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STRONGLY NONLINEAR ELLIPTIC EQUATIONS HAVING NATURAL GROWTH TERMS AND L^1 DATA

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1. INTRODUCTION

IN THIS paper we prove the existence of solutions of nonlinear elliptic equations of the type

$$u \in W_0^{1,p}(\Omega): A(u) + g(x, u, Du) = f \in L^1(\Omega),$$

where A is a Leray–Lions operator and g is a nonlinear lower order term having "natural growth" (of order p) with respect to $|Du|$. With respect to $|u|$, we do not assume any growth restriction, but we assume the "sign-condition" $g(x, s, \xi)s \geq 0$.

It will turn out that for a solution u, $g(x, u, Du) \in L^1(\Omega)$, but, for a general $v \in W^{1,p}(\Omega)$, $g(x, v, Dv)$ can be very singular. If $f \in W^{-1,p'}(\Omega)$ the reader is referred to [1, 3, 5] for existence results and references. If $f \in L^1(\Omega)$ existence results have been proved in [6, 9] (if g does not depend on Du) and in [7] (if g has growth strictly less than p with respect to $|Du|$) when A is linear. The case where A is nonlinear and g does not depend on Du is studied in [2].

The model examples of our equation are

$$-\text{div}(|Du|^{p-2}Du) + \gamma u|u|^r|Du|^p = f, \qquad \gamma > 0$$

$$-\text{div}(a(x)Du) + \gamma u|Du|^2 = f, \qquad \gamma > 0, \qquad p = 2.$$

We shall prove the existence of a solution in $W_0^{1,p}(\Omega)$, but it should be emphasized that for $\gamma = 0$ the existence of u in such a space cannot be expected, if $p \leq N$. In [2] the existence of a solution has been proved in $W_0^{1,q}(\Omega)$ $\forall\, q < ((p - 1)N)/(N - 1)$; (see also [11]).

2. THE MAIN RESULT

Let Ω be a bounded open set of $\mathbb{R}^N$. Let $1 < p < \infty$ be fixed and A be a nonlinear operator from $W_0^{1,p}(\Omega)$ into its dual $W^{-1,p'}(\Omega)$, $1/p + 1/p' = 1$, defined by

$$A(v) = -\text{div}(a(x, v, Dv)),$$

where $a(x, s, \xi): \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function such that

$$\left.\begin{aligned} &\text{there exist } \beta > 0,\ k \in L^{p'}(\Omega),\ \alpha > 0 \text{ such that} \\ &|a(x, s, \xi)| \leq \beta(|s|^{p-1} + |\xi|^{p-1} + k(x)); \\ &[a(x, s, \xi) - a(x, s, \eta)][\xi - \eta] > 0; \qquad \forall\, \xi \neq \eta \\ &a(x, s, \xi)\xi \geq \alpha|\xi|^p. \end{aligned}\right\} \tag{1}$$

Let $g(x, s, \xi): \Omega \times \mathbb{R} \to \mathbb{R}^N \to \mathbb{R}$ be a Carathéodory function such that

$$\left.\begin{array}{l} g(x, s, \xi)s \geq 0 \\ \text{there exist } \sigma > 0, \gamma > 0 \text{ such that } |g(x, s, \xi)| \geq \gamma|\xi|^p; \ |s| \geq \sigma, \\ |g(x, s, \xi)| \leq b(|s|)(|\xi|^p + c(x)), \end{array}\right\} \tag{2}$$

where b is a continuous and increasing real function, $c(x) \in L^1(\Omega)$, $c(x) \geq 0$, and

$$f \in L^1(\Omega). \tag{3}$$

We consider the nonlinear elliptic problem with Dirichlet boundary conditions

$$\left.\begin{array}{ll} A(u) + g(x, u, Du) = f & \text{in } \mathcal{D}'(\Omega) \\ u \in W_0^{1,p}(\Omega), & g(x, u, Du) \in L^1(\Omega). \end{array}\right\} \tag{4}$$

Our objective is to prove the following theorem.

THEOREM 1. Under the assumptions (1)–(3) there exists a solution of (4).

Proof. If f lies in $L^{p'}(\Omega)$, (4) is known to have a weak solution (see [1]). We take a sequence $f_\varepsilon (f_\varepsilon \in L^{p'}(\Omega), \forall \varepsilon > 0)$ which converges to f in $L^1(\Omega)$ with $\|f_\varepsilon\|_{L^1} \leq \|f\|_{L^1}$. Define u_ε to be a solution of the equation

$$\left.\begin{array}{ll} A(u_\varepsilon) + g(x, u_\varepsilon, Du_\varepsilon) = f_\varepsilon & \text{in } \mathcal{D}'(\Omega) \\ u_\varepsilon \in W_0^{1,p}(\Omega), & g(x, u_\varepsilon, Du_\varepsilon) \in L^1(\Omega). \end{array}\right\} \tag{5}$$

Multiplying (5) by $T_k(u_\varepsilon)$ and using (1), (2), we get

$$\alpha \int_\Omega |DT_k(u_\varepsilon)|^p \leq k\|f_\varepsilon\|_{L^1}, \tag{6}$$

where $T_k(v)$, $k \in \mathbb{R}^+$, is the usual truncation in $W_0^{1,p}(\Omega)$. Now we shall prove that

$$\int_{|u_\varepsilon| > t} |g(x, u_\varepsilon, Du_\varepsilon)| \leq \int_{|u_\varepsilon| > t} |f_\varepsilon|, \qquad \text{for any } t \in \mathbb{R}^+. \tag{7}$$

We follow a technique of [8]. Let $\psi_i(s)$ be a sequence of real smooth increasing functions with $\psi_i' \in L^\infty(\mathbb{R})$ and $\psi_i(0) = 0$. The choice of $\psi_i(u_\varepsilon)$ as test function in (5) yields

$$\int_\Omega g(x, u_\varepsilon, Du_\varepsilon)\psi_i(u_\varepsilon) \leq \int_\Omega f_\varepsilon \psi_i(u_\varepsilon). \tag{8}$$

If $\psi_i(s)$ converges to the function $\psi(s)$ defined by

$$\psi(s) = \begin{cases} 1 & \text{if } s \geq t \\ 0 & \text{if } -t < s < t \\ -1 & \text{if } s \leq -t \end{cases}$$

we obtain the estimate (7) which implies

$$\int_{|u_\varepsilon|>t} |Du_\varepsilon|^p \le \frac{1}{\gamma}\int_{|u_\varepsilon|>t} |f_\varepsilon|, \qquad \text{for } t \ge \sigma. \tag{9}$$

Hence from (6) and (9) we get

$$\begin{aligned}\int_\Omega |Du_\varepsilon|^p &= \int_{|u_\varepsilon|\le\sigma} |Du_\varepsilon|^p + \int_{|u_\varepsilon|>\sigma} |Du_\varepsilon|^p \\ &\le \frac{\sigma}{\alpha}\|f_\varepsilon\|_{L^1} + \frac{1}{\gamma}\int_{|u_\varepsilon|>\sigma} |f_\varepsilon| \\ &\le \left(\frac{\sigma}{\alpha} + \frac{1}{\gamma}\right)\|f\|_{L^1}.\end{aligned}$$

Thus we can extract a subsequence, still denoted by u_ε, with

$$u_\varepsilon \rightharpoonup u \qquad \text{in } W_0^{1,p}(\Omega)\text{-weakly, } L^p(\Omega)\text{-strongly and a.e.} \tag{10}$$

Our first objective is to prove that

$$u_\varepsilon^+ \to u^+ \qquad \text{in } W_0^{1,p}(\Omega)\text{-strongly.} \tag{11}$$

Let k be a positive constant greater than σ. We use in (5) $T_k(u_\varepsilon^+ - u^+)^+$ as a test function (where T_k is the truncation at $\pm k$) and we have

$$\langle A(u_\varepsilon), T_k(u_\varepsilon^+ - u^+)^+\rangle + \int_\Omega g(x, u_\varepsilon, Du_\varepsilon)T_k(u_\varepsilon^+ - u^+)^+ = \int_\Omega f_\varepsilon T_k(u_\varepsilon^+ - u^+)^+. \tag{12}$$

Note that where $T_k(u_\varepsilon^+(x) - u^+(x))^+ > 0$, one has $u_\varepsilon^+(x) > 0$, hence $u_\varepsilon(x) > 0$ and from (2) $g(x, u_\varepsilon(x), Du_\varepsilon(x)) \ge 0$. Therefore from (12) we deduce

$$\langle A(u_\varepsilon), T_k(u_\varepsilon^+ - u^+)^+\rangle \le \int_\Omega f_\varepsilon T_k(u_\varepsilon^+ - u^+)^+.$$

Since $u_\varepsilon(x) = u_\varepsilon^+(x)$ on the set $\{x \in \Omega: u_\varepsilon^+(x) > u^+(x)\}$, we can also write

$$\int_\Omega a(x, u_\varepsilon, Du_\varepsilon^+)DT_k(u_\varepsilon^+ - u^+)^+ \le \int_\Omega f_\varepsilon T_k(u_\varepsilon^+ - u^+)^+$$

which implies

$$\lim_{\varepsilon\to 0}\int_\Omega [a(x, u_\varepsilon, Du_\varepsilon^+) - a(x, u_\varepsilon, Du^+)]DT_k(u_\varepsilon^+ - u^+)^+ = 0. \tag{13}$$

We recall again that, where $(u_\varepsilon^+(x) - u^+(x))^+ > 0$, we have $u_\varepsilon^+(x) = u_\varepsilon(x)$. Therefore

$$\begin{aligned}&\int_{u_\varepsilon^+ - u^+ > k} [a(x, u_\varepsilon, Du_\varepsilon^+) - a(x, u_\varepsilon, Du^+)]D(u_\varepsilon^+ - u^+)^+ \\ &\quad\le \int_{u_\varepsilon > k} [a(x, u_\varepsilon, Du_\varepsilon) - a(x, u_\varepsilon, Du^+)]D(u_\varepsilon - u^+) \\ &\quad\le c_1\left\{\int_{u_\varepsilon > k} |Du_\varepsilon|^p + \int_{u_\varepsilon > k} |u_\varepsilon|^p + \int_{u_\varepsilon > k} k(x)^{p'} + \int_{u_\varepsilon > k} |Du^+|^p\right\}\end{aligned} \tag{14}$$

(using (9))

$$\leq c_2\left\{\int_{u_\varepsilon > k} |f_\varepsilon| + \int_{u_\varepsilon > k} |u_\varepsilon|^p + \int_{u_\varepsilon > k} k(x)^{p'} + \int_{u_\varepsilon > k} |Du^+|^p\right\} := R_\varepsilon(k).$$

If k tends to $+\infty$ the right-hand side of (14) tends to zero (uniformly with respect to ε). From this observation and (13) we deduce that

$$\lim_{\varepsilon\to 0}\int_\Omega [a(x, u_\varepsilon, Du_\varepsilon^+) - a(x, u_\varepsilon, Du^+)]D(u_\varepsilon^+ - u^+)^+ = 0. \tag{15}$$

In the next step we study the behaviour of $z_\varepsilon^- := (u_\varepsilon^+ - T_k(u^+))^-$, and we follow the lines of [1].

We use as a test function in (5)

$$v_\varepsilon = \phi_\lambda((u_\varepsilon^+ - T_k(u^+))^-) \tag{16}$$

where

$$\phi_\lambda(s) = s\,e^{\lambda s^2}, \qquad \lambda = \frac{b(k)^2}{4\alpha^2} \tag{17}$$

(see [4]).

Note that if $v_\varepsilon(x) \neq 0$ then $0 \leq u_\varepsilon^+(x) \leq k$. Hence $v_\varepsilon \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ and v_ε is an admissible test function in (2.5). We deduce

$$\int_\Omega a(x, u_\varepsilon, Du_\varepsilon)Dz_\varepsilon^-\phi_\lambda'(z_\varepsilon^-) + \int_\Omega g(x, u_\varepsilon, Du_\varepsilon)\phi_\lambda(z_\varepsilon^-) = \int_\Omega f_\varepsilon\phi_\lambda(z_\varepsilon^-). \tag{18}$$

Now we can follow the proof of [1] because the left-hand side of (18) is exactly the left-hand side of (12) of [1]. On the other hand since $\phi_\lambda(z_\varepsilon^-) \neq 0$, where $0 \leq u_\varepsilon^+(x) \leq k$, we have $\phi_\lambda(z_\varepsilon^-)$ bounded in $L^\infty(\Omega)$, then

$$\int_\Omega f_\varepsilon\phi_\lambda(z_\varepsilon^-) \to \int_\Omega f\phi_\lambda((u^+ - T_k(u^+))^-) \equiv 0.$$

Thus passing to the limit in ε, for k fixed, in (18) we have (as in [1], (17))

$$\overline{\lim_{\varepsilon\to 0}}\int_\Omega - [a(x, u_\varepsilon, Du_\varepsilon^+) - a(x, u_\varepsilon, DT_k(u^+))]D(u_\varepsilon^+ - T_k(u^+))^- \leq 0. \tag{19}$$

We can write the following equalities

$$\int_\Omega - [a(x, u_\varepsilon, Du_\varepsilon^+) - a(x, u_\varepsilon, Du^+)]D(u_\varepsilon^+ - u^+)^-$$

$$= \int_{T_k(u^+) < u_\varepsilon^+ \leq u^+} [a(x, u_\varepsilon, Du_\varepsilon^+) - a(x, u_\varepsilon, Du^+)]D(u_\varepsilon^+ - u^+)$$

$$+ \int_{u_\varepsilon^+ \leq T_k(u^+)} [a(x, u_\varepsilon, Du_\varepsilon^+) - a(x, u_\varepsilon, Du^+)]D(u_\varepsilon^+ - u^+)$$

$$= \int_{k < u_\varepsilon^+ = u_\varepsilon \le u^+} [a(x, u_\varepsilon, Du_\varepsilon^+) - a(x, u_\varepsilon, Du^+)] D(u_\varepsilon^+ - u^+)$$

$$+ \int_\Omega - [a(x, u_\varepsilon, Du_\varepsilon^+) - a(x, u_\varepsilon, DT_k(u^+))] D(u_\varepsilon^+ - T_k(u^+))$$

$$+ \int_\Omega - [a(x, u_\varepsilon, DT_k(u^+)) - a(x, u_\varepsilon, Du^+)] D(u_\varepsilon^+ - T_k(u^+))^-$$

$$+ \int_{u_\varepsilon^+ \le T_k(u^+)} [a(x, u_\varepsilon, Du_\varepsilon^+) - a(x, u_\varepsilon, Du^+)] D(T_k(u^+) - u^+), \tag{20}$$

because

$$\{x \in \Omega: T_k(u^+) < u_\varepsilon^+ \le u^+\} = \{x \in \Omega: k < u_\varepsilon^+ \le u^+\} U \{x \in \Omega: T_k(u^+) < u_\varepsilon^+ \le u^+; u_\varepsilon^+ \le k\}$$

and the last set is empty. Now we study the last four integrals.

The first can be estimated as in (14). It goes to zero as $k \to \infty$, uniformly with respect to ε.

For the second we have the limit (19). For fixed k, the third integral converges to zero (if $\varepsilon \to 0$) and

$$\left| \int_{u_\varepsilon^+ \le T_k(u^+)} [a(x, u_\varepsilon, Du_\varepsilon^+) - a(x, u_\varepsilon, Du^+)] D(T_k(u^+) - u^+) \right| \le c_3 \left(\int_\Omega |D(T_k(u^+) - u^+)|^p \right)^{1/p}$$

which converges to zero, for $k \to +\infty$. Therefore (20) yields

$$\lim_{\varepsilon \to 0} \int_\Omega [a(x, u_\varepsilon, Du_\varepsilon^+) - a(x, u_\varepsilon, Du^+)] D(u_\varepsilon^+ - u^+)^- = 0. \tag{21}$$

From (15) and (21) we deduce that

$$\lim_{\varepsilon \to 0} \int_\Omega [a(x, u_\varepsilon, Du_\varepsilon^+) - a(x, u_\varepsilon, Du^+)] D(u_\varepsilon^+ - u^+) = 0. \tag{22}$$

By a variation of a result of Leray–Lions [10] (for the proof see e.g. [4]), (22) implies

$$u_\varepsilon^+ \to u^+ \qquad \text{in } W_0^{1,p}(\Omega)\text{-strongly}. \tag{23}$$

Now we want to prove that

$$u_\varepsilon^- \to u^- \qquad \text{in } W_0^{1,p}(\Omega)\text{-strongly}. \tag{24}$$

The proof of the convergence (24) is achieved using as test functions $T_k(u_\varepsilon^- - u^-)^+$ and $\phi_\lambda((u_\varepsilon^- - T_k(u^-))^-)$ and working as in the previous steps.

From (23) and (24) we deduce that for some subsequence

$$Du_\varepsilon \to Du \qquad \text{in } L^p(\Omega)\text{-strongly} \tag{25}$$

and

$$Du_\varepsilon \to Du \qquad \text{a.e. in } \Omega. \tag{26}$$

Since $g(x, s, \xi)$ is continuous in (s, ξ) we have

$$g(x, u_\varepsilon(x), Du_\varepsilon(x)) \to g(x, u(x), Du(x)) \qquad \text{a.e.} \tag{27}$$

Thus in order to prove that

$$g(x, u_\varepsilon, Du_\varepsilon) \to g(x, u, Du) \qquad \text{in } L^1(\Omega) \tag{28}$$

it is sufficient to prove that, for any measurable subset E of Ω, we have

$$\lim_{|E| \to 0} \int_E |g(x, u_\varepsilon, Du_\varepsilon)| = 0, \qquad \text{uniformly in } \varepsilon. \tag{29}$$

We can write

$$\int_E |g(x, u_\varepsilon, Du_\varepsilon)| = \int_{E \cap X_m^\varepsilon} |g(x, u_\varepsilon, Du_\varepsilon)| + \int_{E \cap X_m^\varepsilon} |g(x, u_\varepsilon, Du_\varepsilon)|,$$

where

$$X_m^\varepsilon = \{x \in \Omega : |u_\varepsilon(x)| \leq m\}$$

$$Y_m^\varepsilon = \{x \in \Omega : |u_\varepsilon(x)| > m\}.$$

So, using (7), we get

$$\int_E |g(x, u_\varepsilon, Du_\varepsilon)| \leq b(m) \int_E (|Du_\varepsilon|^p + c(x)) + \int_{|u_\varepsilon| > m} |f_\varepsilon|.$$

Now (10), (25) and Vitali's theorem yield (29) (and (28)).

Using (10), (25) and (28) it is easy to pass to the limit in

$$\langle A(u_\varepsilon), v\rangle + \int_\Omega g(x, u_\varepsilon, Du_\varepsilon)v = \int_\Omega fv$$

to obtain

$$\langle A(u), v\rangle + \int_\Omega g(x, u, Du)v = \int_\Omega fv, \tag{30}$$

for any $v \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$.

3. REMARKS

Remark 1. If $f \geq 0$ the solution obtained in the previous section is positive (use $v = -T_k(u^-)$, $k > 0$, in (30)).

Remark 2. If $f \geq 0$, we can use the Monotone Converge theorem (for $k \to +\infty$) in

$$\langle A(u), T_k(u)\rangle + \int_\Omega g(x, u, Du)T_k(u) = \int_\Omega fT_k(u)$$

to obtain

$$\langle A(u), u\rangle + \int_\Omega g(x, u, Du)u = \int_\Omega fu,$$

possibly with $\int_\Omega g(x, u, Du)u = \int_\Omega fu = +\infty$.

Remark 3. Consider $B = \{x \in \mathbb{R}^2 : |x| < 1\}$. If the real number γ belongs to $[\frac{1}{4}, \frac{1}{3})$, then the positive function $u(x) = (-\log|x|)^\gamma$, belongs to $H_0^1(B)$ and $-\Delta u \in L^1(B)$, $-\Delta u \geq 0$, $u|Du|^2 \in L^1(B)$, but $|u|^2|Du|^2 \notin L^1(B)$.

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