Two-phase flows involving capillary barriers in heterogeneous porous media.

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Abstract

We consider a simplified model of a two-phase flow through a heterogeneous porous medium, in which the convection is neglected. This leads to a nonlinear degenerate parabolic problem in a domain shared in an arbitrary finite number of homogeneous porous media. We introduce a new way to connect capillary pressures on the interfaces between the homogeneous domains, which leads to a general notion of solution. We then compare this notion of solution with an existing one, showing that it allows to deal with a larger class of problems. We prove the existence of such a solution in a general case, then we prove the existence and the uniqueness of a regular solution in the one-dimensional case for regular enough initial data.

Keywords. flows in porous media, capillarity, nonlinear PDE of parabolic type.

1 Presentation of the problem

The models of immiscible two-phase flows are widely used in petroleum engineering, particularly in basin modeling, whose aim can be the prediction of the migration of hydrocarbon components at geological time scale in a sedimentary basin.

The heterogeneousness of the porous medium leads to the phenomena of oil-trapping and oil-expulsion, which is modeled with discontinuous capillary pressures between the different geological layers.

The physical principles models and the mathematical models can be found in [4, 5, 12, 27, 13]. The phenomenon of capillary trapping has been completed only in simplified cases (see [7]), and several numerical methods have been developed (see e.g. [15, 14]).

The aim of this paper is to introduce a new notion of weak solution, which allows us to deal with more general cases than those treated in [14], while it is equivalent to the notion of weak solution introduced in [14] on the already treated cases. We will consider a simplified model $(\mathcal{P})$ defined page 5, in which the convection is neglected,

We then give a uniqueness result in the one dimensional case which is inspired from the result in [7] and extends this latter one to more general situations, by requiring weaker assumptions on the solutions and applying to a larger class of initial data.

We have to make some assumptions on the heterogeneous porous medium:

Assumptions 1 (Geometrical assumptions)

1. *The heterogeneous porous medium is represented by a polygonal bounded connected domain $\Omega \subset \mathbb{R}^d$ with $meas_{\mathbb{R}^d}(\Omega) > 0$, where $meas_{\mathbb{R}^n}$ is the Lebesgue's measure of $\mathbb{R}^n$.*
2. *There exists a finite number N of polygonal connected subdomains $(\Omega_i)_{1\le i\le N}$ of Ω such that:*
   - *(a) for all $i \in [\![1, \mathrm{N}]\!]$, $meas_{\mathbb{R}^d}(\Omega_i) > 0$,*
   - *(b) $\displaystyle\bigcup_{i=1}^{N} \overline{\Omega}_i = \overline{\Omega}$,*
   - *(c) for $(i, j) \in [\![1, \mathrm{N}]\!]^2$ with $i \neq j$, $\Omega_i \cap \Omega_j = \emptyset$.*

   Each Ω_i represents an homogeneous porous medium. One denotes, for all $(i, j) \in [\![1, \mathrm{N}]\!]^2$, $\Gamma_{i,j} \subset \Omega$ the interface between the geological layers Ω_i and Ω_j, defined by $\overline{\Gamma}_{ij} = \partial\Omega_i \cap \partial\Omega_j$.

We consider an incompressible and immiscible oil-water flow through Ω, and thus through each Ω_i. Using Darcy's law, the conservation of oil and water phases is given for all $(x, t) \in \Omega_i \times (0, T)$,

$$\left\{\begin{array}{l} \phi_i \partial_t u_i(x,t) - \nabla\cdot\big(\eta_{o,i}(u_i(x,t))(\nabla p_{o,i}(x,t) - \rho_o \mathbf{g})\big) = 0, \\ -\phi_i \partial_t u_i(x,t) - \nabla\cdot\big(\eta_{w,i}(u_i(x,t))(\nabla p_{w,i}(x,t) - \rho_w \mathbf{g})\big) = 0, \\ p_{o,i}(x,t) - p_{w,i}(x,t) = \pi_i(u_i(x,t)), \end{array}\right. \tag{1}$$

where $u_i \in [0, 1]$ is the oil saturation in Ω_i (and therefore $1 - u_i$ the water saturation), $\phi_i \in]0, 1[$ is the porosity of Ω_i, which is supposed to be constant in each Ω_i for the sake of simplicity, $\pi_i(u_i(x, t))$ is the capillary pressure, and $\mathbf{g}$ is the gravity acceleration. The indices o and w respectively stand for the oil and the water phase. Thus, for $\sigma = o, w$, $p_{\sigma,i}$ is the pressure of the phase σ, $\eta_{\sigma,i}$ is the mobility of the phase σ, and ρ_σ is the density of the phase σ.

We have now to make assumptions on the data to explicit the transmission conditions through the interfaces $\Gamma_{i,j}$:

Assumptions 2 (Assumptions on the data)

1. *for all $i \in [\![1, \mathrm{N}]\!]$, $\pi_i \in C^1([0, 1], \mathbb{R})$, with $\pi_i'(x) > 0$ for $x \in]0, 1[$,*

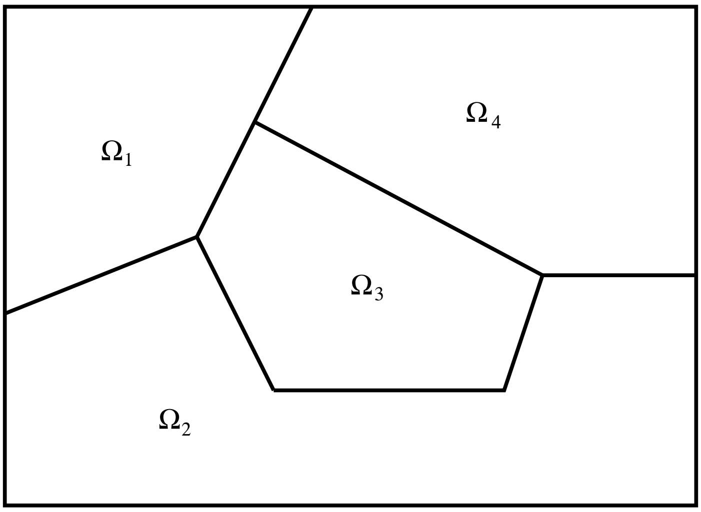

Figure 1: An example for the domain Ω

2. *for all* $i \in [\![1, \mathrm{N}]\!]$, $\eta_{o,i} \in C^0([0,1], \mathbb{R}_+)$ *is an increasing function fulfilling* $\eta_{o,i}(0) = 0$,
3. *for all* $i \in [\![1, \mathrm{N}]\!]$, $\eta_{w,i} \in C^0([0,1], \mathbb{R}_+)$ *is a decreasing function fulfilling* $\eta_{w,i}(1) = 0$,
4. *the initial data* u_0 *belongs to* $L^\infty(\Omega)$, $0 \le u_0 \le 1$.

One denotes $\alpha_i = \lim_{s\to 0} \pi_i(s)$ and $\beta_i = \lim_{s\to 1} \pi_i(s)$. We can now define the monotonous graphs $\tilde{\pi}_i$ by:

$$\tilde{\pi}_i(s) = \begin{cases} \pi_i(s) & \text{if } s \in]0,1[, \\ ]-\infty, \alpha_i] & \text{if } s = 0, \\ [\beta_i, +\infty[& \text{if } s = 1. \end{cases} \tag{2}$$

As it is exposed in [14], the following conditions must be satisfied on the traces of u_i, $p_{\sigma,i}$ and $\nabla p_{\sigma,i}$ on $\Gamma_{i,j} \times (0,T)$, still denoted respectively u_i, $p_{\sigma,i}$ and $\nabla p_{\sigma,i}$ (see [5]):

1. for any $\sigma = o, w$, $(i,j) \in [\![1, \mathrm{N}]\!]^2$ such that $\Gamma_{i,j} \neq \emptyset$, the flux of the phase σ through $\Gamma_{i,j}$ must be continuous:

$$\eta_{\sigma,i}(u_i)(\nabla p_{\sigma,i} - \rho_\sigma \mathbf{g}) \cdot \mathbf{n}_i + \eta_{\sigma,j}(u_j)(\nabla p_{\sigma,j} - \rho_\sigma \mathbf{g}) \cdot \mathbf{n}_j = 0, \tag{3}$$

   where $\mathbf{n}_i$ denotes the outward normal of $\Gamma_{i,j}$ to Ω_i;

2. for any $\sigma = o, w$, $(i,j) \in [\![1, \mathrm{N}]\!]^2$ such that $\Gamma_{i,j} \neq \emptyset$, either p_σ is continuous or $\eta_\sigma = 0$. Since the saturation is itself discontinuous across $\Gamma_{i,j}$, one must express the mobility at the upstream side of the interface. This gives

$$\eta_{\sigma,i}(u_i)(p_{\sigma,i} - p_{\sigma,j})^+ - \eta_{\sigma,j}(u_j)(p_{\sigma,j} - p_{\sigma,i})^+ = 0. \tag{4}$$

The conditions (4) have direct consequences on the behaviour of the capillary pressures on both side of $\Gamma_{i,j}$. Indeed, if $0 < u_i, u_j < 1$, then the partial pressures p_o and p_w have both to be continuous, and so we have the connection of the capillary pressures $\pi_i(u_i) = \pi_j(u_j)$. If $u_i = 0$ and $0 < u_j < 1$, then $p_{o,i} \ge p_{o,j}$ and $p_{w,i} = p_{w,j}$, thus

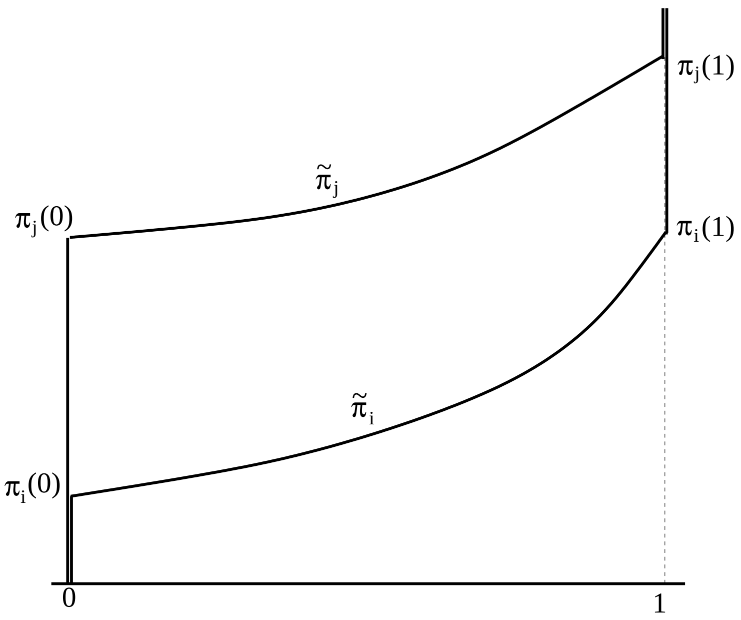

Figure 2: Graphs for the capillary pressures

$\pi_j(u_j) \leq \pi_i(0)$. The same way, $u_i = 1$ and $0 < u_j < 1$ implies $\pi_j(u_j) \geq \pi_i(1)$. If $u_i = 0$, $u_j = 1$, then $p_{o,i} \geq p_{o,j}$ and $p_{w,i} \leq p_{w,j}$, so $\pi_i(0) \geq \pi_j(1)$. Checking that the definition of the graphs $\tilde{\pi}_i$ and $\tilde{\pi}_j$ implies $\tilde{\pi}_i(0) \cap \tilde{\pi}_j(0) \neq \emptyset$, $\tilde{\pi}_i(1) \cap \tilde{\pi}_j(1) \neq \emptyset$, we can claim that (4) leads to:

$$\tilde{\pi}_i(u_i) \cap \tilde{\pi}_j(u_j) \neq \emptyset. \tag{5}$$

We introduce the global pressure in Ω_i

$$\overline{p}_i(x,t) = p_{w,i}(x,t) + \int_0^{u_i(x,t)} \frac{\eta_{o,i}(a)}{\eta_{o,i}(a) + \eta_{w,i}(a)} \pi_i'(a)\mathrm{d}a \tag{6}$$

(see e.g. [3] or [12]), and the global mobility in Ω_i

$$\lambda_i(u_i(x,t)) = \frac{\eta_{o,i}(u_i(x,t))\eta_{w,i}(u_i(x,t))}{\eta_{o,i}(u_i(x,t)) + \eta_{w,i}(u_i(x,t))} \tag{7}$$

which verifies $\lambda_i(0) = \lambda_i(1) = 0$, and $\lambda_i(s) > 0$ for $0 < s < 1$. Taking into account (6) and (7) in (1), and adding the conservation laws leads to, for $(x,t) \in \Omega_i \times (0,T)$:

$$\left\{ \begin{array}{l} \phi_i \partial_t u_i(x,t) - \nabla \cdot \big(\eta_{o,i}(u_i(x,t))(\nabla \overline{p}_i(x,t) - \rho_o \mathbf{g}) - \lambda_i(u_i(x,t)) \nabla \pi_i(u_i(x,t))\big) = 0, \\ -\nabla \cdot \left(\displaystyle\sum_{\sigma = o,w} \eta_{\sigma,i}(u_i(x,t))(\nabla \overline{p}_i(x,t) - \rho_\sigma \mathbf{g}) \right) = 0. \end{array} \right. \tag{8}$$

We neglect the convective effects, so that we focus on the mathematical modeling of flows with discontinuous capillary pressures, which seem to necessary to explain the phenomena of oil trapping. This simplification will allow us to neglect the coupling with the second equation of (8), and we get the simple degenerated parabolic equation in $\Omega_i \times (0,T)$:

$$\phi_i \partial_t u_i(x,t) - \nabla\cdot(\lambda_i(u_i(x,t))\nabla\pi_i(u_i(x,t))) = 0 \quad \text{in } \Omega_i \times (0,T). \tag{9}$$

In this simplified framework, the transmission condition (3) on the fluxes through $\Gamma_{i,j}$ can be rewritten:

$$\lambda_i(u_i(x,t))\nabla(\pi_i(u_i(x,t)))\cdot\mathbf{n}_i + \lambda_j(u_j(x,t))\nabla(\pi_j(u_j(x,t)))\cdot\mathbf{n}_j = 0 \quad \text{on } \Gamma_{i,j}\times(0,T). \tag{10}$$

We suppose furthermore that $u_i(x,0) = u_0(x)$ for $x \in \Omega_i$. In the remainder of this paper, we suppose to take a homogeneous Neumann boundary condition, The existence of a weak solution proven in section 3 can be extended to the case of non-homogeneous Dirichlet conditions. Nevertheless, homogeneous Neumann boundary conditions are needed to prove the theorem 4.1, and thus to prove the conclusion theorem 5.4

Taking into account the equations (5), (9), (10), the boundary condition, and the initial condition, we can write the problem we aim to solve this way: for all $i \in [\![1,\mathrm{N}]\!]$, for all $j \in [\![1,\mathrm{N}]\!]$ such that $\Gamma_{i,j} \neq \emptyset$,

$$\left\{\begin{array}{ll} \phi_i\partial_t u_i - \nabla\cdot(\lambda_i(u_i)\nabla\pi_i(u_i)) = 0 & \text{in } \Omega_i\times(0,T),\\ \tilde\pi_i(u_i)\cap\tilde\pi_j(u_j)\neq\emptyset & \text{on } \Gamma_{i,j}\times(0,T),\\ \lambda_i(u_i)\nabla(\pi_i(u_i))\cdot\mathbf{n}_i + \lambda_j(u_j)\nabla(\pi_j(u_j))\cdot\mathbf{n}_j = 0 & \text{on } \Gamma_{i,j}\times(0,T),\\ \lambda_i(u_i)\nabla(\pi_i(u_i))\cdot\mathbf{n}_i = 0 & \text{on } \partial\Omega_i\cap\partial\Omega\times(0,T),\\ u_i(\cdot,0) = u_0(x) & \text{in } \Omega_i. \end{array}\right. \tag{$\mathcal{P}$}$$

Remark 1.1 *All the results presented in this paper still hold if one not neglects the effect of the gravity and if one assumes that the global pressure is known, that is for problems of the type :*

$$\left\{\begin{array}{ll} \phi_i\partial_t u_i + \nabla\cdot(\mathbf{q}f_i(u_i) + \lambda_i(u_i)(\rho_o-\rho_w)\mathbf{g} - \lambda_i(u_i)\nabla\pi_i(u_i)) = 0 & \textit{in } \Omega_i\times(0,T),\\ \tilde\pi_i(u_i)\cap\tilde\pi_j(u_j)\neq\emptyset & \textit{on } \Gamma_{i,j}\times(0,T),\\ \displaystyle\sum_{k=i,j}(\mathbf{q}f_k(u_k) + \lambda_k(u_k)(\rho_o-\rho_w)\mathbf{g} - \lambda_k(u_k)\nabla\pi_k(u_k))\cdot\mathbf{n}_k = 0 & \textit{on } \Gamma_{i,j}\times(0,T),\\ (\mathbf{q}f_i(u_i) + \lambda_i(u_i)(\rho_o-\rho_w)\mathbf{g} - \lambda_i(u_i)\nabla\pi_i(u_i))\cdot\mathbf{n}_i = 0 & \textit{on } \partial\Omega_i\cap\partial\Omega\times(0,T),\\ u_i(\cdot,0) = u_0(x) & \textit{in } \Omega_i, \end{array}\right.$$

where f_i is supposed to be a $C^1([0,1],\mathbb{R})$-increasing function, λ_i is also supposed to belong to $C^1([0,1],\mathbb{R}_+)$ and $\mathbf{q}$ satisfies

- $\forall i$, $\mathbf{q}\in\left(C^1(\overline{\Omega}_i\times[0,T])\right)^d$,
- $\nabla\cdot\mathbf{q} = 0$ *in* $\Omega_i\times(0,T)$,
- $\mathbf{q}_{|\Omega_i}\cdot\mathbf{n}_i + \mathbf{q}_{|\Omega_j}\cdot\mathbf{n}_j = 0$ *on* $\Gamma_{i,j}\times(0,T)$,
- $\mathbf{q}\cdot\mathbf{n} = 0$.

In order to ensure the uniqueness result stated in theorem 5.1, the technical condition (see [2] or [26]):

$$\forall i, \qquad f_i\circ\varphi_i^{-1}, \lambda_i\circ\varphi_i^{-1} \in C^{0,1/2}([0,\varphi_i(1)],\mathbb{R}).$$

Remark 1.2 *In the modeling of two-phase flows, irreducible saturations are often taken into account. One can suppose that there exists s_i and S_i ($0 < s_i < S_i < 1$) such that $\lambda_i(s) = 0$ if $s \notin (s_i, S_i)$. In such a case, the problem $(\mathcal{P})$ becomes strongly degenerated, but a convenient scaling eliminates this difficulty (at least if $s_i \leq u_0 \leq S_i$ a.e. in Ω_i). Moreover, the dependance of the capillary pressure with regard to the saturation can be weak, at least for saturations not too close to 0 or 1. Thus the effects of the capillarity are often neglected for the study of flows in homogeneous porous media, leading to the Buckley-Leverett equation (see e.g. [19]). Looking for degeneracy of $u \mapsto \pi_i(u)$ is a more complex problem, particularly if the convection is not neglected as above. Suppose for example that $\pi_i(u) = \varepsilon u + P_i$, where P_i are constants, and let ε tend 0. Non-classical shocks can appear at the level of the interfaces $\Gamma_{i,j}$ (see [10]). Thus the notion of entropy solution used by Adimurthi, J. Jaffré, and G.D. Veerappa Gowda [1] is not sufficient to deal with this problem. This difficulty has to be overcome to consider degenerate parabolic problem. But it seems clear that the notion of entropy solution developed by K.H. Karlsen, N.H. Risebro, J.D. Towers [20, 21, 22] is not adapted to our problem.*

2 The notion of weak solution

In this section, we introduce the notion of weak solution to the problem $(\mathcal{P})$, which is more general than the notion of weak solution given in [13, 14]. Indeed, we are able to define such a solution even in the case of an arbitrary finite number of different homogeneous porous media. Furthermore, the notion of weak solution introduced in this paper is still available in cases where the one defined in [14] has no more sense. We finally show that the two notions of solution are equivalent in the case where the notion of weak solution in the sense of [14] is well defined. The existence of a weak solution to problem $(\mathcal{P})$ in a wider case is the aim of the section 3.

One denotes by φ_i the $C^1([0,1], \mathbb{R}_+)$ function which naturally appears in the problem $(\mathcal{P})$ and which is defined by: $\forall s \in [0,1]$,

$$\varphi_i(s) = \int_0^s \lambda_i(a)\pi_i'(a)da. \tag{11}$$

Remark 2.1 *The assumptions on the data insure that $\varphi_i' > 0$ on $]0,1[$, and so we can define an increasing continuous function $\varphi_i^{-1} : [0, \varphi_i(1)] \to [0,1]$.*

We are now able to define the notion of weak solution to the problem $(\mathcal{P})$.

Definition 2.1 (weak solution to the problem $(\mathcal{P})$) *Under assumptions 1 and 2, a function u is said to be a weak solution to the problem $(\mathcal{P})$ if it verifies:*

1. $u \in L^\infty(\Omega \times (0,T))), 0 \leq u \leq 1$ *a.e. in* $\Omega \times (0,T)$,
2. $\forall i \in [\![1, \mathrm{N}]\!]$, $\varphi_i(u_i) \in L^2(0,T; H^1(\Omega_i))$, *where u_i denotes the restriction of u to* $\Omega_i \times (0,T)$,
3. $\tilde{\pi}_i(u_i) \cap \tilde{\pi}_j(u_j) \neq \emptyset$ *a.e. on* $\Gamma_{i,j} \times (0,T)$,

4. for all $\psi \in \mathcal{D}(\overline{\Omega} \times [0,T))$,

$$\sum_{i=1}^{N} \int_{\Omega_i} \int_0^T \phi_i u_i(x,t) \partial_t \psi(x,t) dx dt + \sum_{i=1}^{N} \int_{\Omega_i} \phi_i u_0(x) \psi(x,0) dx \\ - \sum_{i=1}^{N} \int_{\Omega_i} \int_0^T \nabla \varphi_i(u_i(x,t)) \cdot \nabla \psi(x,t) dx dt = 0. \tag{12}$$

The third point of the previous definition, which insures the connection in the graph sense of the capillary pressures on the interfaces between several porous media, is well defined. Indeed, since $\varphi_i(u_i)$ belongs to $L^2(0,T;H^1(\Omega_i))$, it admits a trace still denoted $\varphi_i(u_i)$ on $\Gamma_{i,j} \times (0,T)$. Thanks to the remark 2.1, we can define the trace of u_i on $\Gamma_{i,j} \times (0,T)$.

Remark 2.2 *One can equivalently substitute the condition:*

$$3bis. \quad \breve{\pi}_i(u_i) \cap \breve{\pi}_j(u_j) \neq \emptyset \text{ a.e. on } \Gamma_{i,j} \times (0,T),$$

to the third point of the definition 2.1, where $\breve{\pi}_i$ *is the monotonous graph given by:*

$$\breve{\pi}_i(s) = \begin{cases} \pi_i(s) & \text{if } s \in]0,1[, \\ [\min_j(\alpha_j), \alpha_i] & \text{if } s = 0, \\ [\beta_i, \max_j(\beta_j)] & \text{if } s = 1. \end{cases} \tag{13}$$

We will now quickly show the equivalence between the notion of weak solution to the problem $(\mathcal{P})$ and the notion of weak solution given in [14], in the case where this one is well defined, i.e. $N = 2$ and $\max(\alpha_1, \alpha_2) = \alpha < \beta = \min(\beta_1, \beta_2)$. We denote as in [14] the truncated capillary pressures by $\hat{\pi}_1 = \max(\alpha, \pi_1)$, $\hat{\pi}_2 = \min(\beta, \pi_2)$, and we introduce the problem $(\widetilde{\mathcal{P}})$, which is treated in [14].

$$\begin{cases} \phi_i \partial_t u_i - \nabla \cdot (\lambda_i(u_i) \nabla \pi_i(u_i)) = 0 & \text{in } \Omega_i \times (0,T), \\ \hat{\pi}_1(u_1) = \hat{\pi}_2(u_2) & \text{on } \Gamma_{i,j} \times (0,T), \\ \lambda_1(u_1) \nabla(\pi_1(u_1)) \cdot \mathbf{n}_1 + \lambda_2(u_2) \nabla(\pi_2(u_2)) \cdot \mathbf{n}_2 = 0 & \text{on } \Gamma_{i,j} \times (0,T), \\ \lambda_i(u_i) \nabla(\pi_i(u_i)) \cdot \mathbf{n}_i = 0 & \text{on } \partial\Omega_i \cap \partial\Omega \times (0,T), \\ u_i(\cdot,0) = u_0(x) & \text{in } \Omega_i. \end{cases} \tag{$\widetilde{\mathcal{P}}$}$$

Then it is easy to check that: $\forall (s_1, s_2) \in [0,1]^2$,

$$\hat{\pi}_1(s_1) = \hat{\pi}_2(s_2) \Leftrightarrow \tilde{\pi}_1(s_1) \cap \tilde{\pi}_2(s_2) \neq \emptyset \Leftrightarrow \breve{\pi}_1(s_1) \cap \breve{\pi}_2(s_2) \neq \emptyset. \tag{14}$$

In order to recall the definition of weak solution, we have to introduce the function

$$\Psi : \begin{cases} [\alpha, \beta] \to \mathbb{R} \\ p \mapsto \displaystyle\int_\alpha^p \min_{j=1,2} (\lambda_j \circ \pi_j^{-1}(a)) \mathrm{d}a. \end{cases}$$

Ψ is increasing, and for $i = 1,2$, $\Psi \circ \hat{\pi}_i \circ \varphi_i^{-1}$ is a Lipschitz continuous function.

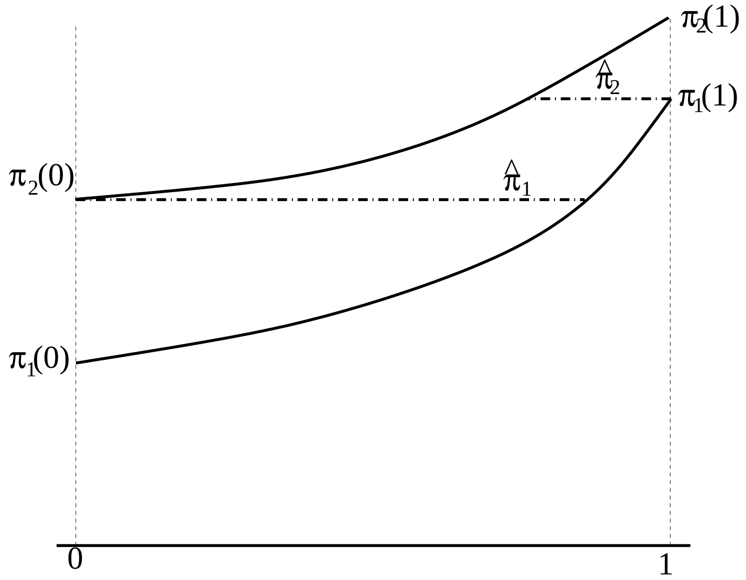

Figure 3: Truncated capillary pressures

Definition 2.2 (weak solution to the problem $(\widetilde{\mathcal{P}})$) *A function u is said to be a weak solution to the problem $(\widetilde{\mathcal{P}})$ if it verifies:*

1. $u \in L^\infty(\Omega \times (0,T))), 0 \leq u \leq 1$ *a.e. in* $\Omega \times (0,T)$,
2. $\forall i \in \{1,2\}, \varphi_i(u_i) \in L^2(0,T;H^1(\Omega_i))$,
3. $w : \Omega \times (0,T) \to \mathbb{R}$, *defined for* $(x,t) \in \Omega_i \times (0,T)$ *by* $w(x,t) = \Psi \circ \hat{\pi}_i(u_i)(x,t)$ *belongs to* $L^2(0,T;H^1(\Omega))$,
4. *for all* $\psi \in \mathcal{D}(\overline{\Omega} \times [0,T))$,

$$\sum_{i=1}^{N} \int_{\Omega_i} \int_0^T \phi_i u_i(x,t) \partial_t \psi(x,t) dxdt + \sum_{i=1}^{N} \int_{\Omega_i} \phi_i u_0(x) \psi(x,0) dx - \sum_{i=1}^{N} \int_{\Omega_i} \int_0^T \nabla \varphi_i(u_i(x,t)) \cdot \nabla \psi(x,t) dxdt = 0.$$

Remark 2.3 *The notion of weak solution to the problem $(\widetilde{\mathcal{P}})$ can be adapted in the case where there are $N > 2$ homogeneous domains, but we have to keep conditions of compatibility on $(\alpha_i)_{1 \leq i \leq N}$ and $(\beta_i)_{1 \leq i \leq N}$.*

Proof of the equivalence of the weak solutions

On the one hand, if u is a weak solution to the problem $(\widetilde{\mathcal{P}})$ in the sense of definition 2.2, then for a.e. $t \in (0,T)$, $w(\cdot,t) \in H^1(\Omega)$,and particularly $w(\cdot,t)$ admits a trace on $\Gamma_{i,j}$, whose value is in the same time $\Psi(\hat{\pi}_i(u_i(\cdot,t)))$ and $\Psi(\hat{\pi}_j(u_j(\cdot,t)))$. Since Ψ in increasing, for a.e $(x,t) \in \Gamma_{i,j} \times (0,T)$, $\hat{\pi}_i(u_i(x,t)) = \hat{\pi}_j(u_j(x,t))$. Using (14), we conclude that any weak solution to the problem $(\widetilde{\mathcal{P}})$ is a weak solution to the problem $(\mathcal{P})$ in the sense of definition 2.1.

On the other hand, if u is a weak solution to the problem $(\mathcal{P})$ in the sense of definition 2.1, then thanks to (14), for almost every $(x,t) \in \Gamma_{i,j} \times (0,T)$,

$$\hat{\pi}_i(u_i(x,t)) = \hat{\pi}_j(u_j(x,t)) \Leftrightarrow \Psi \circ \hat{\pi}_i \circ \varphi_i^{-1}(\varphi_i(u_i(x,t))) = \Psi \circ \hat{\pi}_j \circ \varphi_j^{-1}(\varphi_j(u_j(x,t))). \quad (15)$$

Since $\Psi \circ \hat{\pi}_i \circ \varphi_i^{-1}$ is a Lipschitz continuous function, the second point in definition 2.1 insures us that $\Psi \circ \hat{\pi}_i(u_i)$ belongs to $L^2(0,T,H^1(\Omega_i))$ for $i = 1, 2$, and (15) insures the connection of the traces on $\Gamma_{i,j} \times (0,T)$, then the third point of definition 2.2 is fulfilled and u is a weak solution to the problem $(\widetilde{\mathcal{P}})$. ■

Remark 2.4 *We can define a* function $\tilde{\pi}_i^{-1}, i \in [\![1,\mathrm{N}]\!]$, *which verifies* $\tilde{\pi}_i^{-1} \circ \tilde{\pi}_i(s) = s$ *for any* $s \in [0,1]$. *Using the function defined on* $\mathbb{R}$ *by* $\tilde{\Psi}(p) = \int_{-\infty}^{p} \min_{j=1,2}(\lambda_j \circ \tilde{\pi}_j^{-1}(a))da$, *it is easy to check that we can equivalently substitute the function* $\tilde{\Psi} \circ \pi_i(u_i)$ *to* $\Psi \circ \hat{\pi}_i(u_i)$ *in the third point of definition 2.2. This function is still defined if* $\alpha \geq \beta$, *but it becomes identically* 0, *so the notion of weak solution to the problem* $(\widetilde{\mathcal{P}})$ *is weaker than the notion of weak solution to the problem* $(\mathcal{P})$. *Indeed, in such a case,* $u(x,t) = u_0(x) = a \in]0,1[$ *for any* $(x,t) \in \Omega \times (0,T)$ *is a weak solution to the problem* $(\widetilde{\mathcal{P}})$, *but it does not fulfill the third point in definition 2.1.*

3 Existence of a weak solution

The aim of this section is to prove the following theorem, which claims the existence of a weak solution to the problem $(\mathcal{P})$. This result has already been proven in section 2 in the case $N = 2$ and $\alpha > \beta$, for which the notion of weak solution in the sense of definition 2.1 is equivalent to the notion of weak solution in the sense of definition 2.2.

Theorem 3.1 (Existence of a weak solution) *Under assumptions 1 and 2, there exists a weak solution to problem* $(\mathcal{P})$ *in the sense of definition 2.1.*

Proof

In order to prove the existence of a weak solution to the problem $(\mathcal{P})$ in the sense of the definition 2.1, we build a sequence of solutions to approximated problems (16), which converges, up to a subsequence, toward a weak solution to the problem $(\mathcal{P})$. The approximated problems do not involve capillary barriers, so existence and uniqueness of such approximated solutions is given in [9]. We let the proof of the following technical lemma to the reader.

Lemma 3.2 *There exists sequences* $(\lambda_{i,n})_n$, $(\pi_{i,n})_n$ *belonging to* $(C^\infty([0,1],\mathbb{R}))^{\mathbb{N}}$ *such that, for* $i \in [\![1,\mathrm{N}]\!]$, *and for* n *large enough:*

- $\lambda_{i,n|[0,1/n]\cup[1-1/n,1]} = \dfrac{1}{n^2}$, $\lambda_{i,n}(s) > \dfrac{1}{2n^2}$, *for all* $s \in [0,1]$, $\lambda_{i,n} \to \lambda_i$ *uniformly on* $[0,1]$,
- $\pi_{i,n}(0) = \pi_{j,n}(0) \to -\infty$, $\pi_{i,n}(1) = \pi_{j,n}(1) \to +\infty$, $Kn^{\frac{3}{2}} > \pi'_{i,n} \geq \dfrac{1}{n}$, $\pi_{i,n} \to \pi_i$ *in* $L^1(0,1)$, $\pi_{i,n} \to \pi_i$ *and* $\pi'_{i,n} \to \pi'_i$ *uniformly on any compact set of* $]0,1[$,
- *the function* $\varphi_{i,n} : s \mapsto \int_0^s \lambda_{i,n}(a)\pi'_{i,n}(a)da$ *furthermore fulfills* $\varphi_{i,n}([0,1]) = \varphi_i([0,1])$ *and* $\varphi_{i,n} \to \varphi_i$ *in* $W^{1,\infty}(0,1)$.

We also define the increasing functions:

$$\Psi_n : \begin{cases} [a_n, b_n] \to \mathbb{R} \\ p \mapsto \displaystyle\int_{a_n}^{p} \min_{j \in [\![1,\mathrm{N}]\!]} (\lambda_{j,n} \circ \pi_{j,n}^{-1}(a)) \mathrm{d}a. \end{cases}$$

The conditions on the functions on the intervals $[0, \frac{1}{n}] \cup [1 - \frac{1}{n}, 1]$ insures that for any fixed large n, the functions $(\varphi_{i,n} \circ \pi_{i,n}^{-1} \circ \Psi_n^{-1})'$ are Lipschitz continuous. Then thanks to [9], for all n, the approximated problems:

$$\begin{cases} \phi_i \partial_t u_{i,n} - \nabla \cdot (\lambda_{i,n}(u_{i,n}) \nabla \pi_{i,n}(u_{i,n})) = 0 & \text{in } \Omega_i \times (0, T), \\ \pi_{i,n}(u_{i,n}) = \pi_{j,n}(u_{j,n}) & \text{on } \Gamma_{i,j} \times (0, T), \\ \lambda_{i,n}(u_{i,n}) \nabla(\pi_{i,n}(u_{i,n})) \cdot \mathbf{n}_i + \lambda_{j,n}(u_{j,n}) \nabla(\pi_{j,n}(u_{j,n})) \cdot \mathbf{n}_j = 0 & \text{on } \Gamma_{i,j} \times (0, T), \\ \lambda_{i,n}(u_{i,n}) \nabla(\pi_{i,n}(u_{i,n})) \cdot \mathbf{n}_i = 0 & \text{on } \partial\Omega_i \cap \partial\Omega \times (0, T), \\ u_{i,n}(x, 0) = u_0(x) & \text{in } \Omega. \end{cases} \tag{16}$$

admit a unique weak solution in the sense of definition 3.1 given below, and this solution belongs to $C([0, T], L^p(\Omega))$ for $1 \leq p < +\infty$.

Definition 3.1 (Weak solutions for approximated problems)
A function u_n is said to be a weak solution to the problem (16) if it verifies:

1. $u_n \in L^\infty(\Omega \times (0, T))), 0 \leq u_n \leq 1$ *a.e. in* $\Omega \times (0, T)$,
2. $\forall i \in \{1, 2\}, \varphi_{i,n}(u_{i,n}) \in L^2(0, T; H^1(\Omega_i))$,
3. $w_n : \Omega \times (0, T) \to \mathbb{R}$, *defined on* $\Omega_i \times (0, T)$ *by* $w_n = \Psi_n \circ \pi_{i,n}(u_{i,n})$ *belongs to* $L^2(0, T; H^1(\Omega))$,
4. *for all* $\psi \in \mathcal{D}(\Omega \times [0, T))$,

$$\begin{aligned} \sum_{i=1}^{N} \int_{\Omega_i} \int_0^T \phi_i u_{i,n}(x, t) \partial_t \psi(x, t) dx dt + \sum_{i=1}^{N} \int_{\Omega_i} \phi_i u_0(x) \psi(x, 0) dx \\ - \sum_{i=1}^{N} \int_{\Omega_i} \int_0^T \nabla \varphi_{i,n}(u_{i,n}(x, t)) \cdot \nabla \psi(x, t) dx dt = 0. \end{aligned} \tag{17}$$

The proof of existence of a weak solution given in [9], shows that for all $i \in [\![1, \mathrm{N}]\!]$, for all n, there exists $C_1 > 0$ not depending on n such that, for all $i \in [\![1, \mathrm{N}]\!]$:

$$\|\varphi_{i,n}(u_{i,n})\|^2_{L^2(0,T;H^1(\Omega_i))} \leq C_1 \|\pi_{i,n}\|_{L^1(0,1)}, \tag{18}$$

thus $(\varphi_{i,n}(u_{i,n}))_n$ is a bounded sequence of $L^2(0, T; H^1(\Omega_i))$ using lemma 3.2. A study of the proof of the time translate estimate used in [9, 14], and detailed in [16, lemma 4.6] leads to the existence of C_2 not depending on n such that:

$$\|\varphi_{i,n}(u_{i,n}(\cdot, \cdot + \tau)) - \varphi_{i,n}(u_{i,n}(\cdot, \cdot))\|^2_{L^2(\Omega_i \times (0, T - \tau))} \leq \tau C_2 \|\pi_{i,n}\|_{L^1(0,1)} \|\varphi'_{i,n}\|_{L^\infty(0,1)}. \tag{19}$$

Using lemma 3.2 once again, estimates (18), (19) allow us to apply Kolmogorov's compactness criterion (see e.g. [8]), thus we can claim the relative compactness of the sequence $(\varphi_{i,n}(u_{i,n}))_n$ in $L^2(\Omega_i \times (0, T))$. There exists $f_i \in L^2(0, T; H^1(\Omega_i))$ such that

$$\varphi_{i,n}(u_{i,n}) \to f_i \text{ in } L^2(\Omega_i \times (0, T)),$$

$$\varphi_{i,n}(u_{i,n}) \to f_i \text{ weakly in } L^2(0,T;H^1(\Omega_i)).$$

Let us now recall a very useful lemma, classically called Minty trick, and introduced in this framework by Leray and Lions in the famous paper [24].

Lemma 3.3 (Minty trick) *Let $(\phi_n)_n$ be a sequence of non-decreasing functions with for all n, $\phi_n : \mathbb{R} \to \mathbb{R}$, and let $\phi : \mathbb{R} \to \mathbb{R}$ be a non-decreasing continuous function such that:*

- *$\phi_n \to \phi$ pointwise,*
- *there exists $g \in L^1_{loc}(\mathbb{R})$ such that $|\phi_n| \le g$.*

Let $\mathcal{O}$ be an open subset of $\mathbb{R}^k$, $k \ge 1$. Let $(u_n)_n \in (L^\infty(\mathcal{O}))^{\mathbb{N}}$, let $u \in L^\infty(\mathcal{O})$ and let $f \in L^1(\mathcal{O})$ such that:

- *$u_n \to u$ in the $L^\infty(\mathcal{O})$-weak-$\star$ sense,*
- *$\phi_n(u_n) \to f$ in $L^1(\mathcal{O})$.*

Then

$$f = \phi(u).$$

Since $0 \le u_{i,n} \le 1$, $(u_{i,n})_n$ converges up to a subsequence to u_i in the $L^\infty(\Omega_i \times (0,T))$-weak-$\star$ sense. $(\varphi_{i,n})_n$ converges uniformly toward φ_i on $[0,1]$, and we can easily check, using Minty trick, that $f_i = \varphi_i(u_i) \in L^2(0,T;H^1(\Omega_i))$. Thus we can pass to the limit in the formulation (17) to obtain the wanted weak formulation:

$$\sum_{i=1}^N \int_{\Omega_i} \int_0^T \phi_i u_i(x,t) \partial_t \psi(x,t) dx dt + \sum_{i=1}^N \int_{\Omega_i} \phi_i u_0(x) \psi(x,0) dx$$
$$- \sum_{i=1}^N \int_{\Omega_i} \int_0^T \nabla \varphi_i(u_i(x,t)) \cdot \nabla \psi(x,t) dx dt = 0.$$

The last point needed to achieve the proof of theorem 3.1 is the convergence of the traces of the approximate solutions $(u_{i,n})_n$ on $\Gamma_{i,j} \times (0,T)$ toward the trace of u_i, and to verify that $\tilde{\pi}_i(u_i) \cap \tilde{\pi}_j(u_j) \neq \emptyset$ a.e. on $\Gamma_{i,j} \times (0,T)$.

Since Ω_i has a Lipschitz boundary, there exists an operator P, continuous from $H^1(\Omega_i)$ into $H^1(\mathbb{R}^d)$, and also from $L^2(\Omega_i)$ into $L^2(\mathbb{R}^d)$, such that $Pv_{|\Omega_i} = v$ for all $v \in L^2(\Omega_i)$. Then P is continuous from $H^s(\Omega_i)$ into $H^s(\mathbb{R}^d)$ for all $s \in [0,1]$. One has, for all $v \in H^s(\Omega_i)$,

$$\|v\|_{H^s(\Omega_i)} \le \|Pv\|_{H^s(\mathbb{R}^d)} \le \|Pv\|^s_{H^1(\mathbb{R}^d)} \|Pv\|^{1-s}_{L^2(\mathbb{R}^d)} \le C \|v\|^s_{H^1(\Omega_i)} \|v\|^{1-s}_{L^2(\Omega_i)}.$$

One deduces from the previous inequality and from (19) that for all $s \in]0,1[$, for all $\tau \in]0,T[$, there exists C_3 not depending on n, τ such that

$$\|\varphi_{i,n}(u_{i,n}(\cdot, \cdot + \tau)) - \varphi_{i,n}(u_{i,n}(\cdot,\cdot))\|^2_{L^2(0,T-\tau;H^s(\Omega_i))} \le \tau^{1-s} C_3 \tag{20}$$

For $s_1 > s_2$, H^{s_1} is compactly imbedded in H^{s_2}, and then estimate (20) allows us to claim that the sequence $(\varphi_{i,n}(u_{i,n}))_n$ is relatively compact in $L^2(0,T;H^s(\Omega_i))$ for all $s \in]0,1[$. Particularly, one can extract a subsequence converging toward $\varphi_i(u_i)$ in $L^2(0,T;H^s(\Omega_i))$. We can claim, using once again Minty trick, that the traces of $(\varphi_{i,n}(u_{i,n}))_n$ on $\Gamma_{i,j}$ also converge toward the trace of $\varphi_i(u_i)$, still denoted $\varphi_i(u_i)$ in $L^2(0,T;H^{s-1/2}(\Gamma_{i,j}))$, and particularly for almost every $(x,t) \in \Gamma_{i,j} \times (0,T)$. Since φ_i is increasing, $(u_{i,n}(x,t))_n$ converges almost everywhere on $\Gamma_{i,j} \times (0,T)$ toward $u_i(x,t)$.

Let us now check that $\tilde{\pi}_i(u_i) \cap \tilde{\pi}_j(u_j) \neq \emptyset$ a.e. on $\Gamma_{i,j} \times (0,T)$. For almost every $(x,t) \in \Gamma_{i,j} \times (0,T)$ the sequence $(\pi_{i,n}(u_{i,n}(x,t)))_n$ converges (up to a new extraction) toward $\gamma_i(x,t) \in \overline{\mathbb{R}}$. Since for all n, $\pi_{i,n}(u_{i,n}(x,t)) = \pi_{j,n}(u_{j,n}(x,t))$, one has:

$$\gamma_i(x,t) = \gamma_j(x,t) \text{ a.e. on } \Gamma_{i,j} \times (0,T). \tag{21}$$

If $u_i(x,t) \in]0,1[$, then $\gamma_i(x,t) = \pi_i(u_i(x,t))$. If $u_i(x,t) = 0$, $\gamma_i(x,t) \leq \alpha_i$, and $\gamma_i(x,t) \in \tilde{\pi}_i(0)$. In the same way, if $u_i(x,t) = 1$, $\gamma_i(x,t) \in \tilde{\pi}_i(1)$.

This achieves the proof of theorem 3.1, because relation (21) insures the connection of the traces in the sense of:

$$\tilde{\pi}_i(u_i) \cap \tilde{\pi}_j(u_j) \neq \emptyset \text{ a.e. on } \Gamma_{i,j} \times (0,T).$$

■

4 A regularity result

In this section and in section 5, we show the existence and the uniqueness of a solution with bounded flux to the problem $(\mathcal{P})$ in the one-dimensional case. We make the proofs in the case where there are only two sub-domains $\Omega_1 =]-1,0[$ and $\Omega_2 =]0,1[$, but a straightforward adaptation of them gives the same result for an arbitrary finite number of Ω_i, each one with an arbitrary finite measure. We now state the main result of this section, which claims the existence of a solution with bounded spatial derivatives on $\mathcal{Q}_i$, where $\mathcal{Q}_i = \Omega_i \times (0,T)$. We also set $\mathcal{Q} =]-1,1[\times]0,T[$ and $\Gamma = \{x = 0\}$.

Theorem 4.1 (Existence of a bounded flux solution) *Let $u_0 \in L^\infty(-1,1)$, $0 \leq u_0 \leq 1$ such that:*

- $\varphi_i(u_0) \in W^{1,\infty}(\Omega_i)$,
- $\tilde{\pi}_1(u_{0,1}) \cap \tilde{\pi}_2(u_{0,2}) \neq \emptyset$ *on* Γ.

Then there exists a weak solution u to the problem $(\mathcal{P})$ such that $\partial_x \varphi_i(u_i) \in L^\infty(\mathcal{Q}_i)$.

All the section will be devoted to the proof of the theorem 4.1. As in section 3, we will get this existence result by taking the limit of a sequence of solutions to approximate problems (16) involving no capillary barriers, whose data fulfill the properties stated in lemma 3.2.

Proof

We will now build a sequence of approximate initial data $(u_{0,n})$ adapted to the sequence of approximate problems.

Lemma 4.2 *Let u_0 be chosen as in theorem 4.1, then there exists $(u_{0,n})_n$ such that, for all n,*

- $0 \leq u_{0,n} \leq 1$,
- $\pi_{1,n}(u_{0,n,1}) = \pi_{2,n}(u_{0,n,2})$ *on* Γ.

The sequence $(u_{0,n})_n$ furthermore fulfills:

$$\lim_{n\to\infty} \|u_{0,n} - u_0\|_\infty = 0, \qquad \|\partial_x \varphi_{i,n}(u_{0,n})\|_{L^\infty(\Omega_i)} \le \|\partial_x \varphi_i(u_0)\|_{L^\infty(\Omega_i)}. \tag{22}$$

Proof
Since $\tilde{\pi}_1(u_{0,1}) \cap \tilde{\pi}_2(u_{0,2}) \neq \emptyset$, then there exists $(a_{1,n}, a_{2,n}) \in [0,1]^2$ such that one has $\pi_{1,n}(a_{1,n}) = \pi_{2,n}(a_{2,n})$ and $|a_{1,n} - u_{0,1}| + |a_{2,n} - u_{0,2}| \to 0$. One sets, for $x \in \Omega_i$:

$$u_{0,n}(x) = \varphi_{i,n}^{-1}\left(T_{\varphi_i}\left[\varphi_i(u_0) + \varphi_{i,n}(a_{i,n}) - \varphi_i(u_{0,i})\right]\right)$$

where

$$T_{\varphi_i}(s) = \begin{cases} s & \text{if} \quad s \in [0, \varphi_i(1)] = [0, \varphi_{i,n}(1)], \\ \varphi_{i,n}(1) & \text{if} \quad s > \varphi_i(1), \\ 0 & \text{if} \quad s < 0. \end{cases}$$

Then the sequence $(u_{0,n})$ converges uniformly toward u_0. For all n, $0 \le u_{0,n} \le 1$ and either $\partial_x \varphi_{i,n}(u_{0,n}) = \partial_x \varphi_i(u_0)$, or $\partial_x \varphi_{i,n}(u_{0,n}) = 0$. ■

The approximate problem (16) admits a unique solution u_n thanks to [9], which belongs to $C([0,T], L^1(\Omega))$. Now, in order to get a $L^\infty(\mathcal{Q}_i)$-estimate on the sequence $(\partial_x \varphi_{i,n}(u_n))_n$, we introduce a new family of approximate problems (23) for which the spatial dependence of the data is smooth.

Let $\theta \in C^\infty(\mathbb{R}), 0 \le \theta \le 1$, with $\theta(x) = 0$ if $x < -1$, and $\theta(x) = 1$ if $x > 1$. Let $k \in \mathbb{N}^\star$, one sets:

- $\phi^k(x) = (1 - \theta(kx))\phi_1 + \theta(kx)\phi_2$,
- $\lambda_{n,k}(s, x) = (1 - \theta(kx))\lambda_{1,n}(s) + \theta(kx)\lambda_{2,n}(s)$,
- $\pi_{n,k}(s, x) = (1 - \theta(kx))\pi_{1,n}(s) + \theta(kx)\pi_{2,n}(s)$.

We will now take a new approximation of the initial data.

$$u_{0,n,k}(x) = \begin{cases} u_{0,n}\left(\frac{k}{k-1}\left(x + \frac{1}{k}\right)\right) & \text{if} \quad x < -1/k, \\ u_{0,n}\left(\frac{k}{k-1}\left(x - \frac{1}{k}\right)\right) & \text{if} \quad x > 1/k. \end{cases}$$

In the layer $[-1/k, 1/k]$, $u_{0,n,k}$ is defined by the relation

$$(1 - \theta(kx))\pi_{1,n}(u_{0,n,k}(x)) + \theta(kx)\pi_{2,n}(u_{0,n,k}(x)) = \pi_{1,n}(a_{1,n}) = \pi_{2,n}(a_{2,n}),$$

so that the approximate capillary pressure $\pi_{n,k}(u_{0,n,k}, \cdot)$ is constant through the layer.

Moreover one has either

$$\lambda_{n,k}(u_{0,n,k}, x)\partial_x(\pi_{n,k}(u_{0,n,k}, x)) = \frac{k}{k-1}\partial_x \varphi_{i,n}(u_{0,n}) \qquad \text{if } |x| > \frac{1}{k},$$

or

$$\partial_x(\pi_{n,k}(u_{0,n,k}, x)) = 0 \qquad \text{if } |x| < \frac{1}{k}.$$

So we directly deduce from the definition of $u_{0,n,k}$ the following lemma:

Lemma 4.3 *Let $n \geq 1$, $0 \leq u_{0,n} \leq 1$ with $\varphi_{i,n}(u_{0,n}) \in W^{1,\infty}(\Omega_i)$ and $\pi_{1,n}(u_{0,n,1}) = \pi_{2,n}(u_{0,n,2})$, then there exists a sequence $(u_{0,n,k})_k$ satisfying, for all $k \geq 2$, that $0 \leq u_{0,n,k} \leq 1$ and*

$$\|\lambda_{n,k}(u_{0,n,k},\cdot)\partial_x(\pi_{n,k}(u_{0,n,k},\cdot))\|_\infty \leq 2 \max_{i=1,2}(\|\partial_x\varphi_{i,n}(u_{0,n})\|_\infty),$$

$$u_{0,n,k} \to u_{0,n} \text{ in } L^1(\Omega) \text{ as } k \to +\infty.$$

For any fixed $k \geq 2$ and n large enough, we can now introduce the smooth non-degenerate parabolic problem (23):

$$\left\{\begin{array}{l} \phi^k(x)\partial_t u_{n,k} - \partial_x(\lambda_{n,k}(u_{n,k},x)\partial_x\pi_{n,k}(u_{n,k},x)) = 0, \\ \partial_x u_{n,k}(-1,t) = \partial_x u_{n,k}(1,t) = 0, \\ u_{n,k}(x,0) = u_{0,n,k}(x). \end{array}\right. \tag{23}$$

Moreover, one can furthermore suppose, up to a new regularization, that $u_{0,n,k} \in C^\infty([-1,1])$. Then (23) admits a unique strong solution $u_{n,k} \in C^\infty([0,T]\times[-1,1])$ (see for instance [17, 23]).

Now one sets $f_{n,k}(x,t) = \lambda_{n,k}(u_{n,k},x)\partial_x\pi_{n,k}(u_{n,k},x)$, so the main equation of (23) can be rewritten:

$$\phi^k\partial_t u_{n,k} = \partial_x f_{n,k}.$$

A short calculation shows that $f_{n,k}(x,t)$ is the solution of the problem:

$$\left\{\begin{array}{l} \partial_t f_{n,k} = a_{n,k}\partial^2_{xx} f_{n,k} + b_{n,k}\partial_x f_{n,k}, \\ f_{n,k}(-1,t) = f_{n,k}(1,t) = 0, \\ f_{n,k}(x,0) = \lambda_{n,k}(u_{0,n,k},\cdot)\partial_x(\pi_{n,k}(u_{0,n,k},\cdot)), \end{array}\right. \tag{24}$$

where $a_{n,k}, b_{n,k}$ are the regular functions defined below.

$$a_{n,k} = \lambda_{n,k}(u_{n,k},x)\frac{(\pi_{n,k})'(u_{n,k},x)}{\phi^k(x)} > 0,$$

$$b_{n,k} = (\lambda_{n,k})'(u_{n,k},x)\frac{\partial_x[\pi_{n,k}(u_{n,k},x)]}{\phi^k(x)} + \lambda_{n,k}(u_{n,k},x)\partial_x\left[\frac{(\pi_{n,k})'(u_{n,k},x)}{\phi^k(x)}\right].$$

The fact that $u_{0,n,k}$ is supposed to be regular allows us to write the problem (24) in a strong sense (this is necessary, because this problem can not be written in a conservative form). In particular, $f_{n,k}$ satisfies the maximum principle, and thus

$$\|f_{n,k}\|_{L^\infty((-1,1)\times(0,T))} \leq \|\lambda_{n,k}(u_{0,n,k},\cdot)\partial_x(\pi_{n,k}(u_{0,n,k},\cdot))\|_{L^\infty(-1,1)}.$$

Thanks to the lemmas 4.3 and 4.2, we have a uniform bound on $(f_{n,k})$:

$$\|f_{n,k}\|_{L^\infty((-1,1)\times(0,T))} \leq 2\max_{i=1,2}(\|\partial_x\varphi_i(u_0)\|_\infty). \tag{25}$$

Since the problem (23) is fully non degenerated (recall that $\lambda_{i,n} > \frac{1}{2n^2}$ and $\pi'_{i,n} \geq \dfrac{1}{n}$) it follows that $\partial_x u_{n,k}$ and $\partial_t u_{n,k}$ are uniformly bounded respectively in $L^\infty(\mathcal{Q}_i)$ and in

$L^2(0,T:H^{-1}(\Omega_i))$ with respect to k, then the sequence $(u_{n,k})_k$ converges toward u_n in $L^2(\mathcal{Q}_i)$, and the limit u_n fulfills, thank to estimate (25):

$$\|\partial_x \varphi_{i,n}(u_n)\|_{L^\infty(\mathcal{Q}_i)} \leq 2 \max_{i=1,2}(\|\partial_x \varphi_i(u_0)\|_\infty). \tag{26}$$

One has for all $\psi \in \mathcal{D}([-1,1]\times[0,T[)$,

$$\int_0^T \int_{-1}^1 \phi^k u_{n,k} \partial_t \psi + \int_{-1}^1 \phi^k u_{0,n}^k \psi_0 - \int_0^T \int_{-1}^1 f_{n,k} \partial_x \psi = 0. \tag{27}$$

Thanks to (25),

$$\lim_{k\to+\infty} \int_0^T \int_{-\frac{1}{k}}^{\frac{1}{k}} f_{n,k} \partial_x \psi = 0.$$

One has $u_{n,k} \to u_n$ in the $L^\infty(\mathcal{Q})$-weak- $\star$ and $L^2(\mathcal{Q})$ senses, $u_{0,n,k} \to u_{0,n}$ in $L^1(-1,1)$ thanks to lemma 4.3. Moreover, thanks to estimate (25), $\partial_x \pi_{i,n,k}(u_{n,k}) \to \partial_x \pi_{i,n}(u_{n,k})$ in the $L^\infty(\mathcal{Q})$-weak- $\star$ sense. Thus we can let k tend toward $+\infty$ in (27) to get

$$\int_0^T \sum_{i=1,2} \int_{\Omega_i} \phi_i u_n \partial_t \psi + \sum_{i=1,2} \int_{\Omega_i} \phi_i u_{0,n} \psi_0 - \int_0^T \sum_{i=1,2} \int_{\Omega_i} \lambda_{i,n}(u_n) \partial_x \pi_{i,n}(u_n) \partial_x \psi = 0. \tag{28}$$

Furthermore, using the fact that $\pi_{n,k}(u_{n,k},x)$ belongs to $L^2(0,T;H^1(\Omega))$ and, even more, that $\partial_x(\pi_{n,k}(u_{n,k},x))$ is bounded uniformly in k, we can claim that $\pi_{1,n}(u_{1,n}) = \pi_{2,n}(u_{2,n})$, and so u_n is the unique weak solution to the approximate problem (16) for $u_{0,n}$ as initial data.

When n tends toward $+\infty$, the sequence $(u_n)_n$ converges, up to a subsequence toward a weak solution to the problem $(\mathcal{P})$, as seen in section 3, but the estimate (26) insures that

$$\partial_x \varphi_i(u) \in L^\infty(\mathcal{Q}_i).$$

This achieves the proof of theorem 4.1. ■

5 A uniqueness result

In this section, we give a uniqueness result in the one dimensional case in a framework where the existence results are stronger than the general existence result stated in theorem 3.1. Under a regularity assumption on the initial data u_0, we proved in section 4 the existence of a solution having bounded flux, for which we give a uniqueness result in theorem 5.1 and corollary 5.2. The bound on the flux will be necessary to prove that the contraction property is also available in the neighborhood of the interface $\{x=0\}$. Then we show in theorem 5.4 the existence and uniqueness of the weak solution which is the limit of bounded flux solutions for any initial data u_0 with $0 \leq u_0 \leq 1$. Indeed, the set of initial data giving a bounded flux solution is dense in $L^\infty(\Omega)$ for the $L^1(\Omega)$ topology, and theorem 5.1 has for consequence that the contraction property can be extended to a larger class of solution, defined for all initial data in $L^\infty(\Omega)$. We unfortunately are not able to characterize them differently than by a limit of bounded flux solutions, and we can not either exhibit a weak solution which is not the limit of bounded flux solutions.

Theorem 5.1 (L^1-contraction principle for bounded flux solutions) *Let u, v be two weak solutions to the problem $(\mathcal{P})$ for the initial data u_0, v_0. Then, if $\partial_x\varphi_i(u_i)$ and $\partial_x\varphi_i(v_i)$ belong to $L^\infty(\mathcal{Q}_i)$, we have the following L^1-contraction principle: $\forall t \in [0,T]$,*

$$\sum_{i=1,2}\int_{\Omega_i}\phi_i\left(u(x,t)-v(x,t)\right)^{\pm}dx \leq \sum_{i=1,2}\int_{\Omega_i}\phi_i\left(u_0(x)-v_0(x)\right)^{\pm}dx. \tag{29}$$

The first part of this section is devoted to the proof of the theorem 5.1 which, with theorem 4.1, admits the following straightforward consequence:

Corollary 5.2 (Uniqueness of the bounded flux solution) *For all $u_0 \in L^\infty(-1,1)$ with $0 \leq u_0 \leq 1$, such that, for $i = 1,2$, $\varphi_i(u_0) \in W^{1,\infty}(\Omega_i)$, and $\tilde{\pi}_1(u_{0,1}) \cap \tilde{\pi}_2(u_{0,2}) \neq \emptyset$, there exists a unique weak solution to the problem $(\mathcal{P})$ in the sense of definition 2.1 and such that $\partial_x\varphi_i(u) \in L^\infty(\mathcal{Q}_i)$; moreover $u \in C([0,T], L^p(\Omega))$ for all $1 \leq p < +\infty$.*

Proof

The proof of the theorem 5.1 is based on entropy inequalities, obtained through the method of doubling variables, first introduced by S. Kružkov [18] for first order equations, and then adapted by J. Carrillo [11] for degenerate parabolic problems. Note that in the present setting, we only need doubling with respect to the time–variable, as it is done, for instance by F. Otto [26] for elliptic–parabolic problems (or in [6] for Stefan–type problems).

In the sequel of the proof, we will only give the comparison

$$\sum_{i=1,2}\int_{\Omega_i}\phi_i\left(u(x,t)-v(x,t)\right)^{+}dx \leq \sum_{i=1,2}\int_{\Omega_i}\phi_i\left(u_0(x)-v_0(x)\right)^{+}dx.$$

The comparison with $(\cdot)^-$ instead of $(\cdot)^+$ can be proven exactly the same way.

Let u be a bounded flux solution to the one-dimensional problem, i.e $\partial_x\varphi_i(u) \in L^\infty(\mathcal{Q}_i)$, $i = 1,2$. The weak formulation of definition 2.1 adapted to the one-dimensional framework of the section can be rewritten, for all $\psi \in \mathcal{D}(\overline{\Omega} \times [0,T[)$,

$$\begin{aligned}\int_0^T\sum_{i=1,2}\int_{\Omega_i}\phi_i u(x,t)\partial_t\psi(x,t)dxdt + \sum_{i=1,2}\int_{\Omega_i}\phi_i u_0(x)\psi(x,0)dx \\ -\int_0^T\sum_{i=1,2}\int_{\Omega_i}\partial_x\varphi_i(u)(x,t)\partial_x\psi(x,t)dxdt = 0\end{aligned} \tag{30}$$

This formulation clearly implies, for $i = 1,2$, for all $\psi \in C_c^\infty(\overline{\Omega}_i \times [0,T[)$ with $\psi(0,t) = 0$,

$$\begin{aligned}\int_0^T\int_{\Omega_i}\phi_i u(x,t)\partial_t\psi(x,t)dxdt + \int_{\Omega_i}\phi_i u_0(x)\psi(x,0)dx \\ -\int_0^T\int_{\Omega_i}\partial_x\varphi_i(u)(x,t)\partial_x\psi(x,t)dxdt = 0\end{aligned} \tag{31}$$

Classical computations (see e.g. [6, 11, 26]) on equation (31) lead to the following entropy inequalities: for all weak solutions u, v, for initial data u_0, v_0, for all $\xi \in \mathcal{D}^+(\overline{\Omega}_i \times$

$[0,T[\times[0,T[)$ such that $\xi(0,t,s)=0$,

$$\begin{aligned}\int_0^T\int_0^T\int_{\Omega_i}\phi_i(u(x,t)-v(x,s))^+(\partial_t\xi(x,t,s)+\partial_s\xi(x,t,s))dxdtds\\
+\int_0^T\int_{\Omega_i}\phi_i(u_0(x)-v(x,s))^+\xi(x,0,s)dxds\\
+\int_0^T\int_{\Omega_i}\phi_i(u(x,t)-v_0(x))^+\xi(x,t,0)dxdt\\
-\int_0^T\int_0^T\int_{\Omega_i}\partial_x(\varphi_i(u)(x,t)-\varphi_i(v)(x,s))^+\partial_x\xi(x,t,s)dxdtds\geq 0.\end{aligned}\tag{32}$$

Let us note here an important consequence of the entropy inequality (32) (and of the corresponding one for $(u-v)^-$), namely that u can be proved to satisfy

$$ess-\lim_{t\to 0}\int_{\Omega_i}|u(x,t)-u_0(x)|dx=0\,.\tag{33}$$

Indeed, this follows by taking v as a constant in (32) and using an approximation argument, see e.g. Lemma 7.41 in [25]. We deduce the time continuity at $t=0$ for any solution and in particular for both u and v taken above.

Now, let $\rho\in C_c^\infty(\mathbb{R},\mathbb{R}^+)$ with $supp(\rho)\subset[-1,1]$ and $\int_{\mathbb{R}}\rho(t)dt=1$. One denotes $\rho_m(t)=m\rho(mt)$. Let $\psi\in\mathcal{D}^+([-1,1]\times[0,T[)$ with $\psi(0,\cdot)=0$. For m large enough, $\xi(x,t,s)=\psi(x,t)\rho_m(t-s)$ belongs to $\mathcal{D}^+([-1,1]\times[0,T[\times[0,T[)$, and we can take it as test function in (32). Then summing on $i=1,2$ leads to

$$\begin{aligned}\int_0^T\int_0^T\sum_{i=1,2}\int_{\Omega_i}\phi_i(u(x,t)-v(x,s))^+\partial_t\psi(x,t)\rho_m(t-s)dxdtds\\
+\int_0^T\sum_{i=1,2}\int_{\Omega_i}\phi_i(u_0(x)-v(x,s))^+\psi(x,0)\rho_m(-s)dxds\\
+\int_0^T\sum_{i=1,2}\int_{\Omega_i}\phi_i(u(x,t)-v_0(x))^+\psi(x,t)\rho_m(t)dxdt\\
-\int_0^T\int_0^T\sum_{i=1,2}\int_{\Omega_i}\partial_x(\varphi_i(u)(x,t)-\varphi_i(v)(x,s))^+\partial_x\psi(x,t)\rho_m(t-s)dxdtds\geq 0.\end{aligned}\tag{34}$$

We can now let m tend toward $+\infty$ in (34), and using (33) for u and v, and the theorem of continuity in mean, we get: for all $\psi\in\mathcal{D}^+(\overline{\Omega}\times[0,T[)$ such that $\psi(0,t)=0$,

$$\begin{aligned}\int_0^T\sum_{i=1,2}\int_{\Omega_i}\phi_i(u(x,t)-v(x,t))^+\partial_t\psi(x,t)dxdt\\
+\sum_{i=1,2}\int_{\Omega_i}\phi_i(u_0(x)-v_0(x))^+\psi(x,0)dx\\
-\int_0^T\sum_{i=1,2}\int_{\Omega_i}\partial_x(\varphi_i(u)(x,t)-\varphi_i(v)(x,t))^+\partial_x\psi(x,t)dxdt\geq 0.\end{aligned}\tag{35}$$

We aim now to extend the inequality (35) in the case where $\psi(0,t)\neq 0$, and particularly in the case $\psi(x,t)=\theta(t)$, so that the third term disappears in (35).

To this purpose, let us set here $u_i(t) = u_i(0,t)$ to denote the trace of u_i at the interface Γ (and correspondingly, $v_i(t) = v_i(0,t)$). We introduce the subsets of $(0,T)$:

- $E_{u>v} = \{t \in [0,T] \mid u_1(t) > v_1(t) \text{ or } u_2(t) > v_2(t)\}$,
- $E_{u\le v} = \{t \in [0,T] \mid u_1(t) \le v_1(t) \text{ and } u_2(t) \le v_2(t)\}$,

so that $E_{u\le v}$ is the complement of $E_{u>v}$ in $[0,T]$.

For all $\varepsilon > 0$, one defines $\psi_\varepsilon(x) = \max\left(1 - \frac{|x|}{\varepsilon}, 0\right)$. For all $\theta \in \mathcal{D}^+([0,T[)$, we take $(x,t) \mapsto \theta(t)(1-\psi_\varepsilon(x))$ instead of $\psi(x,t)$ as test-function in (35), thus we get:

$$\int_0^T \sum_{i=1,2} \int_{\Omega_i} \phi_i(u(x,t) - v(x,t))^+ \partial_t\theta(t)(1-\psi_\varepsilon(x))dxdt + \sum_{i=1,2}\int_{\Omega_i} \phi_i(u_0(x) - v_0(x))^+(1-\psi_\varepsilon)(x)\theta(0)dx - \int_0^T \frac{\theta(t)}{\varepsilon}\begin{pmatrix} (\varphi_1(u)(-\varepsilon,t) - \varphi_1(v)(-\varepsilon,t))^+ - (\varphi_1(u_1)(t) - \varphi_1(v_1)(t))^+ \\ +(\varphi_2(u)(\varepsilon,t) - \varphi_2(v)(\varepsilon,t))^+ - (\varphi_2(u_2)(t) - \varphi_2(v_2)(t))^+ \end{pmatrix} dt \ge 0.$$

For almost every $t \in E_{u\le v}$, the function $(\varphi_i(u) - \varphi_i(v))^+(\cdot,t)$ admits a nil trace on $\{x=0\}$, thus the third term in the previous inequality can be reduced to the set $E_{u>v}$ obtaining

$$\int_0^T \sum_{i=1,2} \int_{\Omega_i} \phi_i(u(x,t) - v(x,t))^+ \partial_t\theta(t)(1-\psi_\varepsilon(x))dxdt + \sum_{i=1,2}\int_{\Omega_i} \phi_i(u_0(x) - v_0(x))^+(1-\psi_\varepsilon)(x)\theta(0)dx + \int_{E_{u>v}} \theta(t)\sum_{i=1,2}\int_{\Omega_i} \partial_x(\varphi_i(u)(x,t) - \varphi_i(v)(x,t))^+ \partial_x\psi_\varepsilon(x)dxdt \ge 0. \tag{36}$$

We show now the crucial point of the uniqueness proof, which is the subject of the following lemma.

Lemma 5.3 *For all $\theta \in \mathcal{D}^+([0,T[)$, if u, v are both bounded flux solutions, i.e. if one has $\partial_x\varphi_i(u), \partial_x\varphi_i(v) \in L^\infty(\mathcal{Q}_i)$ one has,*

$$\limsup_{\varepsilon\to 0} \int_{E_{u>v}} \theta(t)\sum_{i=1,2}\int_{\Omega_i} \partial_x(\varphi_i(u)(x,t) - \varphi_i(v)(x,t))^+ \partial_x\psi_\varepsilon(x)dxdt \le 0.$$

Using the weak formulation (30), we can claim that for any regular function $\vartheta \in \mathcal{D}([0,T[)$,

$$\lim_{\varepsilon\to 0}\int_0^T \vartheta(t)\sum_{i=1,2}\int_{\Omega_i} \partial_x(\varphi_i(u) - \varphi_i(v))\partial_x\psi_\varepsilon(x)dxdt = 0. \tag{37}$$

Since for $i = 1,2$, $\partial_x(\varphi_i(u) - \varphi_i(v))$ belongs to $L^\infty(\Omega_i \times (0,T))$, one has

$$\left|\int_0^T \vartheta(t)\sum_{i=1,2}\int_{\Omega_i} \partial_x(\varphi_i(u) - \varphi_i(v))\partial_x\psi_\varepsilon(x)dxdt\right| \le C\|\vartheta\|_{L^1(0,T)},$$

then a density argument allows us to claim that (37) still holds for any $\vartheta \in L^1(0,T)$, and particularly for $\vartheta(t) = \theta(t)1\!\!1_{\mathrm{E_{u>v}}}(\mathrm{t})$. Thus there exists $A(\varepsilon)$ tending to 0 as ε tends to 0 such that

$$\int_{E_{u>v}} \theta(t) \sum_{i=1,2} \int_{\Omega_i} \partial_x(\varphi_i(u)(x,t) - \varphi_i(v)(x,t))\partial_x \psi_\varepsilon(x) dxdt = A(\varepsilon). \tag{38}$$

Splitting up the positive and negative parts of $(\varphi_i(u)(x,t) - \varphi_i(v)(x,t))$, (38) becomes:

$$\begin{aligned} &\int_{E_{u>v}} \theta(t) \sum_{i=1,2} \int_{\Omega_i} \partial_x(\varphi_i(u)(x,t) - \varphi_i(v)(x,t))^+ \partial_x \psi_\varepsilon(x) dxdt \\ = &\int_{E_{u>v}} \theta(t) \sum_{i=1,2} \int_{\Omega_i} \partial_x(\varphi_i(u)(x,t) - \varphi_i(v)(x,t))^- \partial_x \psi_\varepsilon(x) dxdt + A(\varepsilon). \end{aligned} \tag{39}$$

It is at this point that we actually use the monotony of the transmission condition, i.e. condition 3 in Definition 2.1. Indeed, the conditions $\tilde{\pi}_1(u_1(t)) \cap \tilde{\pi}_2(u_2(t)) \neq \emptyset$ and $\tilde{\pi}_1(v_1(t)) \cap \tilde{\pi}_2(v_2(t)) \neq \emptyset$ insure that :

$$u_1 > v_1 \implies u_2 \geq v_2 \qquad \text{and} \qquad u_1 < v_1 \implies u_2 \leq v_2\,. \tag{40}$$

Therefore, recalling the definition of the set $E_{u>v}$ and of ψ_ε, the first term in the right member of (39) is non-positive, and then we conclude

$$\limsup_{\varepsilon \to 0} \int_{E_{u>v}} \theta(t) \sum_{i=1,2} \int_{\Omega_i} \partial_x(\varphi_i(u)(x,t) - \varphi_i(v)(x,t))^+ \partial_x \psi_\varepsilon(x) dxdt \leq 0.$$

This achieves the proof of lemma 5.3, and allows us to take the limit in inequality (36) for $\varepsilon \to 0$. Then for all $\psi \in \mathcal{D}^+([0,T[)$, one gets

$$-\int_0^T \sum_{i=1,2} \int_{\Omega_i} \phi_i(u(x,t) - v(x,t))^+ \partial_t \psi(t) dxdt \leq \sum_{i=1,2} \int_{\Omega_i} \phi_i(u_0(x) - v_0(x))^+ \psi(0) dx. \tag{41}$$

One can also prove exactly the same way that

$$-\int_0^T \sum_{i=1,2} \int_{\Omega_i} \phi_i(u(x,t) - v(x,t))^- \partial_t \psi(t) dxdt \leq \sum_{i=1,2} \int_{\Omega_i} \phi_i(u_0(x) - v_0(x))^- \psi(0) dx. \tag{42}$$

These inequalities still hold for $\psi = (T-t)$, and then if $u_0 = v_0$, one has $u = v$ almost everywhere in $\mathcal{Q}$. Moreover we can take $\psi(t) = 1\!\!1_{[0,\mathrm{s}]}(\mathrm{t})$ as test function in (41) to get the L^1-contraction principle (29) stated in theorem 5.1. ■

In the sequel, we prove that for any u_0 in $L^\infty(-1;1)$, $0 \leq u_0 \leq 1$, there exists a unique weak solution of problem $(\mathcal{P})$ which is the limit of a sequence of bounded flux solutions $(u_n)_n$, i.e. for all $n \geq 1$, $\partial_x \varphi_i(u_n) \in L^\infty(\mathcal{Q}_i)$.

Theorem 5.4 (Existence and uniqueness of the SOLA) *Let $u_0 \in L^\infty(-1,1)$, $0 \leq u_0 \leq 1$, and let $(u_{0,n})_{n\geq 1}$ be a sequence of bounded flux initial data, i.e. for all $n \geq 1$,*

- $0 \leq u_{0,n} \leq 1$,

- $\varphi_i(u_{0,n}) \in W^{1,\infty}(\Omega_i)$,
- $\tilde{\pi}_1(u_{0,n,1}) \cap \tilde{\pi}_2(u_{0,n,2}) \neq \emptyset$,

such that

$$\lim_{n\to+\infty} \|u_{0,n} - u_0\|_{L^1(\Omega)} = 0.$$

Let $(u_n)_{n\geq 1}$ be the sequence of the bounded flux solutions to the problem $(\mathcal{P})$ for $u_{0,n}$ as initial data. Then the sequence $(u_n)_{n\geq 1}$ converges toward u in $C(]0,T[, L^p(-1,1))$, $1 \leq p < +\infty$, where u is a solution to the problem $(\mathcal{P})$, called Solution Obtained as Limit of Approximation (SOLA). Furthermore, if u, v are two SOLAs, for initial data u_0, v_0, one has the following L^1-contraction principle: $\forall t \in [0,T]$,

$$\sum_{i=1}^{N} \int_{\Omega_i} \phi_i(u(x,t) - v(x,t))^{\pm} dx \leq \sum_{i=1}^{N} \int_{\Omega_i} \phi_i(u_0(x) - v_0(x))^{\pm} dx. \tag{43}$$

This particularly leads to the uniqueness of the SOLA.

Proof

Let $(u_{0,n})$ be a regular sequence of initial data converging toward u_0 in $L^1(-1,1)$ - one take e.g. $u_{0,n} \in C_c^\infty(]-1,0[\cup]0,1[)$. Then $(u_{0,n})$ is a Cauchy sequence, and thanks to (29), for all $t \in [0,T]$,

$$\sum_{i=1}^{N} \int_{\Omega_i} \phi_i |u_n(x,t) - u_m(x,t)| dx \leq \sum_{i=1}^{N} \int_{\Omega_i} \phi_i |u_{0,n}(x) - u_{0,m}(x)| dx.$$

Thus $(u_n)_n$ is a Cauchy sequence in $C([0,T]; L^1(\Omega))$ and converges to a function u in $C([0,T]; L^1(\Omega))$. Since $(u_n)_n$ is bounded in $L^\infty(\mathcal{Q})$, one has $u_n \to u$ in $C([0,T]; L^p(-1,1))$.

We now have to check that u is a weak solution to the problem $(\mathcal{P})$. It is easy to check, using to the L^∞-bound of u_n, that $\varphi_i(u_n)$ tends toward $\varphi_i(u)$ in $L^p(\Omega_i \times (0,T))$, for all $p \in [1, +\infty[$. Thanks to (18), the sequence $(\varphi_i(u_n))_n$ is bounded in $L^2(0,T; H^1(\Omega_i))$, and thus $\varphi_i(u_n) \to \varphi_i(u)$ weakly in $L^2(0,T; H^1(\Omega_i))$, and $\varphi_i(u_n)$ converges in $L^2(0,T; H^s(\Omega_i))$, for all $s \in]0,1[$, still toward $\varphi_i(u)$. Particularly, $u_{n,i}(t)$ tends toward $u_i(t)$. Since the set $\{(a,b) \in [0,1]^2 \mid \tilde{\pi}_1(a) \cap \tilde{\pi}_2(b) \neq \emptyset\}$ is closed, we can claim that

$$\tilde{\pi}_1(u_1(t)) \cap \tilde{\pi}_2(u_2(t)) \neq \emptyset \quad \text{for a.e. } t \in [0,T].$$

We can also pass to the limit in the weak formulation in order to conclude that u is a weak solution to the problem $(\mathcal{P})$, achieving this way the existence of a SOLA u.

Let now v be another SOLA, obtained through a sequence $(v_{0,n})_n$ of regular initial data converging toward v_0. Thanks to (29), one has,

$$\sum_{i=1}^{N} \int_{\Omega_i} \phi_i |u_n(x,t) - v_n(x,t)| dx \leq \sum_{i=1}^{N} \int_{\Omega_i} \phi_i |u_{0,n}(x) - v_{0,n}(x)| dx,$$

whose limit as n tends toward $+\infty$ gives the attempted L^1-contraction principle:

$$\sum_{i=1}^{N} \int_{\Omega_i} \phi_i |u(x,t) - v(x,t)| dx \leq \sum_{i=1}^{N} \int_{\Omega_i} \phi_i |u_0(x) - v_0(x)| dx,$$

and so the uniqueness of the SOLA, completing the proof of theorem 5.4. ■

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