T_j[u_n-T_k(u)] \quad\right)^\theta (meas\,\Omega)^{1-\theta}$$

$$+C_3\; meas\{x\in\Omega : |u_n(x)-T_k(u(x))|>j\}^{1-p\theta}.$$

Thus, the use of $T_j[u_n-T_k(u)]$ in (2.1) implies that

$$J_{\Omega,n} \;\le C_4 \left(\int_\Omega f_n\; T_j[u_n-T_k(u)] - \int_\Omega \{a(x,u_n,\nabla T_k(u))\}\nabla T_j[u_n-T_k(u)]\right)^\theta$$

$$+\; C_3\; meas\{x\in\Omega : |u_n(x)-T_k(u(x))|>j\}^{1-p\theta}.$$

We remark that

$$\lim_{n\to\infty}\int_\Omega f_n\; T_j[u_n-T_k(u)] = \int_\Omega f\; T_j[u-T_k(u)] \;=\; \omega_2(k);$$

for $n > j+k$ and for almost every j we have

$$\lim_{n\to\infty}\int_\Omega a(x,u_n,\nabla T_k(u))\nabla T_j[u_n-T_k(u)] = \int_\Omega a(x,u,\nabla T_k(u))\nabla T_j[u-T_k(u)] \;=0;$$

$$\limsup_{n\to\infty} meas\{|u_n(x)-T_k(u(x))|>j\}^{1-p\theta} \;\le\; meas\{|u(x)-T_k(u(x))|\ge j\}^{1-p\theta}$$

$$=\;\omega_3(k).$$

Thus

$$\lim_{n\to\infty} J_{\Omega,n} \le C_1\omega_2(k)^\theta + C_3\omega_3(k).$$

Hence we have proved that

$$\lim_{n\to\infty} [I_{C_k,n} + I_{A_k,n}] \le \omega_1(k) + C_1\omega_2(k)^\theta + C_3\omega_3(k).$$

Therefore

$$\int_\Omega \{[a(x,u_n,\nabla u_n) - a(x,u_n,\nabla u)]\nabla(u_n-u)\}^\theta \;\to\; 0,$$

that is

$$\|\{[a(x,u_n,\nabla u_n) - a(x,u_n,\nabla u)]\nabla(u_n-u)\}^\theta\|_{L^1(\Omega)} \;\to\; 0,$$

which implies (for a suitable subsequence, still denoted by u_n)

$$\{[a(x,u_n,\nabla u_n) - a(x,u_n,\nabla u)]\nabla(u_n-u)\}^\theta \;\to\; 0 \quad \text{almost everywhere},$$

and also (since θ is positive)

$$\{[a(x,u_n,\nabla u_n) - a(x,u_n,\nabla u)]\nabla(u_n-u)\} \;\to\; 0 \quad \text{almost everywhere}.$$

Then, in [25], it is proved that, under our assumptions on the function $a(x,s,\xi)$, the previous limit implies that

$$\nabla u_n(x) \;\to\; \nabla u(x) \qquad \text{almost everywhere}.$$

□

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References

[1] P. Benilan, L. Boccardo, T. Gallouët, R. Gariepy, M. Pierre, J.L. Vazquez: An L^1 theory of existence and uniqueness of nonlinear elliptic equations; Ann. Scuola Norm. Sup. Pisa, 22 (1995), 240–273.

[2] M. Bildhauer: A priori gradient estimates for bounded generalized solutions of a class of variational problems with linear growth; J. Convex Anal. 9 (2002), 117–137.

[3] L. Boccardo: Some nonlinear Dirichlet problems in L^1 involving lower order terms in divergence form. Progress in elliptic and parabolic partial differential equations (Capri, 1994), 43–57, Pitman Res. Notes Math. Ser., 350, Longman, Harlow, 1996.

[4] L. Boccardo: The role of truncates in nonlinear Dirichlet problems in L^1. Nonlinear partial differential equations (Fés, 1994), 42–53, Pitman Res. Notes Math. Ser., 343, Longman, Harlow, 1996.

[5] L. Boccardo, G. Croce, L. Orsina: Nonlinear degenerate elliptic problems with $W_0^{1,1}$ solutions; Manuscripta Math. to appear.

[6] L. Boccardo, A. Dall'Aglio, T. Gallouët, L. Orsina: Nonlinear parabolic equations with measure data; J. Funct. Anal., 147 (1997), 237–258.

[7] L. Boccardo, T. Gallouët: Nonlinear elliptic and parabolic equations involving measure data; J. Funct. Anal., 87 (1989), 149–169.

[8] L. Boccardo, T. Gallouët: Nonlinear elliptic equations with right hand side measures; Comm. Partial Differential Equations, 17 (1992), 641–655.
[9] L. Boccardo, T. Gallouet: Problèmes unilatéraux dans L^1; C. R. Acad. Sc; Paris 311 (1990), 617–619.
[10] L. Boccardo, T. Gallouët, L. Orsina: Existence and uniqueness of entropy solutions for nonlinear elliptic equations with measure data; Ann. Inst. H. Poincaré Anal. Non Linéaire 13 (1996), 539–551.
[11] L. Boccardo, T. Gallouët, J.L. Vazquez: Nonlinear elliptic equations in R^N without growth restrictions on the data; J. Diff. Eq. 105 (1993), 334-363.
[12] E. Bombieri, E. De Giorgi and M. Miranda: Una maggiorazione a priori relativa alle ipersuperfici minimali non parametriche; Arch. Rational Mech. Anal. 32 (1969) 255–267.
[13] H. Brezis: Equations et inéquations non linéaires dans les espaces vectoriels en dualité; Ann. Inst. Fourier (Grenoble), 18 (1968), 115–175.
[14] H. Brezis: Semilinear equations in $\mathbb{R}^N$ without condition at infinity; Appl. Math. Optim. 12 (1984), 271–282.
[15] H. Brezis: Some variational problems of the Thomas-Fermi type; in *Variational inequalities and complementarity problems*, Cottle, Gianessi, and Lions eds., Wiley, New York (1980), 53–73.
[16] H. Brezis, W. Strauss: Semi-linear second-order elliptic equations in L^1; J. Math. Soc. Japan, 25 (1973), 565–590.
[17] F.E. Browder: Existence theorems for nonlinear partial differential equations; Proceedings of Symposia in Pure Mathematics, vol. XVI, American Mathematical Society, Providence (1970), 1–60.
[18] G.R. Cirmi: Regularity of the solutions to nonlinear elliptic equations with a lower order term; Nonlinear Anal. 25 (1995), 569–580.
[19] A. Dall'Aglio: Approximated solutions of equations with L^1 data. Application to the H-convergence of quasi-linear parabolic equations; Ann. Mat. Pura Appl. (4) 170 (1996), 207–240.
[20] T. Gallouët, J.M. Morel: Resolution of a semilinear equation in L^1; Proc. Roy. Soc. Edinburgh, 96 (1984), 275–288.
[21] T. Gallouët, J.M. Morel, The equation $-\Delta u + |u|^{\alpha-1}u = f$, for $0 \leq \alpha \leq 1$; Nonlinear Anal. **11** n.8 (1987), 893–912.
[22] E. Giusti: Minimal surfaces and functions of bounded variation; Monographs in Mathematics, 80. Birkhäuser Verlag, Basel, 1984.
[23] H. Jenkins, J. Serrin: The Dirichlet problem for minimal surface equation in higher dimension; J. Reine Ang. Math. 229 (1968), 170–187.
[24] F. Leoni: Nonlinear elliptic equations in R^N with absorbing zero order terms; Adv. Differential Equations 5 (2000), 681–722.
[25] J. Leray, J.L. Lions: Quelques résultats de Višik sur les problèmes elliptiques semi-linéaires par les méthodes de Minty et Browder; Bull. Soc. Math. France, 93 (1965), 97–107.
[26] M.M. Porzio: An uniqueness result for monotone elliptic problems; C.R. Acad. Sci. Paris, Ser. I 337 (2003), 313–316.
[27] J. Serrin: Pathological solutions of elliptic differential equations; Ann. Scuola Norm. Sup. Pisa Cl. Sci., 18 (1964), 385–387.
[28] G. Stampacchia: Le problème de Dirichlet pour les équations elliptiques du second ordre à coefficients discontinus; Ann. Inst. Fourier (Grenoble), 15 (1965), 189–258.
[29] R. Temam: Solutions généralisées de certaines équations du type hypersurfaces minima; Arch. Rational Mech. Anal. 44 (1971/72), 121–156.

E-mail address: boccardo@mat.uniroma1.it, gallouet@cmi.univ-mrs.fr