appear in Ap-

ationary semi-
l., 18:861–872,

ns. Springer,

ns in semicon-

devices. Boole

ermanium and

s with electro-

L BOCCARDO, T GALLOUËT AND F MURAT

A unified presentation of two existence results for problems with natural growth

1. Introduction.

In this paper we prove the existence of a solution for the following problem

$$\begin{cases} -\operatorname{div} a(x,u,Du) + g(x,u,Du) = f & \text{in } \Omega, \\ u \in W_0^{1,p}(\Omega), \quad g(x,u,Du) \in L^1(\Omega), \end{cases}$$

where $-\operatorname{div} a(x,u,Du)$ is a Leray-Lions operator from $W_0^{1,p}(\Omega)$ into $W^{-1,p'}(\Omega)$ and $g(x,u,Du)$ is a nonlinearity with natural growth ($|g(x,s,\xi)| \leq b(|s|)(c(x)+|\xi|^p)$) which satisfies the sign condition $g(x,s,\xi)s \geq 0$. The right hand side f is assumed to belong either to $W^{-1,p'}(\Omega)$ or to $L^1(\Omega)$; in the latest case we also assume that $|g(x,s,\xi)| \geq \gamma|\xi|^p$ for $|s|$ sufficiently large. This result unifies both the statements and the proofs of results previously obtained in [BMP1], [BBM], [G], [D] and [BG2]. We also prove that there exists a nonnegative solution of the above problem when f and $g(x,s,\xi)$ are nonnegative.

2. Setting of the problem and main result.

Let Ω be a bounded domain in $\mathbb{R}^N$ and p, p' be real numbers such that

$$1 < p,\ p' < +\infty,\ 1/p + 1/p' = 1.$$

Let A be a nonlinear operator from $W_0^{1,p}(\Omega)$ into its dual defined by

$$A(v) = -\operatorname{div}(a(x,v,Dv))$$

where $a(x,s,\xi) : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is a Caratheodory function satisfying the following conditions for almost every $x \in \Omega$ and for all $s \in \mathbb{R}, \xi, \xi^* \in \mathbb{R}^N$

$$|a(x,s,\xi)| \leq \beta[k(x) + |s|^{p-1} + |\xi|^{p-1}] \tag{2.1}$$

$$[a(x,s,\xi) - a(x,s,\xi^*)][\xi - \xi^*] > 0 \quad \text{if } \xi \neq \xi^* \tag{2.2}$$

$$a(x,s,\xi)\xi \geq \alpha|\xi|^p \tag{2.3}$$

where α and β are strictly positive constants and $k(x)$ is a given nonnegative function in $L^{p'}(\Omega)$. Under these hypotheses, A is a bounded, continuous, coercive, pseudomonotone operator of Leray-Lions type from $W_0^{1,p}(\Omega)$ into its dual.

Furthermore let $g(x,s,\xi) : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ be a Caratheodory function such that for almost every $x \in \Omega$ and for all $s \in \mathbb{R}, \xi \in \mathbb{R}^N$

$$g(x,s,\xi)s \geq 0 \tag{2.4}$$

$$|g(x,s,\xi)| \leq b(|s|)(c(x) + |\xi|^p) \tag{2.5}$$

where $b : \mathbb{R} \to \mathbb{R}$ is a continuous and nondecreasing function and $c(x)$ is a given nonnegative function in $L^1(\Omega)$.

Finally we assume one of the following two assumptions:
either

$$f \in W^{-1,p'}(\Omega) \tag{2.6}$$

or

$$\begin{cases} f \in L^1(\Omega) \\ \text{and there exists } \sigma > 0 \quad \text{and } \gamma > 0 \\ \text{such that } |g(x,s,\xi)| \geq \gamma|\xi|^p \text{ when } |s| \geq \sigma. \end{cases} \tag{2.7}$$

We consider the following nonlinear elliptic problem with Dirichlet boundary condition

$$\begin{cases} A(u) + g(x,u,Du) = f \quad \text{in } \mathcal{D}'(\Omega) \\ u \in W_0^{1,p}(\Omega), \quad g(x,u,Du) \in L^1(\Omega) \end{cases} \tag{2.8}$$

We shall prove the following existence theorem:

Theorem 1. *Under the assumptions* (2.1)-(2.5) *and either* (2.6) *or* (2.7), *there exists at least one solution of* (2.8).

The above Theorem unifies in the same statement as well as by the same proof the two results of [BBM], [BMP1] and of [BG2], which are respectively concerned with right hand sides in $W^{-1,p'}(\Omega)$ (hypothesis (2.6)) and in $L^1(\Omega)$ (hypothesis (2.7)).

Note that the solution of (2.8) belongs to $W_0^{1,p}(\Omega)$ even in the case where $f \in L^1(\Omega)$. This seems to be strange since for f in $L^1(\Omega)$ the solution u of

$$A(u) = f \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega$$

is known to belong only to $W_0^{1,q}(\Omega)$ for all $q < N(p-1)/(N-1)$ (cf. [BG1]), but this is due to the second part of hypothesis (2.7).

An example where hypotheses (2.4), (2.5) and where either (2.6) or (2.7) are satisfied is the case where

$$g(x, s, \xi) = d(s)|\xi|^2$$

with $d : \mathbb{R} \to \mathbb{R}, d(s)s \geq 0$ and (if (2.7) is required to hold) $|d(s)| \geq \sigma$ when $|s| \geq \gamma$.

Under assumption (2.6) it is also true that $ug(x, u, Du)$ belongs to $L^1(\Omega)$, which in contrast is in general false (cf. Remark 3 of [BG2]) if we assume hypothesis (2.7).

The proof of Theorem 1 is given in Section 3. It consists in the following steps. We first define approximate equations. We then prove an a priori estimate in $W_0^{1,p}(\Omega)$ for the solutions u_ε of these approximate equations. Finally using a proof somewhat similar to the proof of [BM] and [LM] we prove (this is the main step) that the truncations $T_k(u_\varepsilon)$ are relatively compact in the strong topology of $W_0^{1,p}(\Omega)$, a result which allows us to pass to the limit and to obtain the existence result.

In Section 4 we consider the case where $f \geq 0$ and remark that in this case there exists a nonnegative solution u whenever $g(x, s, \xi) \geq 0$.

3. Proof of Theorem 1.

3.1. Approximation.

In order to prove Theorem 1, we consider the sequence of approximate equations

$$\begin{cases} A(u_\varepsilon) + g_\varepsilon(x, u_\varepsilon, Du_\varepsilon) = f_\varepsilon \quad \text{in } \mathcal{D}'(\Omega) \\ u_\varepsilon \in W_0^{1,p}(\Omega) \end{cases} \tag{3.1}$$

where

$$g_\varepsilon(x, s, \xi) = \frac{g(x, s, \xi)}{1 + \varepsilon |g(x, s, \xi)|} \tag{3.2}$$

and where f_ε is a sequence of smooth functions which converges strongly to f in $W^{-1,p'}(\Omega)$ (if we assume (2.6)) or in $L^1(\Omega)$ (if we assume (2.7)). Note that

$$g_\varepsilon(x, s, \xi)s \geq 0, \quad |g_\varepsilon(x, s, \xi)| \leq |g(x, s, \xi)| \quad \text{and} \quad |g_\varepsilon(x, s, \xi)| \leq 1/\varepsilon.$$

Since g_ε is bounded for any fixed $\varepsilon > 0$, there exists at least one solution u_ε of (3.1) (cf. [LL], [L]), and u_ε belongs to $L^\infty(\Omega)$ (cf. [B]).

3.2. A priori estimates.

If we assume (2.6), the use in (3.1) of the test function u_ε yields (see [BBM] if necessary)

$$\|u_\varepsilon\|_{W_0^{1,p}(\Omega)} \leq C_1 \tag{3.3}$$

$$\int_\Omega u_\varepsilon g_\varepsilon(x, u_\varepsilon, Du_\varepsilon) \leq C_2. \tag{3.4}$$

If we assume (2.7), the use in (3.1) of the test function $T_k(u_\varepsilon)$ (where $T_k(v)$, $k \in \mathbb{R}^+$, is the usual truncation in $W_0^{1,p}(\Omega)$) yields for any $k > 0$ (see [BG2] if necessary)

$$\int_\Omega |DT_k(u_\varepsilon)|^p \leq C_3 k \tag{3.5}$$

$$k \int_{|u_\varepsilon| > k} |g_\varepsilon(x, u_\varepsilon, Du_\varepsilon)| \leq \int_\Omega |f^\varepsilon| \, |T_k(u^\varepsilon)| \leq C_4 k \tag{3.6}$$

which combined with (3.5) and the second part of hypothesis (2.7) yields (3.3) again.

Therefore there exist $u \in W_0^{1,p}(\Omega)$ and a subsequence (still denoted by ε) such that

$$u_\varepsilon \rightharpoonup u \quad \text{weakly in } W_0^{1,p}(\Omega) \tag{3.7}$$

$$u_\varepsilon \rightarrow u \quad \text{a.e.} \tag{3.8}$$

3.3. Strong convergence of $T_k(u_\varepsilon)$.

We already know that for any fixed $k \in \mathbb{R}^+$

$$T_k(u_\varepsilon) \rightharpoonup T_k(u) \quad \text{weakly in } W_0^{1,p}(\Omega). \tag{3.9}$$

We shall prove that this convergence is actually strong. This is the most original part of the present paper.

We shall use in (3.1) the test function

$$v_\varepsilon = \varphi(z_\varepsilon)$$

where

$$\begin{cases} z_\varepsilon = T_k(u_\varepsilon) - T_k(u) \\ \varphi(s) = s\, e^{\lambda s^2}. \end{cases} \tag{3.10}$$

(The use of the test function $\varphi(u^\varepsilon)$ is one of the main tools in the existence proof of [BMP2].) It is easy to see that when $\lambda \geq (b(k)/2\alpha)^2$ the following inequality

$$\varphi'(s) - \frac{b(k)}{\alpha}|\varphi(s)| \geq \frac{1}{2} \tag{3.11}$$

holds for all $s \in \mathbb{R}$.

Since v_ε converges to zero weakly in $W_0^{1,p}(\Omega)$ and weakly $\star$ in $L^\infty(\Omega)$ we have

$$< f, v_\varepsilon > \rightarrow 0, \tag{3.12}$$

if we assume (2.6), and

$$\int_\Omega f_\varepsilon v_\varepsilon \rightarrow 0 \tag{3.13}$$

if we assume (2.7). Thus in both cases we get

$$< A(u_\varepsilon), v_\varepsilon > + \int_\Omega g_\varepsilon(x, u_\varepsilon, Du_\varepsilon) v_\varepsilon \to 0. \tag{3.14}$$

From now on we denote by $\omega_1(\varepsilon), \omega_2(\varepsilon), \ldots$ various sequences of real numbers which converge to zero when ε tends to zero. Since $g_\varepsilon(x, u_\varepsilon, Du_\varepsilon)v_\varepsilon \geq 0$ on the subset $\{x \in \Omega : |u_\varepsilon(x)| \geq k\}$ we deduce from (3.14) that

$$< A(u_\varepsilon), v_\varepsilon > + \int_{\{|u_\varepsilon| \leq k\}} g_\varepsilon(x, u_\varepsilon, Du_\varepsilon) v_\varepsilon \leq \omega_1(\varepsilon). \tag{3.15}$$

We now study the terms in the left hand side of (3.15). We have

$$\left\{\begin{array}{l} < A(u_\varepsilon), v_\varepsilon > \\ = \displaystyle\int_\Omega a(x, u_\varepsilon, Du_\varepsilon) D(T_k(u_\varepsilon) - T_k(u)) \varphi'(z_\varepsilon) \\ = \displaystyle\int_\Omega a(x, T_k(u_\varepsilon), DT_k(u_\varepsilon)) D(T_k(u_\varepsilon) - T_k(u)) \varphi'(z_\varepsilon) \\ \quad + \displaystyle\int_{\{u_\varepsilon > k\}} a(x, k, 0) DT_k(u) \varphi'(k - T_k(u)) \\ \quad + \displaystyle\int_{\{u_\varepsilon < -k\}} a(x, -k, 0) DT_k(u) \varphi'(-k - T_k(u)) \\ \quad - \displaystyle\int_{\{|u_\varepsilon| > k\}} a(x, u_\varepsilon, Du_\varepsilon) DT_k(u) \varphi'(z_\varepsilon) \\ = \displaystyle\int_\Omega [a(x, T_k(u_\varepsilon), DT_k(u_\varepsilon)) - a(x, T_k(u_\varepsilon), DT_k(u))] D(T_k(u_\varepsilon) - T_k(u)) \varphi'(z^\varepsilon) \\ \quad + \displaystyle\int_\Omega a(x, T_k(u_\varepsilon), DT_k(u)) D(T_k(u_\varepsilon) - T_k(u)) \varphi'(z^\varepsilon) + \omega_2(\varepsilon) \\ = \displaystyle\int_\Omega [a(x, T_k(u_\varepsilon), DT_k(u_\varepsilon)) - a(x, T_k(u_\varepsilon), DT_k(u))] D(T_k(u_\varepsilon) - T_k(u)) \varphi'(z^\varepsilon) \\ \quad + \omega_3(\varepsilon). \end{array}\right. \tag{3.16}$$

On the other hand

$$
\left\{
\begin{aligned}
&\Big|\int_{\{|u_\varepsilon|\le k\}} g_\varepsilon(x,u_\varepsilon,Du_\varepsilon)v_\varepsilon\Big| \\
&\le \int_{\{|u_\varepsilon|\le k\}} b(k)(c(x)+|Du_\varepsilon|^p)\,|v_\varepsilon| \\
&= \omega_4(\varepsilon) + b(k)\int_\Omega |DT_k(u_\varepsilon)|^p\,|v_\varepsilon| \\
&\le \omega_4(\varepsilon) + \frac{b(k)}{\alpha}\int_\Omega a(x,T_k(u_\varepsilon),DT_k(u_\varepsilon))DT_k(u_\varepsilon)\,|v_\varepsilon| \\
&= \frac{b(k)}{\alpha}\int_\Omega [a(x,T_k(u_\varepsilon),DT_k(u_\varepsilon)) - a(x,T_k(u_\varepsilon),DT_k(u))] \\
&\qquad\qquad D(T_k(u_\varepsilon) - DT_k(u))|v_\varepsilon| + \omega_5(\varepsilon).
\end{aligned}
\right. \tag{3.17}
$$

Combining (3.15), (3.16) and (3.17) yields

$$
\left\{
\begin{aligned}
&\int_\Omega [a(x,T_k(u_\varepsilon),DT_k(u_\varepsilon)) - a(x,T_k(u_\varepsilon),DT_k(u))]D(T_k(u_\varepsilon)-T_k(u)) \\
&\qquad\qquad (\varphi'(z_\varepsilon) - \frac{b(k)}{\alpha}|\varphi(z_\varepsilon)|) \le \omega_6(\varepsilon).
\end{aligned}
\right. \tag{3.18}
$$

Recalling inequality (3.11) we have

$$
\left\{
\begin{aligned}
&0 \le \int_\Omega [a(x,T_k(u_\varepsilon),DT_k(u_\varepsilon)) - a(x,T_k(u_\varepsilon),DT_k(u))]D(T_k(u_\varepsilon)-T_k(u)) \\
&\le 2\omega_6(\varepsilon) \to 0.
\end{aligned}
\right. \tag{3.19}
$$

Lemma 5 of [BMP2] or Lemma S of [B'] implies

$$
T_k(u_\varepsilon) \to T_k(u) \quad \text{strongly in } W_0^{1,p}(\Omega). \tag{3.20}
$$

3.4. Passing to the limit.

The strong convergence (3.20) implies that for some subsequence

$$
Du_\varepsilon \to Du \quad \text{a.e.} \tag{3.21}
$$

which yields, since $a(x, u_\varepsilon, Du_\varepsilon)$ is bounded in $(L^{p'}(\Omega))^N$

$$a(x, u_\varepsilon, Du_\varepsilon) \rightharpoonup a(x, u, Du) \quad \text{weakly in } (L^{p'}(\Omega))^N \tag{3.22}$$

as well as

$$g_\varepsilon(x, u_\varepsilon, Du_\varepsilon) \to g(x, u, Du) \quad \text{a.e.} \tag{3.23}$$

We now use the classical trick in order to prove that $g_\varepsilon(x, u_\varepsilon, Du_\varepsilon)$ is uniformly equiintegrable. For any measurable subset E of Ω and for any $m \in \mathbb{R}^+$ we have

$$\left\{\begin{aligned} \int_E |g_\varepsilon(x, u_\varepsilon, Du_\varepsilon)| &= \int_{E\cap\{|u_\varepsilon|\le m\}} |g_\varepsilon(x, u_\varepsilon, Du_\varepsilon)| \\ &\quad + \int_{E\cap\{|u_\varepsilon|> m\}} |g_\varepsilon(x, u_\varepsilon, Du_\varepsilon)| \\ \le \int_{E\cap\{|u_\varepsilon|\le m\}} & b(m)(c(x) + |DT_m(u_\varepsilon)|^p) \\ &\quad + \int_{E\cap\{|u_\varepsilon|> m\}} |g_\varepsilon(x, u_\varepsilon, Du_\varepsilon)|. \end{aligned}\right. \tag{3.24}$$

For fixed m the first integral of the right hand side of (3.24) is smaller that $\int_E b(m)(c(x) + |DT_m(u_\varepsilon)|^p)$ and is thus small uniformly in ε when the measure of E is small (recall that $DT_m(u_\varepsilon)$ converges strongly in $(L^p(\Omega))^N$).

We now discuss the behaviour of the second integral of the right hand side of (3.24), which is smaller than $\int_{\{|u_\varepsilon|>m\}} |g_\varepsilon(x, u_\varepsilon, Du_\varepsilon)|$. If we assume (2.6), we have estimate (3.4) and thus

$$\left\{\begin{aligned} \int_{\{|u_\varepsilon|>m\}} |g_\varepsilon(x, u_\varepsilon, Du_\varepsilon)| &\le \int_{\{|u_\varepsilon|>m\}} \frac{1}{|u_\varepsilon|} u_\varepsilon g_\varepsilon(x, u_\varepsilon, Du_\varepsilon) \\ &\le \frac{1}{m} \int_\Omega u_\varepsilon g_\varepsilon(x, u_\varepsilon, Du_\varepsilon) \le \frac{C_2}{m}. \end{aligned}\right. \tag{3.25}$$

If we assume (2.7), we use in (3.1) the test function $S_m(u_\varepsilon)$, where for $m > 1$

$$\begin{cases} S_m(s) = 0 \quad \text{if} \quad |s| \le m - 1, \\ S_m(s) = 1 \quad \text{if} \quad s \ge m, \quad S_m(s) = -1 \quad \text{if } s \le -m, \\ S'_m(s) = 1 \quad \text{if} \quad m - 1 \le |s| \le m. \end{cases}$$

This yields

$$\int_\Omega a(x, u_\varepsilon, Du_\varepsilon) Du_\varepsilon \, S'_m(u_\varepsilon) + \int_\Omega S_m(u_\varepsilon) g_\varepsilon(x, u_\varepsilon, Du_\varepsilon) = \int_\Omega f_\varepsilon S_m(u_\varepsilon)$$

which implies

$$\int_{\{|u_\varepsilon| > m\}} |g_\varepsilon(x, u_\varepsilon, Du_\varepsilon)| \le \int_{\{|u_\varepsilon| \ge m-1\}} |f_\varepsilon|$$

and thus

$$\limsup_{\varepsilon} \int_{\{|u_\varepsilon| > m\}} |g_\varepsilon(x, u_\varepsilon, Du_\varepsilon)| \le \int_{\{|u| > m-1\}} |f| \tag{3.26}$$

In both cases we have proved that the second term of the right hand side of (3.24) is small, uniformly in ϵ and in E, when m is sufficiently large. This completes the proof of the uniform equiintegrability of $g_\varepsilon(x, u_\varepsilon, Du_\varepsilon)$. In view of (3.23) we thus have

$$g_\varepsilon(x, u_\varepsilon, Du_\varepsilon) \to g(x, u, Du) \quad \text{strongly in } L^1(\Omega). \tag{3.27}$$

Using (3.22) and (3.27) it is now easy to pass to the limit in (3.1) to obtain that u is a solution to (2.8). Theorem 1 is proved.

4. The case where f is nonnegative.

In this Section we assume that (2.1), (2.2), (2.3) and (2.5) as well as either (2.6) or (2.7) still hold. We moreover assume that

$$f \ge 0 \tag{4.1}$$

while we replace hypothesis (2.4) by

$$g(x, s, \xi) \ge 0, \qquad g(x, 0, 0) = 0. \tag{4.2}$$

In this case we have the following existence theorem:

Theorem 2. *Under the assumptions* (2.1), (2.2), (2.3), (4.2), (2.5) *and* (4.1), *and either* (2.6) *or* (2.7), *there exists at least one solution u of* (2.8) *such that*

$$u \ge 0. \tag{4.3}$$

Proof of Theorem 2.

The proof consists in repeating the proof of Theorem 1, for the approximate equation

$$\begin{cases} A(u_\varepsilon) + g_\varepsilon(x, u_\varepsilon, Du_\varepsilon) = f_\varepsilon & \text{in } \mathcal{D}'(\Omega) \\ u_\varepsilon \in W_0^{1,p}(\Omega) \end{cases} \tag{4.4}$$

where now

$$\begin{cases} g_\varepsilon(x, s, \xi) = h_\varepsilon(s) \dfrac{g(x, s, \xi)}{1 + \varepsilon\, g(x, s, \xi)} \\ h_\varepsilon(s) = 0 \text{ if } s \leq 0, \quad h^\varepsilon(s) = s/\varepsilon \text{ if } 0 \leq s \leq \varepsilon, \quad h^\varepsilon(s) = 1 \text{ if } s \geq \varepsilon, \end{cases} \tag{4.5}$$

and where f_ε is a sequence of smooth functions which strongly converges to f in $W^{-1,p'}(\Omega)$ or in $L^1(\Omega)$ with

$$f_\varepsilon \geq 0.$$

Use of the test function $-u_\varepsilon^-$ in (4.4) implies that

$$u_\varepsilon \geq 0.$$

The remainder of the proof is identical. It is indeed sufficient to remark that

$$u \geq 0$$

and that for almost every $x \in \Omega$

$$g_\varepsilon(x, s_\varepsilon, \xi_\varepsilon) \to g(x, s, \xi) \quad \text{if } s_\varepsilon \to s,\ \xi_\varepsilon \to \xi$$

since we assumed $g(x, 0, 0) = 0$.

Note that the latest assumption is natural since $g(x, 0, 0)$ has to be nonnegative because of the first part of assumption (4.2). But $g(x, 0, 0)$ can not be assumed to be strictly positive, as it is easily seen in the case $g(x, s, \xi) = g(x)$. Indeed the solution of the equation

$$\begin{cases} -\Delta u + g(x) = f(x) \\ u \in H_0^1(\Omega) \end{cases}$$

does not result in general to be nonnegative when $g \geq 0$ and $f \geq 0$.

Refer
[BBM] A.
eq
H.
[BG1] L.
me
[BG2] L.
nat
573
[BM] L.
Noı
[BMP1] L. I
cert
[BMP2] L. I
non
pp.
[B] H. B
[B'] F.E.
Proc
A.M.
[D] T. D
ral gı
[G] T. G.
une c
gradi
[LL] J. Le
linéai
1965,
[L] J.-L. I
linéair
[LM] P.-L. I
non liı

References.

[BBM] A. Bensoussan, L. Boccardo & F. Murat : On a nonlinear partial differential equation having natural growth terms and unbounded solution. Ann. Inst. H. Poincaré, Analyse non linéaire, 5, 1988, pp. 347-364.

[BG1] L. Boccardo & T. Gallouët : Nonlinear elliptic equations with right hand side measures. Comm. P.D.E., 17, 1992, pp. 641-655.

[BG2] L. Boccardo & T. Gallouët : Strongly nonlinear elliptic equations having natural growth terms and L^1 data. Nonlinear Anal. T.M.A., 19, 1992, pp. 573-579.

[BM] L. Boccardo & F. Murat : Almost everywhere convergence of the gradients. Nonlinear Anal. T.M.A., 19, 1992, pp. 581-597.

[BMP1] L. Boccardo, F. Murat & J.-P. Puel : Existence de solutions non bornées pour certaines équations quasi-linéaires. Portugaliae Math., 41, 1982, pp. 507-534.

[BMP2] L. Boccardo, F. Murat & J.-P. Puel : Existence of bounded solutions for nonlinear elliptic unilateral problems. Annali Mat. Pura Appl., 152, 1988, pp. 183-196.

[B] H. Brezis : Problèmes unilatéraux. J. Math. Pures Appl., 51, 1972, pp. 1-168.

[B'] F.E. Browder : Existence theorems for nonlinear partial differential equations. Proceedings of Symposia in Pure Mathematics, 16, S.S. Chern & S. Smale ed., A.M.S., Providence, 1970, pp. 1-60.

[D] T. Del Vecchio : Strongly nonlinear problems with Hamiltonian having natural growth. Houston J. Math., 16, 1990, pp. 7-24.

[G] T. Gallouët : Equations semilinéaires elliptiques avec, pour la non linéarité, une condition de signe et une dépendance sous quadratique par rapport au gradient. Ann. Fac. Sci. Toulouse, 2, 1988, pp. 161-169.

[LL] J. Leray & J.-L. Lions : Quelques résultats de Visik sur les problèmes non linéaires par la méthode de Minty-Browder. Bull. Soc. Math. France, 93, 1965, pp. 97-107.

[L] J.-L. Lions : Quelques méthodes de résolution des problèmes aux limites non linéaires. Dunod, Paris, 1969.

[LM] P.-L. Lions & F. Murat : Sur les solutions renormalisées d'équations elliptiques non linéaires. C. R. Acad. Sc. Paris and article. To appear.