COMPACTNESS OF MINIMIZING SEQUENCES

LUCIO BOCCARDO AND THIERRY GALLOUET

ABSTRACT. We consider a minimization problem of a functional in the space $W_0^{1,p}(\Omega)$, where $1 < p < +\infty$ and Ω is a bounded open set of $I\!R^N$. We prove the compactness, in the space $W_0^{1,p}(\Omega)$, under convenient hypotheses, of a minimizing sequence. The main difficulty is to prove the convergence in measure of the gradient of the minimizing sequence. Furthermore, considering a sequence of minimization problems in the space $W_0^{1,p}(\Omega)$, we prove some convergence results of the sequence of minimizers to the minimizer of the limit problem.

Dedicado al Patriarca por su primavera [1]

1. Introduction and main results

We deal with integral problems where the functional are defined as

$$J(v) = \int_\Omega j\big(x, v, \nabla v\big) - \int_\Omega fv, \tag{1}$$

where Ω is a bounded domain of $I\!R^N$, $N \geq 1$, and $j : \Omega \times I\!R \times I\!R^N \to I\!R$ is a Carathéodory function, that is, measurable with respect to x in Ω for every (s, ξ) $I\!R \times I\!R^N$, and continuous with respect to (s, ξ) in $I\!R \times I\!R^N$ for almost every x in Ω.
We assume that there exist $g \in L^1(\Omega)$ and real positive constants α, β such that for almost every x in Ω, for every s in $I\!R$, for every ξ and η in $I\!R^N$ we have

$$\alpha|\xi|^p \leq j\big(x, s, \xi\big), \tag{2}$$

$$j\big(x, s, \xi\big) \leq \beta(|\xi|^p + |s|^p) + g(x), \tag{3}$$

$$f(x) \in L^m(\Omega), \quad m \geq (p^\star)', \tag{4}$$

where $1 < p$, $(p^\star)'$ is the Sobolev conjugate of p, if $1 < p < N$, it is any number greater than 1 if $p = N$, and $m = 1$ if $p > N$.
Thus $J(v)$ is well defined in $W_0^{1,p}(\Omega)$.

Key words and phrases. minimizing sequences, compactness.

[1] (see also [2])

Theorem 1. *We assume (2), (3), (4) and*

$$j\big(x, s, \xi\big) \text{ is strictly convex with respect to } \xi, \tag{5}$$

for a.e. $x \in \Omega$ *and all* $s \in I\!R$. *Then the minimizing sequences of* J, *defined in* (1), *are compact in* $W_0^{1,p}(\Omega)$. *Furthermore, if* u *is a limit of a minimizing sequence, then it is a minimizer of* J.

The situation, described in Theorem 1 is known in the Calculus of Variations, in some simple cases, where it is easy to prove that a weakly convergent minimizing sequence is also strongly convergent (see Remark 4). Our approach use deeply Real Analysis techniques and it is slightly close a method used in [4].
Moreover, we point out some relationships with the results of the papers [5], [8], [7]. In [5], is proved that, under some assumptions on the strictly convex function $j : I\!R^M \to I\!R$, if $(u_n)_{n\in I\!N}$ and u are functions in $L^1(\Omega, I\!R^M)$, the sequence (u_n) converges weakly in $\mathcal{D}'$ (convergence assumption weaker than the assumption of the previous papers) and $\limsup \int_\Omega j(u_n) \le \int_\Omega j(u)$, then (u_n) converges strongly in $L^1(\Omega, I\!R^M)$.
Theorem 1 is also true if Hypothesis (4) is replaced by $f \in W^{-1,p'}(\Omega)$ with $p' = p/(p-1)$ and, in (1), $\int_\Omega fv$ is replaced by the duality product between f and v. We prove Theorem 1 in Section 2.
An adaptation of the proof of Theorem 1 gives the following result on the convergence of the sequence of minimizers associated to a sequence of data $(f_n)_{n\in I\!N}$. We denote by $< \cdot, \cdot >$ the duality product between $W^{-1,p'}(\Omega)$ and $W_0^{1,p}(\Omega)$.

Theorem 2. *We assume (2), (3) and (5). We assume furthermore that* j *does not depend of its second argument. Let* $(f_n)_{n\in I\!N}$ *be a sequence of* $W^{-1,p'}(\Omega)$ *and* f *such that*

$$f_n \text{ converges to } f \text{ in } W^{-1,p'}(\Omega), \text{ as } n \to \infty. \tag{6}$$

Let u *be the minimizer (in* $W_0^{1,p}(\Omega)$*) of* $\int_\Omega j(x, \nabla v) - < f, v >$ *and, for all* n, *let* u_n *be the minimizer (in* $W_0^{1,p}(\Omega)$*) of* $\int_\Omega j(x, \nabla u_n) - < f_n, v >$. *Then the sequence* $\{u_n\}$ *converges to* u *in* $W_0^{1,p}(\Omega)$.

In Theorem 2, the existence of u (and of u_n for all n) is an easy consequence of (2), (3), (5). In order to prove the uniqueness of u (and of u_n for all n) we also use the fact that j does not depend on its second argument. Indeed, let $v, w \in W_0^{1,p}(\Omega)$ such that $v \neq w$. Let $A = \{\nabla v \neq \nabla w\}$. One has, thanks to (5),

$$j(\cdot, \frac{1}{2}\nabla v + \frac{1}{2}\nabla w) < \frac{1}{2} j(\cdot, \nabla v) + \frac{1}{2} j(\cdot, \nabla w) \text{ a.e. on } A,$$

Then, since the measure of A is positive, this gives $J(\frac{1}{2}\nabla v + \frac{1}{2}\nabla w) < \frac{1}{2}J(v) + \frac{1}{2}J(w)$ and proves the uniqueness of the minimizers in Theorem 2.
Finally, the proof of the convergence of u_n to u in $W_0^{1,p}(\Omega)$ is given in Section 3.
A natural question consists to replace in Theorem 2 the hypothesis 6 by the hypothesis

$$f_n \text{ converges to } f \text{ weakly in } W^{-1,p'}(\Omega), \text{ as } n \to \infty. \tag{7}$$

If $p = 2$, the conclusion of Theorem 2 becomes that $u_n \to u$ weakly in $W_0^{1,p}(\Omega)$. This is quite easy to prove, thanks to fact that the Euler-Lagrange equation of this minimization problem is linear. If $p \neq 2$, this result is not true. A counter example is given in Section 4. However, we have a convergence result of u_n to u, with an additional hypothesis on the sequence $(f_n)_{n \in \mathbb{N}}$. This is given in Theorem 3, whose proof is also in Section 3.

Theorem 3. *We assume (2), (3), (5) and that j does not depend of its second argument. Let $(f_n)_{n \in \mathbb{N}}$ be a sequence of $W^{-1,p'}(\Omega)$ and f satisfying Hypothesis (7). We assume furthermore that f_n and f are functions satisfying (4) and*

$$f_n \text{ converges to } f \text{ weakly in } L^1(\Omega), \text{ as } n \to \infty. \tag{8}$$

Let u be the minimizer (in $W_0^{1,p}(\Omega)$) of $\int_\Omega j(x, \nabla v) - \int_\Omega fv$ and, for all n, let u_n be the minimizer (in $W_0^{1,p}(\Omega)$) of $\int_\Omega j(x, \nabla u_n) - \int_\Omega f_n v$.
Then the sequence $\{u_n\}$ converges to u in $W_0^{1,s}(\Omega)$ for all $1 \leq s < p$ (in particular $\nabla u_n \to \nabla u$ in measure) and weakly in $W_0^{1,p}(\Omega)$.

Theorem 3 is interesting only in the case $p \leq N$. Indeed, in the case $p > N$, Hypothesis (7) gives the convergence of f_n to f in $W^{-1,p'}(\Omega)$ and therefore Theorem 2 gives that u_n converges to u in $W_0^{1,p}(\Omega)$.

2. Compactness of minimizing sequences

In this section we prove Theorem 1. The assumptions (2), (4) imply that, for all $v \in W_0^{1,p}(\Omega)$,

$$J(v) \geq \alpha \int_\Omega |\nabla v|^p - C_1 \|v\|_{W_0^{1,p}(\Omega)}, \tag{9}$$

for some positive constant $C_1 > 0$, only depending on f. Since $p > 1$, $J(v)$ is bounded from below. Let $I = \inf\{J(v),\ v \in W_0^{1,p}(\Omega)\}$. Thus $I \in \mathbb{R}$.

Let $\{u_n\}$ be a minimizing sequence, that is $J(u_n) \to I$ as $n \to \infty$. The inequality (9) and $p > 1$ imply that the sequence $\{u_n\}$ is bounded in $W_0^{1,p}(\Omega)$. This ensures the existence of a subsequence (not relabelled) and a function $u \in W_0^{1,p}(\Omega)$ such that

$$u_n \text{ converges weakly to } u \text{ in } W_0^{1,p}(\Omega). \tag{10}$$

Moreover, thanks to the assumptions on the function $j\big(x, s, \xi\big)$, a classic semicontinuity result, due to Ennio De Giorgi (see [6], [3]), we have

$$\int_\Omega j\big(x, u, \nabla u\big) \leq \liminf_{n\to+\infty} \int_\Omega j\big(x, u_n, \nabla u_n\big).$$

Then, since $\lim_{n\to\infty} \int f u_n = \int f u$, one has $J(u) = I$ (and u is a minimizer of J) and also

$$\int_\Omega j\big(x, u_n, \nabla u_n\big) \to \int_\Omega j\big(x, u, \nabla u\big).$$

Moreover, once more, the semicontinuity theorem says that

$$\int_\Omega j\big(x, u, \nabla u\big) \leq \liminf_{n\to+\infty} \int_\Omega j\Big(x, u_n, \frac{\nabla u_n + \nabla u}{2}\Big). \tag{11}$$

Furthermore, since $u_n \to u$ in $L^p(\Omega)$, assumption (3) imply

$$\int_\Omega j\big(x, u_n, \nabla u\big) \to \int_\Omega j\big(x, u, \nabla u\big), \tag{12}$$

so that

$$\limsup_{n\to+\infty} \int_\Omega \Big[\frac{1}{2} j\big(x, u_n, \nabla u_n\big) + \frac{1}{2} j\big(x, u_n, \nabla u\big) - j\big(x, u_n, \frac{\nabla u_n + \nabla u}{2}\big)\Big] \leq 0.$$

Thus, using the convexity of j with respect to its third argument, we have

$$\lim_{n\to+\infty} \int_\Omega \Big[\frac{1}{2} j\big(x, u_n, \nabla u_n\big) + \frac{1}{2} j\big(x, u_n, \nabla u\big) - j\Big(x, u_n, \frac{\nabla u_n + \nabla u}{2}\Big)\Big] = 0. \tag{13}$$

Now, following [4], we will prove that

$$\nabla u_n \to \nabla u \text{ in measure.} \tag{14}$$

Let $\lambda > 0$ and $\varepsilon > 0$. We want to prove that there exists ν such that

$$n > \nu \Rightarrow \text{measure } (\{x \in \Omega : |\nabla u_n(x) - \nabla u(x)| > \lambda\}) \leq 2\varepsilon. \tag{15}$$

Since the sequence $(u_n)_{n\in I\!N}$ is bounded in $W_0^{1,p}(\Omega)$, we first remark that there exists $k>0$ such that

$$\text{measure }(\mathcal{G}_n)\le\varepsilon,\quad \forall\, n\in I\!N,$$

where

$$\mathcal{G}_n=\{|\nabla u_n(x)|>k\}\cup\{|\nabla u(x)|>k\}\cup\{|u_n(x)|>k\}.$$

We now define

$$\mathcal{K}=\{(s,\xi,\eta)\in I\!R^{2N+1}:\ |s|\le k,\ |\xi|\le k,\ |\eta|\le k,\ |\xi-\eta|\ge\lambda\}.$$

The set $\mathcal{K}$ is compact and the function

$$\gamma(x)=\min_{(s,\xi,\eta)\in\mathcal{K}}\Big[\frac{1}{2}\,j\big(x,s,\xi\big)+\frac{1}{2}\,j\big(x,s,\eta\big)-j\big(x,s,\frac{\xi+\eta}{2}\big)\Big],$$

which is defined in Ω, satisfies the assumptions of Lemma 5, thanks to (5). Setting

$$\mathcal{L}_n=\{|\nabla u_n(x)|\le k,|\nabla u(x)|\le k,|u_n(x)|\le k,|\nabla u_n(x)-\nabla u(x)|\ge\lambda\},$$

we note that, if $x\in\mathcal{L}_n$,

$$\gamma(x)\le\Big[\frac{1}{2}\,j\big(x,u_n,\nabla u_n\big)+\frac{1}{2}\,j\big(x,u_n,\nabla u\big)-j\Big(x,u_n,\frac{\nabla u_n+\nabla u}{2}\Big)\Big].$$

Then, we have

$$\int_{\mathcal{L}_n}\gamma(x)\ \le\int_{\mathcal{L}_n}\Big[\frac{1}{2}\,j\big(x,u_n,\nabla u_n\big)+\frac{1}{2}\,j\big(x,u_n,\nabla u\big)-j\big(x,u_n,\frac{\nabla u_n+\nabla u}{2}\big)\Big]$$

$$\le\int_{\Omega}\Big[\frac{1}{2}\,j\big(x,u_n,\nabla u_n\big)+\frac{1}{2}\,j\big(x,u_n,\nabla u\big)-j\big(x,u_n,\frac{\nabla u_n+\nabla u}{2}\big)\Big]$$

and it now follows from (13) that $\int_{\mathcal{L}_n}\gamma(x)\to 0$, which implies that measure $(\mathcal{L}_n)\to 0$ (thanks to assumption (5) and Lemma 5). Then, there exists $\nu\in I\!N$ such that

$$n\ge\nu\Rightarrow\text{measure }(\mathcal{L}_n)\le\varepsilon.$$

and, for $n>\nu$,

$$\text{measure }(\{x\in\Omega:|\nabla u_n(x)-\nabla u(x)|>\lambda\})$$
$$\le\text{measure }(\mathcal{G}_n)+\text{measure }(\mathcal{L}_n)\le 2\varepsilon.$$

So we proved (15) which gives

$$\nabla u_n(x)\ \text{ converges in measure to }\ \nabla u(x). \tag{16}$$

Now we follow the classical proof of the Lebesgue Theorem (with the convergence in measure). Since

$$|\nabla u_n - \nabla u|^p \le \frac{2^p}{\alpha}\Big[j\big(x,u_n,\nabla u_n\big)+j\big(x,u,\nabla u\big)\Big], \tag{17}$$

we have

$$\frac{2^p}{\alpha}\Big(j\big(x,u_n,\nabla u_n\big)+j\big(x,u,\nabla u\big)\Big) - |\nabla u_n - \nabla u|^p \ge 0.$$

We can apply Fatou's Lemma (with respect to the convergence in measure) and we obtain

$$\frac{2^{p+1}}{\alpha}\int_\Omega j\big(x,u,\nabla u\big) \le \frac{2^{p+1}}{\alpha}\int_\Omega j\big(x,u,\nabla u\big) - \limsup_{n\to+\infty}\int_\Omega |\nabla u_n - \nabla u|^p.$$

This gives

$$\limsup_{n\to+\infty}\int_\Omega |\nabla u_n - \nabla u|^p \le 0,$$

so that $u_n \to u$ in $W_0^{1,p}(\Omega)$, as $n \to +\infty$, and concludes the proof of Theorem 1.

REMARK 4. *If $j\big(x,s,\xi\big) = |\xi|^p$, the proof of Theorem 1 follows easily from the Clarkson inequality and* (13).

3. Convergence of a sequence of minimizers

In this section, we first give a proof of Theorem 2. It follows closely the proof of Theorem 1. We first remark that the sequence $(u_n)_{n\in\mathbb{N}}$ is bounded in $W_0^{1,p}(\Omega)$. Then, up to subsequence, $u_n \to u^\star$ weakly in $W_0^{1,p}(\Omega)$, as $n \to +\infty$.
We now prove that $u^\star = u$ (this will give in particular that the whole sequence $(u_n)_{n\in\mathbb{N}}$ converges to u, thanks to the uniqueness of the minimizer).
As in Theorem 1, the semicontinuity result gives

$$\int_\Omega j(x,\nabla u^\star) \le \liminf_{n\to+\infty}\int_\Omega j(x,\nabla u_n)$$

Furthermore, thanks to (6), one has $\int_\Omega f_n u_n \to \int_\Omega f u^\star$ as $n \to +\infty$ and then

$$\int_\Omega j(x,\nabla u^\star) - \int_\Omega f u^\star \le \liminf_{n\to+\infty}\int_\Omega j(x,\nabla u_n) - \lim_{n\to+\infty}\int_\Omega f_n u_n.$$

Since u_n is the minimizer associated to f_n, one has, for all $v \in W_0^{1,p}(\Omega)$,

$$\int_\Omega j(x,\nabla u_n) - \int_\Omega f_n u_n \le \int_\Omega j(x,\nabla v) - \int_\Omega f_n v. \tag{18}$$

Passing to the lim inf in this inequality, we obtain that $u^\star = u$ (recall that u is the minimizer of the functional J). Taking $v = u$ in (18), we also obtain that

$$\int_\Omega j(x, \nabla u_n) - \int_\Omega f_n u_n \to \int_\Omega j(x, \nabla u) - \int_\Omega f u \tag{19}$$

and then

$$\int_\Omega j(x, \nabla u_n) \to \int_\Omega j(x, \nabla u).$$

We now follow exactly the proof of Theorem 1. It gives that

$$\nabla u_n(x) \text{ converges in measure to } \nabla u(x),$$

and finally that $u_n \to u$ in $W_0^{1,p}(\Omega)$ as $n \to \infty$. This concludes the proof of Theorem 2. Note that, in this proof, a main tool was the fact that $\int_\Omega f_n u_n \to \int_\Omega f u^\star$. A tool which is no longer true if Hypothesis (6) on the sequence $(f_n)_{n\in I\!N}$ is replaced by Hypothesis (7).

We now give the proof of Theorem 3. We begin as in the previous proof, The sequence $\{u_n\}$ is bounded in $W_0^{1,p}(\Omega)$. Then, up to a subsequence, we have $u_n \to u^*$ weakly in $W_0^{1,p}(\Omega)$, as $n \to +\infty$. We can also assume that $u_n \to u^\star$ a.e.. The new difficulty with respect to the previous proof is that we do not have $\int_\Omega f_n u_n \to \int_\Omega f u^\star$.
We fix $h \in I\!R^+$ and let $T_h(s) = \max\{\min\{k, s\} - k\}$, $s \in I\!R$. Using the minimality of u_n and u_m, we get (here and in (24) we follow some techniques of [1])

$$\int_\Omega j(x, \nabla u_n) \le \int_\Omega j\Big(x, \nabla(u_n - \frac{T_h(u_n - u_m)}{2})\Big) + \int_\Omega f_n \frac{T_h(u_n - u_m)}{2},$$

and

$$\int_\Omega j(x, \nabla u_m) \le \int_\Omega j\Big(x, \nabla(u_m - \frac{T_h(u_m - u_n)}{2})\Big) + \int_\Omega f_m \frac{T_h(u_m - u_n)}{2}.$$

Adding these equations, this gives

$$\int_{|u_m - u_n| \le h} \Big(j(x, \nabla u_m) + j(x, \nabla u_n) - 2j(x, \frac{\nabla u_m + \nabla u_n}{2})\Big) \le \frac{1}{2}\int_\Omega (f_m - f_n) T_h(u_m - u_n).$$

Since $f_n \to f$ weakly in $L^1(\Omega)$ and $u_n \to u^\star$ a.e., for all $\delta > 0$ there exists $\bar\nu(\delta) > 0$ such that $n, m > \bar\nu(\delta)$ implies (recall that h is fixed)

$$\int_{|u_m - u_n| \le h} \Big\{\frac{1}{2} j(x, \nabla u_m) + \frac{1}{2} j(x, \nabla u_n) - j\Big(x, \frac{\nabla u_m + \nabla u_n}{2}\Big)\Big\} < \delta. \tag{20}$$

Let $\lambda > 0$ and $\epsilon > 0$. We want to prove that there exists $\nu(\epsilon, \lambda)$ such that

$$m,\, n > \nu(\epsilon,\lambda) \Rightarrow \text{measure } (\{|\nabla u_n - \nabla u_m| > \lambda\}) < \epsilon. \tag{21}$$

First of all, there exists $k > 0$ such that

$$\text{measure } (\{|\nabla u_n| > k\}) < \epsilon, \text{ measure } (\{|\nabla u_m| > k\}) < \epsilon,$$

uniformly w.r.t. n, m (thanks to the $W_0^{1,p}(\Omega)$ bound on $(u_n)_{n\in I\!N}$).
We define

$$\mathcal{K} = \{(\xi,\eta) \in I\!R^{2N} : |\xi| \le k,\ |\eta| \le k,\ |\xi - \eta| \ge \lambda\},$$

$$\gamma(x) = \min_{(\xi,\eta)\in\mathcal{K}} \Big[\frac{1}{2}\, j(x,\xi) + \frac{1}{2}\, j(x,\eta) - j\Big(x, \frac{\xi+\eta}{2}\Big)\Big],$$

$$A_{n,m} = \{|\nabla u_n| \le\ k\} \cap \{|\nabla u_m| \le\ k\} \cap \{|u_n - u_m| \le\ h\} \cap \{|\nabla u_n - \nabla u_m| \ge \lambda\}.$$

Thanks to (20), one has, for $n, m > \bar\nu(\delta)$,

$$\int_{A_{n,m}} \gamma(x) \le \int_{|u_m - u_n| \le h} \Big\{\frac{1}{2} j(x,\nabla u_m) + \frac{1}{2} j(x,\nabla u_n) - j\Big(x, \frac{\nabla u_m + \nabla u_n}{2}\Big)\Big\} < \delta.$$

Now, thanks to Lemma 5, we choose δ such that

$$\int_A \gamma(x) \le \delta \text{ implies measure } (A) < \epsilon.$$

Thus

$$\text{measure } (A_{n,m}) < \epsilon \text{ if } n, m > \bar\nu = \bar\nu(\delta). \tag{22}$$

Now we note that the convergence in measure of the sequence $\{u_n\}$ implies that there exists $\tilde\nu > 0$ such that, for $n, m > \tilde\nu$,

$$\text{measure } \{|u_n - u_m| > h\} \le \epsilon.$$

On the other hand we have

$$\{|\nabla u_n - \nabla u_m| \ge \lambda\} \subset \{|\nabla u_n| > k\} \cup \{|\nabla u_m| > k\} \cup \{|u_n - u_m| > h\} \cup A_{n,m}.$$

Then for $n, m > \nu = \max(\bar\nu, \tilde\nu)$ one has

$$\text{measure } \{|\nabla u_n - \nabla u_m| \ge \lambda\} \le 4\epsilon.$$

This gives that the sequence $(\nabla u_n)_{n\in I\!\!N}$ converges in measure. Then, there exists a function $\xi(x)$ such that

$$\nabla u_n \text{ converges in measure to } \xi. \tag{23}$$

Since we already know that ∇u_n converges weakly to $\nabla u^\star$ in $L^p(\Omega)$, we then conclude that $\xi(x) = \nabla u^\star(x)$ and ∇u_n converges to $\nabla u^\star$ in $L^q(\Omega)$ for all $1 \le q < p$.

It remains to show that $u^\star = u$. The minimality of u_n gives for all n and all $w \in W_0^{1,p}(\Omega)$

$$\int_\Omega j(x,\nabla u_n) - \int_\Omega f_n u_n \le \int_\Omega j(x,\nabla w) - \int_\Omega f_n w$$

Let $v \in W_0^{1,p}(\Omega)$, Taking $w = u_n - T_i[u_n - v]$ we obtain

$$\int_{|u_n - v|< i} j(x,\nabla u_n) \le \int_{|u_n - v|< i} j(x,\nabla v) + \int_\Omega f_n T_i[u_n - v]. \tag{24}$$

Here we pass to the limit with Fatou Lemma, the weak convergence of f_n to f in $L^1(\Omega)$ and the convergence in measure of u_n to $u^\star$. This gives

$$\int_{|u^* - v|< i} j(x,\nabla u^*) \le \int_{|u^* - v|< i} j(x,\nabla v) + \int_\Omega f T_i[u^* - v].$$

Let $i \to \infty$. Then

$$\int_\Omega j(x,\nabla u^*) \le \int_\Omega j(x,\nabla v) + \int_\Omega f[u^* - v],$$

which implies that $u^* = u$. Finally, Thanks to the uniqueness of the minimizer, all the sequence u_n converges to u.

4. Counter example

We give in this section a counter example to Theorem 3 if the sequence $(f_n)_{n\in I\!\!N}$ does not satisfy Hypothesis (8).

We take $N = 1$, $\Omega =]0,1[$, $p = 3$ and $j(x,\nabla v) = |v'|^3$. The functional to minimize in $W_0^{1,p}(\Omega)$ if defined for $f \in W^{-1,p'}(\Omega)$ by

$$J(v) = \int_\Omega j(x,\nabla v) - < f, v > .$$

For any f in $W^{-1,p'}(\Omega)$, the minimizer of J in $W_0^{1,p}(\Omega)$ is the function u in $W_0^{1,p}(\Omega)$ such that $-(|u'|u')' = f$ in the sense of distribution in $(0,1)$.

For $n \in N^\star$, we define the function u_n as follow. The function u_n is a continuous function satisfying $u_n(0) = 0$ and $u_n(1/2 + x) = u_n(1/2 -$

$x)$ for all $x \in [0,1/2]$. On the interval $[0,1/2]$ it is defined, for $i \in \{0,\dots,n-1\}$, by

$$u_n'(x) = +1 \text{ if } x \in]\frac{i}{2n}, \frac{i}{2n} + \frac{1}{3n}[,$$
$$u_n'(x) = -1 \text{ if } x \in]\frac{i}{2n} + \frac{1}{3n}, \frac{i}{2n} + \frac{1}{3n} + \frac{1}{6n}[.$$

The sequence $(u_n)_{n\in I\!N^\star}$ is bounded in $W_0^{1,\infty}(\Omega)$. It converges weakly in $W_0^{1,p}(\Omega)$ (and even weakly in $W_0^{1,q}(\Omega)$ for all $q < +\infty$) to the function u defined by

$$u(x) = \frac{x}{3} \text{ if } x \in [0, \frac{1}{2}],$$
$$u(x) = -\frac{x-1}{3} \text{ if } x \in [\frac{1}{2}, 1].$$

Since u is continuous and $u'(x) = \frac{1}{3}$ if $x \in (0, \frac{1}{2})$ and $u'(x) = -\frac{1}{3}$ if $x \in (\frac{1}{2}, 1)$, one has

$$-(|u'|u')' = \frac{2}{9}\delta_{\frac{1}{2}}.$$

We now take $f_n = -(|u_n'|u_n')'$. We set $g_n = |u_n'|u_n'$. It is quite easy to see that g_n weakly converges in any $L^q(\Omega)$-space, $q < +\infty$, to the function g defined by

$$g(x) = \frac{1}{3} \text{ if } x \in (0, \frac{1}{2}),$$
$$g(x) = -\frac{1}{3} \text{ if } x \in (\frac{1}{2}, 1).$$

This gives that $f_n \to f$ weakly in $W^{-1,p'}(\Omega)$ (and even weakly in $W^{-1,q}(\Omega)$ for any $q < +\infty$) with

$$f = \frac{2}{3}\delta_{\frac{1}{2}}.$$

Since $-(|u'|u')' \neq f$, this concludes our counter example.

Appendix A

Lemma 5. *Let (X,μ) be a measurable space, μ a positive measure with $\mu(X) < +\infty$,*

$$\gamma : X \longrightarrow [0, +\infty]$$

a measurable function such that $\mu(\{x \in X : \gamma(x) = 0\}) = 0$. Then, for every $\epsilon > 0$ there exists $\delta > 0$ such that the statement

$$A \text{ measurable subset of } X, \quad \int_A \gamma(x)\, d\mu < \delta$$

implies $\mu(A) < \epsilon$.

PROOF. For every $B > 0$, we have

$$B\,\mu(\{A \cap \{B < \gamma(x)\}\}) = \int_{A\cap\{B<\gamma(x)\}} B\,d\mu \leq \int_A \gamma(x)$$

Then

$$\mu\big(\{A \cap \{B < \gamma(x)\}\}\big) \leq \frac{1}{B} \int_A \gamma(x). \tag{25}$$

On the other hand, since the sequence $\{x \in X : \gamma(x) \leq \frac{1}{m}\}$ is decreasing, we have

$$\mu\Big(\{\gamma(x) \leq \frac{1}{m}\}\Big) \to \mu\big(\{\gamma(x) = 0\}\big). \tag{26}$$

Fix $\epsilon > 0$. Thus, there exists m_ϵ such that

$$\mu\Big(\Big\{\gamma(x) \leq \frac{1}{m_\epsilon}\Big\}\Big) \leq \frac{\epsilon}{2}.$$

Then, with $B = \frac{1}{m_\epsilon}$

$$\begin{aligned}\mu(A) &\leq \mu\Big(A \cap \Big\{\gamma(x) > B\Big\}\Big) + \mu\Big(\Big\{\gamma(x) \leq \frac{1}{m_\epsilon}\Big\}\Big)\\ &\leq \frac{1}{B} \int_A \gamma(x) + \frac{\epsilon}{2}.\end{aligned}$$

We choose $\delta = B\frac{\epsilon}{2}$ to conclude. □

References

[1] L. Boccardo: T–minima: An approach to minimization problems in L^1. Contributi dedicati alla memoria di Ennio De Giorgi. Ricerche di Matematica, 49 (2000), 135–154.

[2] L. Boccardo: Marcinkiewicz estimates for solutions of some elliptic problems with nonregular data; Ann. Mat. Pura Appl. 188 (2009), 591–601.

[3] L. Boccardo and G. Croce: Elliptic partial differential equations. Existence and regularity of distributional solutions; De Gruyter Studies in Mathematics 55, De Gruyter, Berlin, 2014.

[4] L. Boccardo and T. Gallouët: Nonlinear elliptic equations with right hand side measures; Comm. Partial Differential Equations 17 (1992), 641–655.

[5] H. Brezis: Convergence in $\mathcal{D}'$ and in L^1 under strict convexity; Boundary value problems for partial differential equations and applications, RMA Res. Notes Appl. Math., 29, Masson, Paris, 1993.

[6] E. De Giorgi: Semicontinuity theorems in the calculus of variations. With notes by U. Mosco, G. Troianiello and G. Vergara and a preface by Carlo Sbordone. Dual English-Italian text. Quaderni dell' Accademia Pontaniana [Notebooks of the Accademia Pontaniana], 56. Accademia Pontaniana, Naples, 2008.

[7] D.R. Moreira, E.V. Teixeira: On the behavior of weak convergence under nonlinearities and applications; Proc. Amer. Math. Soc. 133 (2005), 1647–1656.

[8] A. Visintin: Strong convergence results related to strict convexity; Comm. Partial Differential Equations 9 (1984), 439–466.

L. B., Dipartimento di Matematica, Sapienza Università di Roma, Piazza A. Moro 2, Roma, Italia
E-mail address: boccardo@mat.uniroma1.it

T. G., Institut de Mathematiques de Marseille
39 rue Joliot Curie, 13453 Marseille cedex 13, France
E-mail address: thierry.gallouet@univ-amu.fr