NONLINEAR ELLIPTIC EQUATIONS IN $\mathbf{R}^N$ WITHOUT GROWTH RESTRICTIONS ON THE DATA

by

Lucio Boccardo *
Dipartimento di Matemàtica
Università di Roma I
P.le Aldo Moro - 00185 Roma, Italy

Thierry Gallouët *
Département de Mathématiques
Université de Chambéry, Savoie
Bp 1104 - 73011 Chambéry Cedex, France

and

Juan Luis Vazquez **
Departamento de Matemáticas
Universidad Autónoma de Madrid
28049 Madrid, Spain

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ABSTRACT

We show existence and regularity of solutions in $\mathbf{R}^N$ to nonlinear elliptic equations of the form $-\operatorname{div} A(x, Du) + g(x, u) = f$, when f is just a locally integrable function, under appropriate growth conditions on A and g but not on f. Roughly speaking, in the model case $-\Delta_p(u) + |u|^{s-1}u = f$, with $p > 2 - (1/N)$, existence of a nonnegative solution in $\mathbf{R}^N$ is guaranteed for every nonnegative $f \in L^1_{\text{loc}}(\mathbf{R}^N)$ if and only if $s > p - 1$.

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Introduction

Consider the nonlinear elliptic equation

$$-\mathrm{div}\,(|Du|^{p-2}Du) + |u|^{s-1}u = f\,, \tag{E1}$$

posed in $\mathbf{R}^N$, $N \geq 1$. We prove in this paper that, under the assumptions

$$p > 2 - \frac{1}{N} = p_0 \qquad \text{and} \qquad s > p-1\,, \tag{1}$$

for every $f \in L^1_{loc}(\mathbf{R}^N)$ there exists a function $u \in W^{1,1}_{\mathrm{loc}}(\mathbf{R}^N)$ such that $|u|^s \in L^1_{\mathrm{loc}}(\mathbf{R}^N)$, $|Du|^{p-1} \in L^1_{\mathrm{loc}}(\mathbf{R}^N)$ and (E1) holds in the sense of distributions. The important point is that a global weak solution exists with no extra assumption on the locally integrable data f (we will use the word *global* to stress the fact that the solution is defined in the whole space, $\mathbf{R}^N$). Moreover, $f \geq 0$ implies that $u \geq 0$, inequalities between integrable functions being understood almost everywhere.

Actually, our methods apply to more general equations of the form

$$-\mathrm{div}\,A(x, Du) + g(x,u) = f\,, \tag{E2}$$

where A is a nonlinear elliptic operator with power growth in $|Du|$ of order $p-1$ and quite general dependence on x, and $g(x,u)$ is of the same sign as u and grows in u like $|u|^s$. A precise statement of the assumptions on A and g is given in Section 1.

As a precedent to this work, Brezis [Br] proved not only existence but also uniqueness of such a solution for equation (E1) when $p=2$. The uniqueness of solutions for this and related elliptic equations involving the p-Laplacian operator $\Delta_p(u) = \mathrm{div}\,(|Du|^{p-2}Du)$ is still an open problem (there are some recent results obtained by Bénilan [Be]). It is to be noticed that in the general context of equation (E2) the results are new even in the case $p=2$, i.e. when we consider a linear elliptic operator with discontinuous coefficients.

As for the restrictions on the exponents, the condition $p > p_0$ is natural in the theory of the p-Laplacian operator in the L^1-framework, as was explained in [BG]. But it is not essential and, as a matter of fact, we obtain results even for $p \leq p_0$. The essential restriction is thus $s > p-1$. Indeed, we also prove that for $s \leq p-1$ and radially symmetric $f \geq 0$, a restriction on the growth of f as $|x| \to \infty$ is necessary in order for (E1) to admit a nonnegative radially symmetric solution. A similar growth condition was investigated by Gallouët and Morel [GM] in the case $p=2$. Observe that this condition makes the result essential nonlinear: the case $p=2$, $s=1$ is excluded.

In order to state in detail our results we introduce for $1 < p \leq N$ and $s > 0$ the numbers

$$q_0 = \frac{(p-1)N}{N-1} \tag{2}$$

and its Sobolev conjugate

(3) $$r_0 = \begin{cases} \frac{(p-1)N}{N-p} & \text{if} \quad p < N \\ +\infty & \text{if} \quad p = N\,. \end{cases}$$

Clearly, $q_0 > p-1$ and $q_0 > 1$ precisely if $p > p_0$. Also, $q_0 = p$ for $p = N$, otherwise $q_0 < p$ and $q_0 < N$. We prove the following results

Theorem 1. *Let $f \in L^1_{\rm loc}(\mathbf{R}^N)$, let p and s satisfy the conditions* (1) *and assume besides that $p \le N$. Then there exists at least a global solution u of equation* (E2) *which belongs to $W^{1,q}_{\rm loc}(\mathbf{R}^N)$ for every $q \in [1, q_0)$. Consequently, $u \in L^r_{\rm loc}(\mathbf{R}^N)$ for every $r \in [1, r_0)$. Furthermore, if f is nonegative, so is u.*

The first ingredient of our proof consists in obtaining certain a priori local bounds: in the case of equation (E1) we estimate the L^1-norm of $|u|^s$ and some suitable L^r-norm of $|Du|$ in a ball $B_R = B_R(0)$, $R > 0$, in terms only of R, p, s, N and the L^1-norm of f in B_{2R}; no other information about the solution or the data is needed. When dealing with equation (E2) the bounds depend also on the local norms of functions appearing in the structure assumptions. The extra difficulty we face at this stage with respect to the method of [Br] consists in obtaining explicit gradient estimates, which are not necessary in the linear case $p = 2$.

The proof of Theorem 1 contains another delicate step in passing to the limit in the sequence of approximate problems; it consists in showing that the gradients Du_n of the approximate solutions converge almost everywhere.

It is worth remarking that the spaces we obtain are optimal if we do not take into account the special structure of the term $g(x,u)$: they correspond to the best regularity of solutions of $\Delta_p(u) \in L^1_{\rm loc}$. In particular, the exponents can be easily obtained from Sobolev-type embedding formulas, which our result justifies.

When $p > N$ the investigation is simpler since we can show that u is in fact locally bounded. We then have

Theorem 2. *Let $f \in L^1_{\rm loc}(\mathbf{R}^N)$ and let $s > p-1$ and $p > N$. Then there exists at least a global solution $u \in L^\infty_{\rm loc}(\mathbf{R}^N)$ of equation* (E2) *which belongs to $W^{1,p}_{\rm loc}(\mathbf{R}^N)$. Furthermore, if f is nonnegative, so is u.*

Observe that $|Du| \in L^p_{\rm loc}$ is the regularity one expects from variational methods, which cannot in principle be applied when the second member f is merely integrable.

Our methods work even in cases where $1 < p \le p_0$. Let us introduce for $p > 1$ and $s > 0$ the number

(4) $$q_1 = \frac{ps}{s+1}$$

which is larger than 1 when $s(p-1) > 1$ and is always less than p. Then we have

Theorem 3. *Let $1 < p \leq p_0$ and $s(p-1) > 1$. Then for every $f \in L^1_{\rm loc}(\mathbf{R}^N)$ there exists at least a global solution u of equation* (E2) *which belongs to $W^{1,q}_{\rm loc}(\mathbf{R}^N)$ for every $q \in [1, q_1)$. Furthermore, if f is nonnegative, then so is u.*

Remark that for $p < 2$ we have $1/(p-1) > p-1$, hence $s(p-1) > 1$ implies $q_1 < s$. On the other hand, notice that if we want to define (in a weak sense) $\Delta_p(u)$ for a function $u \in W^{1,q}_{\rm loc}(\mathbf{R}^N)$, with $q < q_1$, we need $q_1 > p-1$. This happens precisely for $s > p-1$, which is satisfied in Theorem 3.

Another interesting theme of our investigation is the phenomenon that we will call *improved regularity* . It can be explained as follows: according to its definition, a solution u satisfies $|u| \in L^s_{\rm loc}(\mathbf{R}^N)$; this fact can be exploited to obtain a better information for $|Du|$, using a sort of nonlinear interpolation. Of course, this will have a sense when $p < N$ and $s > r_0$, otherwise the information $u \in L^s_{\rm loc}$ is irrelevant in view of the above results. We then have

Theorem 4. *Let $f \in L^1_{\rm loc}(\mathbf{R}^N)$ and assume that $p_0 < p < N$ and $s > r_0$. Then the solution constructed in Theorem 1 satisfies $|Du| \in L^q_{\rm loc}(\mathbf{R}^N)$ for every $q \in [1, q_1)$.*

We remark that $s > r_0$ is equivalent to $q_1 > q_0$, hence the regularity improvement. Moreover, we have $q_1 \to p$ as $s \to \infty$. No extra assumptions are made on f. It is also interesting to notice that for $s > r_0$ the Sobolev exponent associated to q_1,

$$q_1^* = q_1 N/(N - q_1)\,, \tag{5}$$

is smaller than s, i.e. the improved regularity obtained for $|Du|$ is compatible with the Sobolev embedding and the fact that the best information we have on u is $u \in L^s_{\rm loc}$.

Summing up, we obtain the following (p-s)-diagram

Fig. 1. Diagram

The basic a priori estimate is obtained in Section 2 and the results proved in Sections 3 and 4, while Section 5 contains the construction of the counterexample to existence if $s < p - 1$.

Let us recall here a different but related result: an a priori estimate in $L^\infty_{\rm loc}(\mathbf{R}^N)$ for local solutions of (E1) holds for every $s > p - 1$ if we assume regularity on f, namely $f \in L^\infty_{\rm loc}(\mathbf{R}^N)$, as a consequence of the results of [V]. In that paper the condition $s > p-1$ is shown to be necessary and sufficient for such an estimate to hold.

A final Section 6 is devoted to exploit our regularity improvement technique to equation (E2) posed in a bounded domain $\Omega \subset \mathbf{R}^N$ with $f \in L^1(\Omega)$ and homogeneous Dirichlet conditions. Existence and regularity for this problem has been studied in [BG]. The improved exponents correspond to those of the unbounded case. Roughly speaking, our basic lemma says in the case of equation (E1) that, with the definitions of p_0, q_1 and r_0 given above, the fact that a function $u \in W^{1,1}_{\rm loc}(\mathbf{R}^N)$ satisfies

$$\Delta_p(u) \in L^1(\Omega) \text{ and } |u|^s \in L^1(\Omega)$$

implies that

$$u \in W^{1,r}_0(\Omega) \text{ for all } 1 < r < q_1 \text{ if } p_0 < p < N \text{ and } s > r_0.$$

To end this introduction let us remark that similar techniques can be applied to parabolic equations like

$$u_t - \operatorname{div}(a(x,t,Du)) + g(x,t,u) = f\,, \tag{6}$$

posed in $Q = \{(x,t) : x \in \mathbf{R}^N,\ 0 < t < \infty\}$. Details of this adaptation are given in [BGV].

1. Structure assumptions

In subsequent sections we will study equations of the form

$$-\operatorname{div} A(x,Du) + g(x,u) = f\,. \tag{E2}$$

Here $u(x)$ and $f(x)$ are scalar functions of $x \in \mathbf{R}^N$ and Du denotes the gradient of u. Given a function f in $L^1_{\rm loc}(\mathbf{R}^N)$ we look for a global weak solution to (E2), i.e. a function $u \in W^{1,1}_{\rm loc}$ such that both $A(x,Du)$ and $g(x,u)$ are well defined in $L^1_{\rm loc}(\mathbf{R}^N)$ and (1.1) is satisfied in $\mathcal{D}'(\mathbf{R}^N)$.

Both A and g have to satisfy certain structural assumptions which are modelled on equation (E1) of previous section. More precisely, A satisfies:

(A1) $A(x,\xi) : \mathbf{R}^N \times \mathbf{R}^N \to \mathbf{R}^N$ is measurable in $x \in \mathbf{R}^N$ for any fixed $\xi \in \mathbf{R}^N$ and continuous in $\xi \in \mathbf{R}^N$ for a.e. $x \in \mathbf{R}^N$.

(A2) There exist two constants $p > 1$ and $c > 0$ such that for all ξ and a.e. x

$$A(x,\xi) \cdot \xi \geq c|\xi|^p\,.$$

(A3) There exist functions $b(x)$, locally in $L^{p'}$ ($p' = p/(p-1)$), and $d(x)$, locally bounded, such that for all ξ and a.e. x

$$|A(x,\xi)| \le b(x) + d(x)|\xi|^{p-1}\,.$$

(A4) There exist a real number $\lambda > 1$ and a measurable function $\beta(x,\xi,\eta)$ such that for a.e. $x \in \mathbf{R}^N$ and all $(\xi,\eta) \in \mathbf{R}^N \times \mathbf{R}^N$

$$(A(x,\xi) - A(x,\eta))\cdot(\xi-\eta) \ge \frac{|\xi-\eta|^\lambda}{\beta(x,\lambda,\eta)}$$

Moreover, β has to satisfy

$$0 \le \beta(x,\xi,\eta) \le d(x)(|\xi|^\gamma + |\eta|^\gamma) + e(x)^{\lambda-1}\,,$$

for some function $e \in L^1_{\rm loc}(\mathbf{R}^N)$ and some number γ which satisfies: $0 \le \gamma \le \lambda - 1$ if $p \le p_0$, $0 \le \gamma \le (\lambda-1)q_0$ if $p_0 < p \le N$, $0 \le \gamma \le (\lambda-1)p$ if $p > N$.

Hypotheses (A1)-(A3) are classical in the study of nonlinear operators in divergence form, see [LL], [L]. Hypothesis (A4) is more technical (see [BG]). The model example of a function satisfying (A1)-(A4) is of course $A(x,\xi) = |\xi|^{p-2}\xi$ (in this case (A4) is satisfied with $\lambda = p$, $\gamma = 0$ if $p \ge 2$, and with $\lambda = 2$, $\gamma = 2-p$ if $1 < p < 2$).

The assumptions on g are the following:

(G1) $g(x,\sigma) : \mathbf{R}^N \times \mathbf{R} \to \mathbf{R}$ is measurable in $x \in \mathbf{R}^N$ for any fixed $\sigma \in \mathbf{R}$ and continuous in σ for a.e. x.

(G2) There exists $s > 0$ such that for all σ and almost every x

$$g(x,\sigma)\,\mathrm{sign}\,(\sigma) \ge |\sigma|^s\,.$$

(G3) For all $t > 0$ the function

$$G_t(x) = \sup_{|\sigma|\le t} |g(x,\sigma)|$$

is locally integrable over $\mathbf{R}^N$.

Remark. We have formulated our growth assumptions on A and g in terms of powers for convenience. In that sense, it is worth noticing that our results hold with minor modifications for more general growth assumptions. Thus, condition (G2) can be replaced by

$$g(x,\sigma)\mathrm{sign}\,(\sigma) \ge h(\sigma)$$

with $h : \mathbf{R} \to \mathbf{R}$ a continuous odd function. Then Theorem 1 is still valid if we replace the condition $s > p-1$ by the following assumptions on h: $h(t)t^{1-p}$ is increasing in $\mathbf{R}^+$ and

$$\int^{\infty} \frac{dt}{t^{\frac{p}{2p-1}} h(t)^{\frac{1}{2p-1}}} < \infty \,.$$

This condition does not seem to be optimal. The exact condition is probably

$$\int^{\infty} \frac{dt}{t^{1/p} h(t)^{1/p}} < \infty \,,$$

which is weaker; this latter condition has been proved to work for equation (E1) with $p = 2$ in [GM].

2. Local estimates

We want to solve (E2) in $\mathbf{R}^N$ for general $f \in L^1_{\mathrm{loc}}(\mathbf{R}^N)$. Our idea is to begin by solving (E2) in the balls $B_R = \{x \in \mathbf{R}^N,\ |x| < R\}$ with suitable data f_n which approximate f. If we obtain estimates which are independent of R and the approximation f_n, we can then pass to the limits $R \to \infty$ and $f_n \to f$ to obtain a solution of the original problem. We will assume in this section that conditions (A1)-(A4) and (G1)-(G2) hold. In fact, only a weak version of (A4) is necessary at this stage.

To begin our program we observe the well-known fact that for any $f \in L^{p'}(B_R)$ with $p' = p/(p-1)$ there exists a unique $u \in W^{1,p}_0(B_R)$ solution of

$$-\mathrm{div}\,(A(x,Du)) + g(x,u) = f \quad \text{in} \quad \mathcal{D}'(B_R). \tag{2.1}$$

Indeed one has $g(x,u) \in L^1(B_R)$ and

$$\int_{B_R} A(x,Du)\cdot Dv\,dx + \int_{B_R} g(x,u)v\,dx = \int_{B_R} fv\,dx \tag{2.2}$$

holds for all $v \in W^{1,p}_0(B_R) \cap L^\infty(B_R)$ (see for instance [W], [BB]). For these solutions we obtain local estimates for u and Du with convenient dependence on the data.

LEMMA 2.1. *Assume that $s > p-1$, consider a radius $R > 0$ and a function $f \in L^{p'}(B_R)$ and let $0 < 2r < R$. If $u \in W^{1,p}(B_R)$ is the solution of (2.1), then we can estimate*

$$\|u\|_{L^s(B_r)} \le \|g(x,u)\|_{L^1(B_r)} \le C\,, \tag{2.3}$$

where the constant C depends only on the parameters p and s, the radius r, the norm $\|f\|_{L^1(B_{2r})}$ of the data, and the structure conditions. These latter appear through the ellipticity constant c and the norms $\|b\|_{L^1(B_{2r})}$ and $\|d\|_{L^1(B_{2r})}$ and $\|G_1\|_{L^1(B_r)}$. In particular, the estimate is independent of R. Moreover, for every $m > 0$ there exists C_m depending on the same arguments plus m, such that

$$\int_{B_r} \frac{|Du|^p}{(|u|+1)^{m+1}}\,dx \le C_m\,. \tag{2.4}$$

Demostracin. For $m > 0$ we define the function $\phi = \phi_m : \mathbf{R} \to \mathbf{R}$ by

$$\left\{ \begin{array}{lll} \phi(\sigma) & = m \int_0^\sigma \frac{dt}{(t+1)^{m+1}} & \text{if} \quad \sigma \geq 0 \\ \phi(\sigma) & = -\phi(-\sigma) & \text{if} \quad \sigma < 0 \end{array} \right. \tag{2.5}$$

Notice that $|\phi(\sigma)| \leq \int_0^\infty m(t+1)^{-(m+1)}\, dt = 1$. We apply identity (2.2) to our solution u taking as test function

$$v = \phi_m(u)\, \theta^\alpha\,, \tag{2.6}$$

where m and α are two numbers such that

$$0 < m < \frac{s}{p-1} - 1\,, \qquad \alpha \geq \frac{ps}{s-(m+1)(p-1)}\,,$$

(observe that $\alpha > p$); θ is a smooth function with compact support in B_{2r}, such that $0 \leq \theta \leq 1$ and $\theta \equiv 1$ on B_r and $|D\theta| \leq 2/r$.

Thanks to the structure assumptions (A2), (A3), we obtain

$$\begin{aligned} & c\int |Du|^p \phi'(u)\theta^\alpha\, dx + \int g(x,u)\phi(u)\theta^\alpha\, dx = \\ & \int f\phi(u)\theta^\alpha\, dx - \alpha\int A(x,Du)\phi(u)\theta^{\alpha-1}\cdot D\theta\, dx \leq \\ & C_1 + C_2\int |Du|^{p-1}\theta^{\alpha-1} dx\,, \end{aligned}$$

with $C_1 = \|f\|_{L^1(B_{2r})} + (2\alpha/r)\|b\|_{L^1(B_{2r})}$ and $C_2 = (2\alpha/r)\|d\|_{L^\infty(B_{2r})}$. In order to estimate the last term we use Young's inequality

$$ab \leq \frac{a^n}{n} + \frac{b^{n'}}{n'}\,, \tag{2.7}$$

which is true for every $a, b > 0$ and every pair of conjugate numbers $n, n' > 1$. Therefore, taking $n = p'$ we may write

$$\begin{aligned} & |Du|^{p-1}\theta^{\alpha-1} \leq \varepsilon\frac{|Du|^p\theta^\alpha}{p'(|u|+1)^{m+1}} + \\ & \frac{1}{p\varepsilon^{p-1}}\theta^{\alpha-p}(|u|+1)^{(m+1)(p-1)}\,, \end{aligned}$$

where ε can be any positive number. Choosing $\varepsilon \leq cp'm/(2C_2)$ and recalling that $\phi'(u) = m(|u|+1)^{-(m+1)}$ we arrive at

$$\begin{aligned} & \frac{cm}{2}\int \frac{|Du|^p}{(|u|+1)^{m+1}}\theta^\alpha\, dx + \int g(x,u)\phi(u)\theta^\alpha\, dx \leq \\ & C_1 + C_3\int (|u|+1)^{(m+1)(p-1)}\theta^{\alpha-p}\, dx \end{aligned} \tag{2.8}$$

with $C_3 = C_2/(p\varepsilon^{p-1}) = k(p,m)c^{1-p}C_2^p$. We designate by $k(\dots)$, various constants depending on numerical parameters to be specified, never on R.

The last term in (2.8) has to be transformed. Let $\mu = (m+1)(p-1)$. Since $m < \frac{s}{p-1} - 1$, one has $\mu < s$, and then, using again the inequality (2.7) with $n = s/\mu$, we get that for any $\delta > 0$

$$\theta^{\alpha-p}(|u|+1)^{\mu} \leq \frac{\delta\mu}{s}\theta^{\alpha}(|u|+1)^{s} + \frac{s-\mu}{s\delta^{\mu/(s-\mu)}}\theta^{\alpha-\frac{ps}{s-\mu}}\,.$$

Since the exponent $\alpha - \frac{ps}{s-\mu}$ is nonnegative we conclude that

$$\theta^{\alpha-p}(|u|+1)^{(m+1)(p-1)} \leq \frac{2^s\delta\mu}{s}\theta^{\alpha}|u|^{s} + k(\mu,s)(\delta^{\frac{\mu}{\mu-s}} + \delta)\,. \tag{2.9}$$

In order to absorb the resulting term in $|u|^s$ into the term in the first member of (2.9) involving $g(x,u)$ we observe that $g(x,\sigma)\phi(\sigma)$ is always nonnegative. Moreover, thanks to (G2) we have for $\sigma \geq 1$ $g(x,\sigma)\phi(\sigma) \geq |\sigma|^s\phi(1)$ ($\phi(1) = 1 - 2^{-m}$). Summing up, we get for all u

$$|u|^s \leq g(x,u)\frac{\phi(u)}{\phi(1)} + 1\,. \tag{2.10}$$

Using this inequality in (2.9) and setting $\delta = s\phi(1)/(2^{s+1}\mu C_3)$ we transform (2.8) into

$$\frac{cm}{2}\int \frac{|Du|^p}{(|u|+1)^{m+1}}\theta^{\alpha}dx + \frac{1}{2}\int g(x,u)\phi(u)\theta^{\alpha}dx \leq C_4\,, \tag{2.11}$$

with $C_4 = C_1 + k(m,s,\mu)r^N(C_3^{\frac{s}{s-\mu}} + 1)$. Since both terms in the left-hand side of (2.11) are nonnegative, the bound C_4 applies to each of them. Together with (G2), (G3) and the definitions of ϕ and θ, this gives

$$\int_{B_r} |u|^s dx \leq \int_{B_r} |g(x,u)|dx \leq C_5, \tag{2.12}$$

where $C_5 = \|G_1\|_{L^1(B_r)} + 2C_4/\phi(1)$, which gives estimate (2.3), and

$$\int_{B_r} \frac{|Du|^p}{(|u|+1)^{m+1}}dx \leq \frac{2C_4}{cm}\,, \tag{2.13}$$

which gives (2.4) for $0 < m < \frac{s}{p-1} - 1$. Since for $(|u|+1)^{-1} \leq 1$, once (2.4) holds for all small $m > 0$ it holds for every $m > 0$.   #

Remark. We may write the constant C of estimate (2.3) in the form

$$C = a_0\|f\|_{L^1(B_{2r})} + a_1\|b\|_{L^1(B_{2r})} + a_2\|d\|^{\beta}_{L^1(B_{2r})} + \|G_1\|_{L^1(B_r)} + a_3\,, \tag{2.14}$$

where a_0 and β are positive constants depending only on p, s, a_1 and a_3 depend also on r, and a_2 depends on p, s, r and c. A similar expression holds for C_m, but now all the coefficients depend also on c and m and the term in $\|G_1\|$ does not appear.   #

Estimate (2.4) gives an indirect control on the gradient Du of our solutions. We can obtain a direct estimate by means of Sobolev's inequality.

LEMMA 2.2. *Assume that $p > 1$, $s > 0$ and u is a function in $W^{1,1}(B_r)$, $r > 0$, such that*

$$\int_{B_r} |u|^s dx \leq C_1, \tag{2.15}$$

for a some C_1, and that for all $m > 0$

$$\int_{B_r} \frac{|Du|^p}{(|u|+1)^{m+1}}\, dx \leq C_2\,, \tag{2.16}$$

where C_2 may depend on m. Then one has

i) If $p_0 < p \leq N$, then for any $1 \leq q < q_0$ $|Du| \in L^q(B_r)$ and $u \in L^{q^*}(B_r)$, where $q^* = qN/(N-q)$. Observe that $q_0^* = r_0$.

ii) If $p > N$, then u is bounded in B_r and $|Du| \in L^p(B_r)$.

In both cases the corresponding norms can be estimated in terms of C_1 and $C_2(m)$, where m must be small enough. In case ii) we only need (2.16) to be true for some $m < p-1$.

Demostracin. Part i) We assume that $p \in (p_0, N]$, so that in particular $N > 1$. Let $1 \leq q \leq q_0$ ($q_0 \in (1, N]$ has been defined in formula (2) of the Introduction). We choose m such that $0 < m < m_0 = (q_0 - q)\frac{N-1}{N-q}$ (note that $m_0 > 0$ and that $m_0 \to 0$ as $q \to q_0$).

Since $q < q_0 \leq p$ one has, by Hölder's inequality,

$$\int_{B_r} |Du|^q dx \leq (\int_{B_r} \frac{|Du|^p}{(|u|+1)^{m+1}} dx)^{q/p} (\int_{B_r} (|u|+1)^{(m+1)\frac{q}{p-q}} dx)^{\frac{p-q}{p}} \tag{2.17}$$

We have chosen m_0 so that $m < m_0$ is equivalent to $(m+1)\frac{q}{p-q} < q^*$. Hence we may write for every $\varepsilon > 0$,

$$(|u|+1)^{(m+1)\frac{q}{p-q}} \leq \varepsilon |u|^{q^*} + c(\varepsilon)\,.$$

Choosing $\varepsilon > 0$ small enough and using (2.16) we arrive at

$$\int_{B_r} |Du|^q dx \leq \frac{1}{2^{q+1}} (\int_{B_r} |u|^{q^*} dx)^{\frac{p-q}{p}} + C_3. \tag{2.18}$$

Since $q < N$ one also has the classical Sobolev-type inequality

$$\|u - \tilde{u}_r\|_{L^{q^*}(B_r)} \leq \|Du\|_{L^q(B_r)}\,,$$

where $\tilde{u}_r = (\text{meas}\,(B_r))^{-1} \int_{B_r} u dx$. This implies

$$\|u\|_{L^{q^*}(B_r)} \leq (\int_{B_r} |Du|^q dx)^{1/q} + (\text{meas}\, B_r)^{1/q^*} |\tilde{u}_r|\,. \tag{2.19}$$

Now, by (2.15), we have when $s \geq 1$

$$|\tilde{u}_r| \leq \frac{1}{\text{meas}\,(B_r)} \int_{B_r} |u|\, dx \leq C_4\,,$$

while for $0 < s < 1$

$$|\tilde{u}_r| \leq \frac{1}{\text{meas}\,(B_r)} (\int_{B_r} |u|^{\beta\eta} dx)^{1/\eta} (\int_{B_r} |u|^{(1-\beta)\eta'} dx)^{1/\eta'} \leq C_5 (\int_{B_r} |u|^{q^*} dx)^{\frac{1-s}{q^*-s}}\,,$$

with $\eta = \frac{q^*-s}{q^*-1} > 1$ and $\beta = \frac{s}{\eta} < 1$, so that $\beta\eta = s$ and $(1-\beta)\eta' = q^*$. In both cases, one has

$$|\tilde{u}_r| \leq C_6 \|u\|^{\delta}_{L^{q^*}(B_r)}, \tag{2.20}$$

for some $0 \leq \delta < 1$. (2.18) , (2.19) and (2.20) yield

$$\|u\|_{L^{q^*}(B_r)} \leq \frac{1}{2} (\int_{B_r} |u|^{q^*} dx)^{\frac{p-q}{pq}} + C_7 \|u\|^{\delta}_{L^{q^*}(B_r)} + C_8\,. \tag{2.21}$$

Since $\delta < 1$ and $\frac{p-q}{pq} < \frac{1}{q^*}$ if $p < N$, $\frac{p-q}{pq} = \frac{1}{q^*}$ if $p = N$, (2.21) implies that $\|u\|_{L^{q^*}(B_r)} \leq C_9$. It then follows from (2.18) that $\|Du\|_{L^q(B_r)} \leq C_{10}$. This ends Part i).

Part ii) We now choose m such that $0 < m < p-1$. Let $\phi(u) = (|u|+1)^{\frac{p-m-1}{p}}$. Then

$$|D\phi(u)| = \frac{(p-m-1)|Du|}{p(|u|+1)^{\frac{m+1}{p}}}.$$

The bound (2.16) is equivalent to $D\phi(u) \in L^p(B_r)$. Since $p > N$ we obtain Lipschitz continuity of $\phi(u)$ in B_r. Using (2.15) we then obtain an L^∞-bound for $\phi(u)$, and therefore for u, in B_r. The bound for $\|Du\|$ follows then immediately from (2.16). This proves part ii) of Lemma 2.2. #

The regularity of u can be improved when s is very large by using in a more essential way the estimate $|u|^s \in L^1(B_r)$. This is a kind of interpolation result.

LEMMA 2.3. *Under the above hypotheses* (2.15), (2.16), *if moroever* $s(p-1) > 1$, *then* $|Du| \in L^q(B_r)$ *for any* $1 < q < q_1$. *The norm* $\|Du\|_{L^q}$ *depends on* C_1 *and* $C_2(m)$*with* m *small enough.*

Proof. Let $1 < q < q_1 = \frac{ps}{s+1}$. We choose

$$0 < m \leq m_1 = \frac{s(p-q)}{q} - 1 = \frac{(q_1 - q)}{q}(s+1)\,.$$

We have $m_1 > 0$ since $q < q_1$ and $m_1 \to 0$ as $q \to q_1$. Since $q < q_1 < p$ we can write

$$|Du|^q \leq \frac{|Du|^p}{(|u|+1)^{m+1}} + C(|u|+1)^{\frac{m+1}{p-q}q}. \tag{2.22}$$

Now, $m < m_1$ implies that $(m+1)\frac{q}{p-q} \leq s$, and we deduce from (2.15), (2.16) and (2.22) that

$$\int_{B_r} |Du|^q dx \leq C_3. \qquad \#$$

3. Proof of Theorem 1

We assume that A and g satisfy hypotheses (A) and (G), that $p_0 < p \leq N$ and $s > p-1$. We have data $f \in L^1_{loc}(\mathbf{R}^N)$. For $n \geq 1$ we set $f_n = \inf(|f|, n)\text{sign}\,(f)$, and we let $u_n \in W_0^{1,p}(B_n)$ be the unique solution of

$$-\text{div}\,(A(x, Du_n)) + g(x, u_n) = f_n \quad \text{in } \mathcal{D}'(B_n)\,. \tag{3.1}$$

Then for all $v \in W_0^{1,p}(B_n) \cap L^\infty(B_n)$

$$\int_{B_n} A(x, Du_n) \cdot Dv dx + \int_{B_n} g(x, u_n) v dx = \int_{B_n} f_n v\, dx\,. \tag{3.2}$$

Let $r > 0$ and $1 < q < q_0$. By Section 2 (Lemmas 2.1 and 2.2) we know that for any r with $n > 2r$ there exist constants C_i independent of n such that

$$\|g(x, u_n)\|_{L^1(B_r)} \leq C_1$$
$$\|Du\|_{L^q(B_r)} \leq C_2$$
$$\|u_n\|_{L^{q^*}(B_r)} \leq C_3\,.$$

Therefore, and up to extraction of a subsequence if necessary, we may assume that

(3.3) the sequence u_n converges to a function u weakly in $W^{1,q}(B_r)$ for any $1 \leq q < q_0$ and any $r > 0$ and $u_n \to u$ almost everywhere in $\mathbf{R}^N$, while

(3.4) $g(x, u_n)$ converges to $g(x, u)$ a.e. in $\mathbf{R}^N$.

Furthermore, one has

$$f_n \to f \qquad \text{in } L^1(B_r) \text{ for any } r > 0.$$

In order to prove that u is a solution of (E2) two difficulties remain:

(i) Passing to the limit in the nonlinear term $g(x, u_n)$.

(ii) Passing to the limit in div $(A(x, Du_n))$ when A is nonlinear with respect to its second argument.

In order to successfully perform both limit processes, we shall first obtain local equi-integrability of $g(x, u_n)$ and a.e. convergence of Du_n to Du.

Let $t > 0$, and $r > 0$. We define $\phi : \mathbf{R} \to \mathbf{R}$ by

$$
\begin{cases} \phi(s) & = \inf((s-t)^+, 1), \qquad s \geq 0 \\ \phi(s) & = -\phi(-s), \qquad s < 0 \,. \end{cases} \tag{3.5}
$$

Let θ be a cutoff function as in (2.6), Section 2. We take $v = \phi(u_n)\theta$ in formula (3.2) to obtain

$$
\int_{E_{n,t+1} \cap B_r} |g(x, u_n)| \, dx \leq \int_{E_{n,t} \cap B_{2r}} |f_n| \, dx + C_1 \int_{E_{n,t} \cap B_{2r}} (|Du_n|^{p-1} + b) \, dx,
$$

where $E_{n,t}$ denotes the set $\{(x,t) : |u_n(x,t)| \geq t\}$. Here and in the sequel the C_i denote different constants which do not depend on n.

Using the $L^q(B_{2r})$-bound on $|Du_n|$ for some $q \in (p-1, q_0)$ we obtain for some $\delta > 0$,

$$
\int_{E_{n,t+1} \cap B_r} |g(x, u_n)| dx \leq \int_{E_{n,t} \cap B_{2r}} (|f| + |b|) \, dx + C_2 (\text{meas}\, (E_{n,t} \cap B_{2r}))^\delta \,. \tag{3.6}
$$

Since, due to the $L^1(B_{2r})$-bound on u_n, $\text{meas}\,(E_{n,t} \cap B_{2r}) \to 0$ as $t \to +\infty$ uniformly with respect to n, and $f, b \in L^1(B_{2r})$, we deduce from (3.6) that, given $\varepsilon > 0$ there exists t_0 such that for all $r < n/2$ and $t \geq t_0$

$$
\int_{E_{n,t+1} \cap B_r} |g(x, u_n)| \, dx \leq \varepsilon \tag{3.7}
$$

and t_0 is independent of n.

(3.7) and (G3) give equi-integrability of $g(x, u_n)$ on B_r: for all $\varepsilon > 0$ there exists $\eta > 0$ such that whenever a subset $A \subset B_r$ has measure less than η, then $\int_A |g(x, u_n)| \leq \varepsilon$.

From this, (3.4) and Vitali's theorem we obtain the convergence

$$
g(x, u_n) \to g(x, u) \quad \text{in } L^1(B_r) \quad \text{for all } r > 0 \,. \tag{3.8}
$$

It remains to prove a.e. convergence of Du_n to Du. Let $r > 0$ and $\varepsilon > 0$. We define another function ψ by

$$
\begin{cases} \psi(s) & = \inf(s, \varepsilon) \qquad s \geq 0 \\ \psi(-s) & = -\psi(s) \qquad s \geq 0 \,. \end{cases} \tag{3.9}
$$

Applying (3.2) both to u_n and u_m with test function $v = \psi(u_n - u_m)\theta$, and with $n, m \geq 4r$, subtracting and using (A3), we obtain (with $h_n = f_n - g(x, u_n)$)

$$
\int_{\{|u_n - u_m| \leq \varepsilon\} \cap B_r} (A(x, Du_n) - A(x, Du_m)) \cdot (Du_n - Du_m) \, dx \leq \varepsilon \left[\int_{B_{2r}} (|h_n| + |h_m|) \, dx + \right.
$$
$$
\left. + C_3 \int_{B_{2r}} (|Du_n|^{p-1} + |Du_m|^{p-1} + 2b) \, dx \right]
$$

Using (A3), the $L^1(B_{2r})$-bound on f_n and $g(x,u_n)$, and the $L^q(B_{2r})$-bound on Du_n for some $q \in [p-1, q_0)$, we obtain

$$\int_{\{|u_n-u_m|\le\varepsilon\}\cap B_r} (A(x,Du_n) - A(x,Du_n)) \cdot (Du_n - Du_m)\,dx \le \varepsilon C_4.$$

Applying (A4) this gives

$$\int_{\{|u_n-u_m|\le\varepsilon\}\cap B_r} \frac{|Du_n - Du_m|^\lambda}{\beta(x,Du_n,Du_m)}\,dx \le \varepsilon C_4. \tag{3.10}$$

Now

$$\begin{aligned}\int_{B_r} |Du_n - Du_m|\,dx \le \left(\int_{\{|u_n-u_m|\le\varepsilon\}\cap B_r} \frac{|Du_n - Du_m|^\lambda}{\beta(x,Du_n,Du_m)}\,dx\right)^{1/\lambda} \cdot \\ \left(\int_{B_r} \beta(x,Du_n,Du_m)^{\frac{1}{\lambda-1}}\,dx\right)^{1-\frac{1}{\lambda}} + \int_{\{|u_n-u_m|\ge\varepsilon\}\cap B_r} |Du_n - Du_m|\,dx\,.\end{aligned} \tag{3.11}$$

Recall the assumption (A4) on β and observe that, since $\gamma \le (\lambda-1)q_0$ we have a uniform $L^q(B_r)$-bound on $|Du_n|$ for some $q \ge \frac{\gamma}{\lambda-1}$. On the other hand, the fact that $u_n \to u$ in $L^1_{loc}(\mathbf{R}^N)$ implies that $\operatorname{meas}(\{|u_n - u_m| \ge \varepsilon\} \cap B_r) \to 0$ as $n, m \to +\infty$. Using these observations, we deduce from (3.10) and (3.11) that for every $r > 0$

$$\int_{B_r} |Du_n - Du_m|\,dx \to 0, \quad \text{as} \quad n, m \to +\infty\,. \tag{3.12}$$

This implies that, possibly after extraction of a subsequence

$$Du_n \to Du \quad a.e. \text{ in } \mathbf{R}^N\,,$$

and for any $q \in (1, q_0)$

$$Du_n \to Du \quad \text{in } L^q_{loc}(\mathbf{R}^N)\,. \tag{3.13}$$

Therefore

$$A(x,Du_n) \to A(x,Du) \quad \text{in} \quad L^1_{\mathrm{loc}}(\mathbf{R}^N)\,. \tag{3.14}$$

(3.8) and (3.14) allow us to claim that u is a solution of (E2). Furthermore if $f \ge 0$ a.e. on has $u_n \ge 0$ a.e. for any $n \in \mathbf{N}^*$ and therefore $u \ge 0$ a.e. The proof of Theorem 1 is complete.

4. Other existence results and improved regularity

We consider equation (E2) under the conditions (A) and (G).

Proof of Theorem 2. Let $p > N$ and let u_n be as in the proof of Theorem 1, namely one has $u_n \in W_0^{1,p}(B_n)$ and

$$\int_{B_n} A(x, Du_n) \cdot Dv\, dx + \int_{B_n} g(x, u_n) v\, dx = \int_{B_n} f_n v\, dx, \tag{4.1}$$

holds for all $v \in W_0^{1,p}(B_n) \cap L^\infty(B_n)$. By Lemmas 2.1 and 2.2, we have for any $r > 0$ such that $n > 2r$,

$$\|u_n\|_{L^\infty(B_r)} \le C_1\,, \qquad \|Du_n\|_{L^p(B_r)} \le C_1\,.$$

Then, and up of an extraction subsequence, we can assume that for any $r > 0$

$$u_n \to u \text{ weakly in } W^{1,p}(B_r) \text{ and also a.e. in } \mathbf{R}^N. \tag{4.2}$$

Together with (G3) and the dominated convergence theorem, this implies that

$$g(x, u_n) \to g(x, u) \quad \text{in } L^1_{\mathrm{loc}}(\mathbf{R}^N)\,. \tag{4.3}$$

We prove a.e. convergence of Du_n to Du exactly as in Theorem 1 (Notice that $\frac{\gamma}{\lambda-1} \le p$, and that the L^p-bound on Du_n gives an L^1-bound on $|Du_n|^{p-1}$). Then we have $u \in W^{1,p}_{loc}(\mathbf{R}^N)$ and

$$A(x, Du_n) \to A(x, Du) \quad \text{in } L^1_{\mathrm{loc}}(\mathbf{R}^N)\,. \tag{4.4}$$

From (4.1), (4.3) and (4.4) we deduce that u is a solution of (E2). #

Proof of Theorem 3. We now assume that $1 < p \le p_0$ and $s(p-1) > 1$. We begin with the construction of the solutions u_n as in Theorems 1 and 2. By Lemmas 2.1 and 2.3 (notice that $s > p - 1$) we have for any $r > 0$, $1 < q < q_1$ and $n > 2r$, the estimate

$$\|Du_n\|_{L^q(B_r)} \le C_2,$$

and then, since $q < s$ and using the $L^1(B_r)$-bound on $|u_n|^s$,

$$\|u_n\|_{L^q(B_r)} \le C_3.$$

Then we can assume (up to extraction of a subsequence) that for every $r > 0$ and every $q \in (1, q_1)$

$$u_n \to u \qquad \text{weakly in } W^{1,q}(B_r) \text{ and } a.e. \text{ in } \mathbf{R}^N\,, \tag{4.5}$$

and

$$g(x, u_n) \to g(x, u) \qquad a.e. \text{ in } \mathbf{R}^N. \tag{4.6}$$

Furthermore, one has

$$f_n \to f \text{ in } L^1_{\text{loc}}(\mathbf{R}^N)\,. \tag{4.7}$$

We prove local equi-integrability of $g(x, u_n)$ as in the proof of Theorem 1 ; we have only to notice that $p-1 < 1$ and use the $L^1(B_{2r})$-bound on $|Du_n|$, the $L^1(B_{2r})$-bound on $|u_n|$, (4.7) and (G3). With (4.6), this gives

$$g(x, u_n) \to g(x, u) \text{ in } L^1_{\text{loc}}(\mathbf{R}^N)\,. \tag{4.8}$$

We prove a.e. convergence of Du_n to Du much as in Theorem 1. We use the $L^1(B_r)$-bound on $|Du_n|$, and the fact that $\gamma/(s-1) \leq 1$. We then use the fact that meas $(\{|u_n - u_m| \geq \varepsilon\} \cap B_r) \to 0$ as $n, m \to +\infty$ (this is due to the convergence of u_n to u in $L^1_{loc}(\mathbf{R}^N)$), and finally the $L^q(B_r)$-bound on $|Du_n|$ with $1 < q < q_1$, to deduce that $\displaystyle\int_{B_r} |Du_n - Du_m| \to 0$ as $n, m \to +\infty$. Then one has

$$A(x, Du_n) \to A(x, Du) \text{ in } L^1_{\text{loc}}(\mathbf{R}^N)\,, \tag{4.9}$$

as a consequence of a.e. convergence of Du_n to Du and the $L^q(B_r)$-bound on $A(x, Du_n)$ for some $q > 1$ and any $r > 0$.

We then conclude that u is a solution of (E2). #

We end the section with the question of improved regularity. We assume that $p_0 < p < N$ and $s > r_0$ and use the estimate of $|u|^s$ in L^1_{loc} to obtain better regularity for $|Du|$, as stated in Theorem 4.

Proof of Theorem 4. It is an easy consequence of Lemma 2.3 since $s > r_0$ and $p > p_0$ implies $s > 1/(p-1)$.

Let u_n be as in the proof of Theorem 1. Recalling Section 3 we know that (up to extraction of a subsequence) u_n converges to a solution u of (E2). Part iii) of Lemma 2.2 gives an $L^q(B_r)$-bound on Du_n for any $q \in [1, q_1)$ and any $r > 0$. We then conclude that $Du \in L^q_{loc}(B_r)$ for any $q \in [1, q_1)$ and $r > 0$. This ends the proof. #

5. Necessary growth condition on f if $s \leq p - 1$

In this section we consider for $f \in L^1_{\text{loc}}(\mathbf{R}^N)$, $f \geq 0$ the model equation

$$-\Delta_p u + |u|^{s-1} u = f \quad \text{in } \mathcal{D}'(\mathbf{R}^N) \tag{5.1}$$

and show that when $0 < s \leq p-1$ one needs some growth condition on f in order to obtain existence of a nonnegative solution for (5.1). For simplicity we consider the case where f and u are radially symmetric. In this sense the condition $s > p-1$ is optimal in Theorem 1. In particular, if f is a nonnegative radially symmetric function, the sequence $(u_n)_n$ defined in the proof of Theorem 1 is a nondecreasing sequence of radially symmetric nonnegative funtions (we recall that u_n is the weak solution of (5.1) with Dirichlet boundary condition $u_n = 0$). The eventual convergence of this sequence will provide a radially symmetric nonnegative solution of (5.1). The following proposition asserts the nonexistence of such a solution for some $f's$ with fast growth as $|x| \to \infty$ if $0 < s \leq p - 1$.

Proposition 5.1. *Let $p > 1$ and $0 < s \leq p-1$. Let f be a nonnegative and radially symmetric function belonging to $L^1_{\rm loc}(\mathbf{R}^N)$. Assume that (5.1) has a nonnegative radially symmetric solution u (in the sense defined in Section 1, that is $u \in W^{1,1}_{\rm loc}(\mathbf{R}^N), |Du|^{p-1} \in L^1_{\rm loc}(\mathbf{R}^N), u^s \in L^1_{\rm loc}(\mathbf{R}^N)$ and (5.1) is satisfied in the sense of distributions in $\mathbf{R}^N$). Then there exists C_1 (not depending on r) such that*

$$\frac{1}{r^N}\int_{B_r} f\,dx \leq C_1\, r^{\frac{ps}{p-s-1}},$$

for all $r \geq 1$ if $0 < s < p-1$, while for $s = p-1$

$$\frac{1}{r^N}\int_{B_r} f\,dx \leq e^{Cr}, \quad \textit{for all } r \geq 1\,.$$

Proof. Under the hypotheses of Proposition 5.1, we consider u and f as functions of $r = |\,x|$. One has $u \in W^{1,1}_{\rm loc}(0,\infty)$. Thus u is continuous on $(0,\infty)$, a.e. differentiable and one has $u(b) - u(a) = \int_a^b u'(r)dr$ for any $0 < a < b < \infty$. Equation (5.1) gives

$$-(|u'|^{p-2}u'r^{N-1})' + u^s r^{N-1} \quad \text{in } \mathcal{D}'(\prime,\infty).$$

(Notice that $|u'|^{p-2}u'\,r^{N-1} \in L^1_{\rm loc}(0,\infty)$, $u^s r^{N-1} \in L^1_{\rm loc}(0,\infty)$ and $f\,r^{N-1} \in L^1_{\rm loc}(0,\infty)$). Then one has

$$|u'|^{p-2}u'\,r^{N-1} \in W^{1,1}_{\rm loc}(0,\infty).$$

Thus $|u'|^{p-2}u'r^{N-1}$ is continuous in $(0,\infty)$ and a.e. differentiable and

$$-|u'(r)|^{p-2}u'(r)r^{N-1} + |u'(1)|^{p-2}u'(1) = \int_1^r (f(\sigma) - u^s(\sigma))\sigma^{N-1}d\sigma \tag{5.2}$$

for all $r \geq 1$. Setting $C_2 = |u'(1)|^{p-1}$, and using $f \geq 0$ this gives

$$u'(r) \leq \left(\frac{1}{r^{N-1}}(C_2 + \int_1^r u^s(\sigma)\sigma^{N-1}d\sigma)\right)^{1/p-1}. \tag{5.3}$$

for all $r \geq 1$. Integrating (5.3) between 1 and $r > 1$, we obtain

$$u(r) \leq u(1) + z(r) \quad \text{for all } r \geq 1 \tag{5.4}$$

with

$$z(r) = \int_1^r [\frac{1}{t^{N-1}}(C_2 + \int_1^t u^s(\sigma)\sigma^{N-1}d\sigma)]^{\frac{1}{p-1}}dt\,. \tag{5.5}$$

z is a nondecreasing, continuously differentiable function on $[1,\infty)$. One has $z(1) = 0$, and

$$z'(r) = \left(\frac{1}{r^{N-1}}(C_2 + \int_1^r u^s(\sigma)\sigma^{N-1}d\sigma)\right)^{\frac{1}{p-1}} \tag{5.6}$$

for all $r > 1$. By (5.6), z' is a.e. differentiable on $(1, \infty)$, and with (5.4), one has

$$(r^{N-1}(z'(r))^{p-1})' = u^s(r)r^{N-1} \leq (u_1 + z(r))^s r^{N-1}$$

for a.e. $r > 1$ (with $u_1 = u(1)$) . Multiplying by z' (notice that $z' \geq 0$) we deduce that

$$\frac{z'(r)}{r^{N-1}}(r^{N-1}(z'(r))^{p-1})' \leq (u_1 + z(r))^s z'(r)$$

for a.e. $r > 1$, i.e.

$$(p-1)z'(r)^{p-1}z''(r) + \frac{N-1}{r}(z'(r))^p \leq (z(r) + u_1)^s z'(r) \tag{5.7}$$

for a.e. $r > 1$. Since $\frac{N-1}{r}(z'(r))^p \geq 0$, we deduce from (5.7) that

$$\frac{p-1}{p}((z'(r))^p)' \leq \left(\frac{(z(r)+u_1)^{s+1}}{s+1}\right)'$$

for a.e. $r > 0$. The function $(z')^p$ belongs to $W^{1,1}(1, r)$ for any $r > 1$ (see (5.6)). We can integrate the preceding inequality, and obtain

$$\frac{p-1}{p}[z'(r)^p - z'(1)^p] \leq \frac{(z(r)+u_1)^{s+1}}{s+1} - \frac{u^{s+1}}{s+1}$$

for all $r \geq 1$. Therefore, there exists C_3 (depending on p, $z'(1), u_1, s$, but not on r), such that

$$(z'(r))^p \leq C_3(z(r))^{s+1} + C_3 \tag{5.8}$$

for all $r \geq 1$. We now distinguish the two cases: $s < p-1$ and $s = p-1$.

First Case: $s < p - 1$. From (5.8) we deduce

$$z(r) \leq C_4 r^{\frac{p}{p-s-1}} \tag{5.9}$$

for all $r \geq 1$. Thus by (5.4), if $r \geq 1$

$$u(r) \leq u_1 + z(r) \leq C_5 r^{\frac{p}{p-s-1}} . \tag{5.10}$$

We set

$$U(r) = \frac{1}{r^{N-1}} \int_1^r u^s(\sigma)\sigma^{N-1} d\sigma, \quad F(r) = \frac{1}{r^{N-1}} \int_1^r f(\sigma)\sigma^{N-1} d\sigma .$$

(5.2) gives for $r \geq 1$

$$0 \leq F(r) \leq U(r) + C_2 - |u'(r)|^{p-2}u'(r) ,$$

from which we get

$$F(r)^{\frac{1}{p-1}} \le (2^{\frac{1}{p-1}}+1)(U(r)+C_2)^{\frac{1}{p-1}} - 2^{\frac{1}{p-1}}u'(r)\,.$$

Integrating between 1 and r and recalling that $u(r) \ge 0$, we obtain

$$\int_1^r F(\sigma)^{\frac{1}{p-1}}d\sigma \le (2^{\frac{1}{p-1}}+1)\int_1^r (U(\sigma)+C_2)^{\frac{1}{p-1}}d\sigma + 2^{\frac{1}{p-1}}u_1$$

for all $r \ge 1$. With (5.10) and the definition of U, this gives

(5.11) $$\int_1^r (F(\sigma))^{\frac{1}{p-1}}d\sigma \le C_6 r^\delta$$

with $\delta = (\frac{ps}{p-s-1}+1)\frac{1}{p-1}+1$. From (5.11) we can deduce the conclusion of Proposition 5.1 (in the case $s < p-1$). Indeed (5.11) yields, for instance,

$$r^{-\frac{N-1}{p-1}}\int_1^r (\int_1^\sigma f(t)t^{N-1}dt)^{1/p-1}d\sigma \le C_6 r^\delta$$

for all $r \ge 1$, and then, since $\sigma \to \int_1^\sigma f(t)t^{N-1}dt$ is nondecreasing,

$$r^{-\frac{N-1}{p-1}}\frac{r}{2}(\int_1^{r/2} f(t)t^{N-1}dt)^{\frac{1}{p-1}} \le C_6 r^\delta$$

for all $r \ge 2$. Thus, we have

$$(\int_1^r f(t)t^{N-1}dt)^{\frac{1}{p-1}} \le C_7 r^{\delta-1+\frac{N-1}{p-1}}$$

i.e.

$$\int_1^r f(t)t^{N-1}dt \le C_8 r^{(\delta-1)(p-1)+N-1}$$

for all $r \ge 1$. Since $(\delta-1)(p-1)+N-1 = \frac{ps}{p-s-1}+N$, we obtain Proposition 5.1 when $s < p-1$.

Second case: $s = p-1$. In this case (5.8) gives

(5.12) $$z(r) \le e^{C_9 r}\,, \quad \text{for all } r \ge 1\,.$$

Using the same method as for the case $s < p-1$, we easily deduce that $u(r) \le e^{C_{10}r}$, and then

$$\int_1^r f(t)t^{N-1}dt \le e^{C_{11}r}\,, \quad \text{for all } r \ge 1.$$

This gives Proposition 5.1 in the case $s = p-1$. #

In the preceding proposition we have proved the nonexistence of a nonnegative, radially symmetric solution u of (5.1), when f is a nonnegative radially symmetric funtion, "too rapidly" increasing as $r \to \infty$. The hypothesis of nonnegativity of u is essential for this result. Indeed, it is well known that, for instance in the case $p = 2$, $s = p - 1 = 1$, the equation $-\Delta u + u = f$ in $\mathcal{D}'(\mathbf{R}^N)$ has infinitely many solutions for any $f \in L^1_{\text{loc}}(\mathbf{R}^N)$. Our proposition shows that these solutions have changing sign when $f \geq 0$ a.e. and f is "too rapidly" increasing as $r \to \infty$.

The hypothesis of radial symmetry of u can be eliminated when u is regular. In fact, if $p > p_0 = 2 - \frac{1}{N}$ and we assume some additional regularity on u (which happens for instance if f is regular) we prove below that the existence of a nonnegative solution u of (5.1) (with $f \geq 0$ radially symmetric) implies the existence of a nonnegative radially symmetric solution of (5.1). If moreover $s \leq p - 1$, we can then apply Proposition 5.1 to deduce that a growth restriction on f is necessary for existence of any nonnegative solution. Our result is

Proposition 5.2. *Let $p > p_0$, $s > 0$ and let f be a nonnegative radially symmetric function belonging to $L^1_{\text{loc}}(\mathbf{R}^N)$. Assume that* (5.1) *has a nonnegative solution $u \in W^{1,p}_{\text{loc}}(\mathbf{R}^N)$ with $u^s \in L^1_{\text{loc}}(\mathbf{R}^N)$. Then* (5.1) *has a nonnegative radially symmetric solution* (*in the sense defined in Section* 1).

Proof. One has $f \geq 0$, f radially symmetric, $f \in L^1_{\text{loc}}(\mathbf{R}^N)$. Let $u \in W^{1,p}_{\text{loc}}(\mathbf{R}^N)$, $u \geq 0$ a.e., such that $u^s \in L^1_{\text{loc}}(\mathbf{R}^N)$, and

$$-\text{div}\,(|Du|^{p-2}Du) + u^s = f \quad \text{ in } \mathcal{D}'(\mathbf{R}^N)\,.$$

Then for any $n \in \mathbf{N}^*$ one has (using a density argument)

$$\int_{B_n} |Du|^{p-2}Du.D\phi\, dx + \int_{B_n} u^s \phi dx = \int_{B_n} f\phi dx \tag{5.13}$$

for all $\phi \in W^{1,p}_{\circ}(B_n) \cap L^\infty(B_n)$. Let $f_n = \inf(|f|, n)\text{sign}\,(f)$ and let $u_n \in W^{1,p}_0(B_n)$ be the (unique) solution of

$$-\text{div}\,(|Du_n|^{p-2}Du_n) + u_n^s = f_n \quad \text{ in } \mathcal{D}'(B_n). \tag{5.14}$$

We recall that existence and uniqueness are well known (see, for instance, [W], [BB]), and one also has:

$$u_n \geq 0 \quad \text{a.e. in } B_n \tag{5.15}$$

$$u_n^s \in L^1_{\text{loc}}(\mathbf{R}^N) \tag{5.16}$$

$$u_n \text{ radially symmetric} \tag{5.17}$$

$$\int_{B_n} |Du_n|^{p-2}Du_n D\phi dx + \int_{B_n} u_n^s \phi dx = \int_{B_n} f_n\phi dx, \tag{5.18}$$

for all $\phi \in W_0^{1,p}(B_n) \cap L^\infty(B_n)$. Let $\psi \in C^1(\mathbf{R}, \mathbf{R})$, such that $\psi(s) = 0$ if $s \leq 0, \psi'(s) > 0$ if $s > 0$ and ψ and ψ' are bounded. By classical results one has $\psi(u_n - u) \in W_0^{1,p}(B_n)$ (in particular, since $u \geq 0$ and $u_n = 0$ on ∂B_n), and $D\psi(u_n - u) = \psi'(u_n - u)D(u_n - u)$.

Choosing $\phi = \psi(u_n - u)$ in (5.13) and (5.18), we then obtain (notice that $f_n \leq f$)

$$\int_{B_n} (|Du_n|^{p-2}Du_n - |Du|^{p-2}Du)\psi'(u_n - u)(Du_n - Du)dx + \int_{B_n} (u_n^s - u^s)\psi(u_n - u)dx \leq 0 \tag{5.19}$$

Then, we necessarily have

$$u_n \leq u \quad \text{a.e. in } B_n\,. \tag{5.20}$$

By a similar argument we also have

$$u_n \leq u_{n+1} \quad \text{a.e. in } B_n\,. \tag{5.21}$$

With (5.20) and (5.21) we deduce that

$$u_n \to v \quad \text{a.e. in } \mathbf{R}^N \text{ and in } L^p_{\text{loc}}(\mathbf{R}^N)\,. \tag{5.22}$$

Since $0 \leq u_n^s \leq u^s$, we also have (by the Dominated Convergence Theorem)

$$u_n^s \to v^s \quad \text{in } L^1_{\text{loc}}(\mathbf{R}^N)\,. \tag{5.23}$$

Furthermore, one has $v > 0$ a.e. and v is radially symmetric, like u_n. It remains to prove that $v \in W^{1,1}_{\text{loc}}(\mathbf{R}^N), |Dv|^{p-1} \in L^1_{\text{loc}}(\mathbf{R}^N)$ and v is a solution of (5.1). (We have to pass to the limit in (5.18)). Firstly, we can use the $L^p_{\text{loc}}(\mathbf{R}^N)$-bound on u_n (which is due to (5.20) and the fact $u \in L^p_{\text{loc}}(\mathbf{R}^N)$) in order to obtain local estimates on $|Du_n|$ by a way similar to that of Lemma 2.1 and Lemma 2.2 of Section 2.

Indeed, let $\phi : \mathbf{R} \to \mathbf{R}$ be as in Lemma 2.1 (see (2.5)) for some $0 < m < 1/(p-1)$. Let $r > 0$ and θ be the cutoff function of Lemma 2.1 and let $\gamma \geq p$. If for $n > 2r$ we replace in (5.18) ϕ by $\phi(u_n)\,\theta^\gamma$, we obtain

$$\int |Du_n|^p \phi'(u_n)\theta^\gamma dx + \int u_n^s \phi(u_n)\theta^\gamma dx \leq C_1 + C_2 \int |Du_n|^{p-1}\theta^{\gamma-1}dx$$

(the C_i's do not depend on n). As in Lemma 2.1 we then deduce that

$$\frac{1}{2}\int |Du_n|^p \frac{\theta^\gamma}{(u_n+1)^{m+1}}dx + \int u_n^s \phi(u_n)\theta^\gamma dx \leq C_1 + C_3 \int \theta^{\gamma-p}(u_n+1)^{(m+1)(p-1)}dx$$

since $(m+1)(p-1) \leq p$, $\gamma \geq p$, we use the $L^p(B_{2r})$-estimate on u_n to deduce

$$\int_{B_r} \frac{|Du_n|}{(u_n+1)^{m+1}}dx \leq C_4 \tag{5.24}$$

Thus, by Lemma 2.2, we deduce if $p_0 < p \le N$, for any $1 \le q < q_0$,

$$\int_{B_r} |Du_n|^q dx \le C_5 \,, \tag{5.25}$$

while for $p > N$

$$\int_{B_r} |Du_n|^p dx \le C_5. \tag{5.26}$$

Therefore, we have

$$u_n \to v \quad \text{weakly in } W^{1,q}(B_r) \tag{5.27}$$

for any $r > 0$ with any $q \in [1, q_0)$ if $p_0 < p \le N$, and $q = p$ if $p > N$. Then we prove the a.e. convergence of Dv as in Theorem 1 and 2. This gives

$$|Du_n|^{p-2}Du_n \to |Dv|^{p-2}Dv \text{ in } L^1_{\text{loc}}(\mathbf{R}^N)\,. \tag{5.28}$$

With (5.28) and (5.23) we can pass to the limit in (5.18) and obtain

$$-\text{div}\,(|Dv|^{p-2}Dv) + v^s = f \text{ in } \mathcal{D}'(\mathbf{R}^N)\,.$$

This proves Proposition 5.2. #

6. Problem on a Bounded Domain

Let Ω be a bounded open set of $\mathbf{R}^N$, $N \ge 1$. We are interested in the following problem

$$-\text{div}\,(A(x, Du)) + g(x, u) = f \text{ in } \Omega \tag{6.1}$$

$$u = 0 \text{ on } \partial\Omega\,, \tag{6.2}$$

where $f \in L^1(\Omega)$. The map A verifies the set of hypotheses (A1')-(A4') obtained from (A1)-(A4) of Section 1 by replacing $x \in \mathbf{R}^N$ by $x \in \Omega$ and the spaces $L^r_{\text{loc}}(\mathbf{R}^N)$, $1 \le r \le \infty$ by $L^r(\Omega)$. The function g verifies the set of hypotheses (G1')-(G3') obtained from (G1)-(G3) in the same way (in (G3') we assume that $G_t \in L^1(\Omega)$). We recall that the model example is $-\Delta_p u + |u|^{s-1}u = f$.

Existence of solutions for such a problem is proved in [BG]:

Theorem ([BG]). *Let A and g be as above, and let $p_0 = 2 - (1/N) < p \le N$. Then for any $f \in L^1(\Omega)$ there exists u, solution of* (6-1)-(6.2) *in the following sense: $u \in W^{1,q}_0(\Omega)$ for any $1 \le q < q_0 = (p-1)\frac{N}{N-1}$, $g(x, u) \in L^1(\Omega)$ and*

$$\int_\Omega A(x, Du) \cdot Dv\, dx + \int_\Omega g(x, u) v dx = \int_\Omega f v\, dx, \tag{6.3}$$

for all $v \in W^{1,r}_0(\Omega)$ for some $r > N$.

Remarks. 1) In this existence result we only need a weaker condition than (G2'), namely (G'2) $g(x, \sigma)\sigma \ge 0$, for any $\sigma \in \mathbf{R}$ and a.e. $x \in \Omega$.

2) A similar existence result is true in the case $p > N$; then $u \in W_0^{1,p}(\Omega)$ (see [W], [BB]). #

Here we want to improve on this existence theory by extending to problem (6.1)-(6.2) two results proved precedently for equation (E2) in $\mathbf{R}^N$, namely the improved regularity result when $p_0 < p < N$ and $s > r_0$ (Theorem 4), and the existence for $1 < p \leq p_0$ when $s(p-1) > 1$ (Theorem 3). Though no essential differences appear, we will explain it in some detail for the reader's convenience. Let us begin with the question of regularity improvement.

Theorem 5. *Let A and g satisfy hypotheses* (A1')-(A4') *and* (G1')-(G3') *respectively, and let $p_0 < p < N$ and $s > r_0 = N(p-1)/(N-p)$. Then for any $f \in L^1(\Omega)$ there exists u satisfying* (6.3) *and such that $u \in W_0^{1,q}(\Omega)$ for any $1 \leq q < q_1 = ps/(s+1)$.*

We recall that $q_1 > q_0$ since $s > r_0$, see the Introduction.

Proof. The proof of the existence Theorem of [BG] recalled above relies on estimates for some approximate solutions of (6.1)-(6.2). Passing to the limit in these estimates gives the existence of a solution u of (6.3) with the additional properties (see [BG]).

$$\int_\Omega |g(x,u)|dx \leq C_1 \tag{6.4}$$

$$\int_{B_n} |Du|^p dx \leq C_2, \quad \text{for all } n \in \mathbf{N}, \tag{6.5}$$

where C_2 does not depend on n and $B_n = \{x \in \Omega,\ n \leq |u(x)| \leq n+1\}$. From (6.5) we deduce, for any $m > 0$,

$$\begin{aligned}\int_\Omega \frac{|Du|^p}{(|u|+1)^{m+1}}dx &= \sum_{n=0}^{\infty} \int_{B_n} \frac{|Du|^p}{(|u|+1)^{m+1}}dx \leq \\ &\leq C_2 \sum_{n=0}^{\infty} \frac{1}{(n+1)^{m+1}} = C_3\,.\end{aligned} \tag{6.6}$$

(C_3 depends on m). From (6.4) and (G2') we deduce that

$$\int_\Omega |u|^s dx \leq C_4. \tag{6.7}$$

We have for any $q < p$

$$|Du|^q \leq \frac{|Du|^p}{(|u|+1)^{m+1}} + (|u|+1)^{\frac{m+1}{p-q}\cdot q}.$$

If $q < \frac{ps}{s+1}$ we can choose $m > 0$ such that $\frac{m+1}{p-q}q \leq s$; we then obtain

$$|Du|^q \leq \frac{|Du|^p}{(|u|+1)^{m+1}} + (|u|+1)^s\,. \tag{6.8}$$

From (6.6), (6.7) and (6.8) we deduce $|Du|^q \in L^1(\Omega)$ for any $1 \le q < q_1$. This proves Theorem 5. #

As in Section 4 we can use certain growth assumptions on the nonlinear term $g(x,u)$ in order to obtain existence results in the case $1 < p < p_0$. Indeed we obtain the following theorem.

Theorem 6. *Let A and g be as above and let $1 < p \le p_0$ and $s(p-1) > 1$. Then for any $f \in L^1(\Omega)$ there exists u, solution of (6.1)-(6.2) in the following sense : $u \in W_0^{1,q}(\Omega)$ for any $1 \le q < q_1$, $g(x,u) \in L^1(\Omega)$ and*

$$-div\,(A(x,Du)) + g(x,u) = f \quad in\ \mathcal{D}'(\Omega). \tag{6.9}$$

Notice that, since $s > p-1$ we have $q_1 > p-1$ and then $A(x,Du)$ is well defined in $L^1(\Omega)$).

Proof. Let $f_n = \inf(|f|, n)\mathrm{sign}\,(f)$. Since $f_n \in W^{-1,p'}(\Omega)$ it is known (see, for instance, [W], [BB]), that there exists u_n such that $u_n \in W_0^{1,p}(\Omega)$, $g(x,u_n) \in L^1(\Omega)$ and

$$\int_\Omega A(x,Du_n)Dv dx + \int_\Omega g(x,u_n)v dx = \int_\Omega f v dx, \tag{6.10}$$

for all $v \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$. Taking in (6.10) $v = \phi_\varepsilon(u_n)$ with $\phi_\varepsilon(\sigma) = \inf(|\sigma|/\varepsilon, 1)\,\mathrm{sign}\,(\sigma)$, we obtain when $\varepsilon \to 0$,

$$\int_\Omega |g(x,u_n)| dx \le \int_\Omega |f| dx = C_1\,. \tag{6.11}$$

Then by (G2') we also have

$$\int_\Omega |u_n|^s dx \le C_1\,. \tag{6.12}$$

Let $k \in \mathbf{N}$. Taking in (6.10) $v = \phi(u_n)$ with

$$\begin{cases} \phi(\sigma) = \inf((\sigma - k)^+, 1) & \text{if} \quad \sigma \ge 0 \\ \phi(\sigma) = -\phi(\sigma) & \text{if} \quad \sigma < 0 \end{cases}$$

we obtain

$$\int_{B_k} |Du_n|^p dx \le \frac{1}{C}\int_\Omega |f| dx = C_2 \tag{6.13}$$

with $B_k = \{x \in \Omega, k \le u_n(x) \le k+1\}$. From (6.13), we deduce for any $m > 0$ (as in Theorem 5)

$$\begin{aligned} &\int_\Omega \frac{|Du_n|^p}{(|u_n|+1)^{m+1}} dx = \sum_{k=0}^{\infty} \int_{B_k} \frac{|Du_n|^p}{(|u_n|+1)^{m+1}} dx \le \\ &\le C_2 \sum_{k=0}^{\beta} \frac{1}{(k+1)^{m+1}} = C_3 \end{aligned} \tag{6.14}$$

(with C_3 depending on m, not on n). Let $1 \le q < q_1$. As in Lemma 2.3 we take $m \in (0, m_1)$ where $m_1 = s(p-q)/q - 1$ ($m_1 > 0$ thanks to $q > q_1$).

$$|Du_n|^q \le \frac{|Du_n|^p}{(|u_N|+1)^{m+1}} + (|u_n|+1)^{\frac{m+1}{p-q}q} \le \frac{|Du_n|^p}{(|u_n|+1)^{m+1}} + (|u_n|+1)^s$$

with (6.12) and (6.14) this gives

$$\int_\Omega |Du_n|^q dx \le C_4 \tag{6.15}$$

Therefore (since $u = 0$ on $\partial\Omega$)

$$\|u_n\|_{W_0^{1,q}(\Omega)} \le C_5 \tag{6.16}$$

(C_5 depends on q, $1 \le q < q_1$, not on n). Then we can assume (up to extraction of a subsequence)

$$u_n \to u \tag{6.17}$$

a.e. in $W_0^{1,q}(\Omega)$ for any $1 \le q < q_1$. Also,

$$g(x, u_n) \to g(x, u) \qquad a.e. \tag{6.18}$$

Using techniques similar to those of the proof of Theorem 1 (see also [BG]) we prove

$$g(x, u_n) \to g(x, u) \text{ in } L^1_{\mathrm{loc}}(\Omega), \tag{6.19}$$

and with (6.11), (6.18) and Fatou's Lemma,

$$g(x, u) \in L^1(\Omega)\,. \tag{6.20}$$

Using (A4') we also prove the a.e. convergence of Du_n to Du. Indeed one has for $\varepsilon > 0$, for any $n, m \in \mathbf{N}$,

$$\int_{\{|u_n - u_m| \le \varepsilon\}} (A(x, Du_n) - A(x, Du_m))(Du_n - Du_m) dx \le \varepsilon \int_\Omega (|h_n| + |h_m|) dx \le \varepsilon C_6$$

with $h_n = f_n - g(x, u_n)$. With (A4') this gives

$$\int_{\{|u_n - u_m| \le \varepsilon\}} \frac{|Du_n - Du_m|^s}{\beta(x, Du_n, Du_m)} dx \le \varepsilon C_6.$$

Thus we have,

$$\int_\Omega |Du_n - Du_m| dx \le \varepsilon^{1/s} C_7 \Big(\int_\Omega \beta(x, Du_n, Du_m)^{\frac{1}{s-1}} dx\Big)^{1-\frac{1}{s}} + \int_{\{|u_n - u_m| > \varepsilon\}} |Du_n - Du_m| dx.$$

Using $\frac{\gamma}{s-1} \le 1$, the L^q-bound on $|Du_n|$ for some $1 < q < q_1$ and the fact that meas $\{|u_n - u_m| \ge \varepsilon\} \to 0$ as $n, m \to \infty$ (this is due to the $L^1(\Omega)$ convergence of u_n to u), we conclude that

$$\int_\Omega |Du_n - Du_m| dx \to 0 \text{ as } n, m \to \infty. \tag{6.21}$$

This proves (up to extraction of a subsequence) that

$$Du_n \to Du \qquad a.e.$$

Thus $Du_n \to Du$ in $L^q(\Omega)$ for any $1 \le q < q_1$, and therefore (since $p - 1 < 1$),

$$A(x, Du_n) \to A(x, Du) \text{ in } L^1(\Omega)\,. \tag{6.22}$$

(6.22), (6.19) and $f_n \to f$ in $L^1(\Omega)$ imply that u satisfies (6.9). This completes the proof. #

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