A CONVERGENT FINITE ELEMENT-FINITE VOLUME SCHEME FOR THE COMPRESSIBLE STOKES PROBLEM PART II – THE ISENTROPIC CASE

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Abstract. In this paper, we propose a discretization for the (nonlinearized) compressible Stokes problem with an equation of state of the form $p = \rho^\gamma$ (where p stands for the pressure and ρ for the density). This scheme is based on Crouzeix-Raviart approximation spaces. The discretization of the momentum balance is obtained by the usual finite element technique. The discrete mass balance is obtained by a finite volume scheme, with an upwinding of the density, and two additional stabilization terms. We prove *a priori* estimates for the discrete solution, which yield its existence. Then the convergence of the scheme to a solution of the continuous problem is established. The passage to the limit in the equation of state requires the a.e. convergence of the density. It is obtained by adapting at the discrete level the "effective viscous pressure lemma" of the theory of compressible Navier-Stokes equations.

1. Introduction

Let Ω be a bounded open set of $\mathbb{R}^d$, $d = 2$ or 3, with a Lipschitz continuous boundary, and $\gamma > 1$. For $\boldsymbol{f} \in \mathrm{L}^2(\Omega)^d$ and $M > 0$, we consider the following problem:

$$-\Delta \boldsymbol{u} + \boldsymbol{\nabla} p = \boldsymbol{f} \text{ in } \Omega, \;\; \boldsymbol{u} = 0 \text{ on } \partial\Omega, \tag{1.1a}$$

$$\operatorname{div}(\rho \boldsymbol{u}) = 0 \;\text{ in } \Omega, \; \rho \geq 0 \;\text{ in } \Omega, \; \int_\Omega \rho(x)\, \mathrm{d}\boldsymbol{x} = M, \tag{1.1b}$$

$$p = \rho^\gamma \text{ in } \Omega. \tag{1.1c}$$

Definition 1.1. Let $\boldsymbol{f} \in \mathrm{L}^2(\Omega)^d$ and $M > 0$. A weak solution of Problem (1.1) is a function $(\boldsymbol{u}, p, \rho) \in \mathrm{H}^1_0(\Omega)^d \times \mathrm{L}^2(\Omega) \times \mathrm{L}^{2\gamma}(\Omega)$ satisfying:

$$\int_\Omega \boldsymbol{\nabla} \boldsymbol{u} : \boldsymbol{\nabla} \boldsymbol{v}\, \mathrm{d}\boldsymbol{x} - \int_\Omega p \operatorname{div}(\boldsymbol{v})\, \mathrm{d}\boldsymbol{x} = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}\, \mathrm{d}\boldsymbol{x} \text{ for all } \boldsymbol{v} \in (\mathrm{H}^1_0(\Omega))^d, \tag{1.2a}$$

$$\int \rho \boldsymbol{u} \cdot \boldsymbol{\nabla} \varphi\, \mathrm{d}\boldsymbol{x} = 0 \text{ for all } \varphi \in \mathrm{W}^{1,\infty}(\Omega), \tag{1.2b}$$

$$\rho \geq 0 \text{ a.e. in } \Omega, \; \int_\Omega \rho\, \mathrm{d}\boldsymbol{x} = M, \; p = \rho^\gamma \text{ a.e. in } \Omega. \tag{1.2c}$$

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The main objective of this paper is to present a numerical scheme for the computation of an approximate solution of Problem (1.1) and to prove the convergence (up to a subsequence, since, up to now, no uniqueness result is available for the solution of (1.1)) of this approximate solution towards a weak solution of (1.1) (*i.e.* a solution of (1.2)) as the mesh size goes to 0.

The proposed numerical scheme combines low order finite element and finite volume techniques, and is very close to a scheme which was implemented for the solution of barotropic Navier-Stokes equations in [6] and further extended to two-phase flows in [9]; the resulting code is today currently used at the French *Institut de Radioprotection et de Sûreté Nucléaire* (IRSN) for "real-life" studies in the nuclear safety field. Up to now, stability (in the sense of conservation of the entropy) is known for these schemes, and numerical experiments show convergence rates close to one in natural energy norms. Our goal is now to prove their convergence. This work is a step in this direction, and follows a previous analysis ([7], part I of the present paper) restricted to the linear equation of state $p = \rho$. The additional difficulty tackled here is to prove the a.e. convergence for the density, which necessitates to adapt P.L. Lions' "effective viscous pressure trick" [11] at the discrete level. Finally, for the sake of simplicity, we use here a simplified form of the diffusion term ($-\Delta \boldsymbol{u}$) but it is clear from the subsequent developments that the presented theory holds for any linear elliptic operator (and in particular for the usual form of the viscous term for compressible constant viscosity flows).

This paper is organized as follows. In Section 2, we present a simple way to prove a known preliminary result, namely the convergence (up to a subsequence) of the weak solution of (1.1) with $\boldsymbol{f}_n$ and M_n (instead of $\boldsymbol{f}$ and M) towards a weak solution of (1.1) as $n \to \infty$, assuming that $\boldsymbol{f}_n$ weakly converges to $\boldsymbol{f}$ in $(L^2(\Omega))^d$ and M_n converges to M in $\mathbb{R}$. Then, after introducing the discretization (Section 3) and the proposed scheme (Section 4), we adapt this proof in Section 5 to prove the convergence of the scheme to a weak solution of Problem (1.1). Finally, for the sake of completeness, Sections A and B gathers some simple proofs of known lemmas.

Remark 1.2 (Forcing term involving the density). Instead of taking a given function $\boldsymbol{f}$ in (1.1a), it is possible, in order to take the gravity effects into account, to take $\boldsymbol{f} = \rho\, \boldsymbol{g}$ with $\boldsymbol{g} \in \mathrm{L}^\infty(\Omega)^d$. The convergence results given below are still true.

In this paper, we use the following notations:

$$\text{if } d=2,\ \operatorname{curl}(\boldsymbol{v}) = \frac{\partial \boldsymbol{v}_2}{\partial \boldsymbol{x}_1} - \frac{\partial \boldsymbol{v}_1}{\partial \boldsymbol{x}_2},\ \text{and, if } d=3,\ \operatorname{curl}(\boldsymbol{v}) = \begin{bmatrix} \dfrac{\partial \boldsymbol{v}_3}{\partial \boldsymbol{x}_2} - \dfrac{\partial \boldsymbol{v}_2}{\partial \boldsymbol{x}_3} \\ \dfrac{\partial \boldsymbol{v}_1}{\partial \boldsymbol{x}_3} - \dfrac{\partial \boldsymbol{v}_3}{\partial \boldsymbol{x}_1} \\ \dfrac{\partial \boldsymbol{v}_2}{\partial \boldsymbol{x}_1} - \dfrac{\partial \boldsymbol{v}_1}{\partial \boldsymbol{x}_2} \end{bmatrix}.$$

where $\boldsymbol{v}$ is a vector-valued function. With these notations, if $\boldsymbol{v} \in \mathrm{H}^2(\Omega)^d$ and $\boldsymbol{w} \in \mathrm{H}^1(\Omega)^d$, the following identity holds:

(1.3)
$$\int_\Omega \boldsymbol{\nabla} \boldsymbol{v} : \boldsymbol{\nabla} \boldsymbol{w} \,\mathrm{d}\boldsymbol{x} = \int_\Omega \operatorname{curl}(\boldsymbol{v}) \cdot \operatorname{curl}(\boldsymbol{w}) \,\mathrm{d}\boldsymbol{x} + \int_\Omega \operatorname{div}(\boldsymbol{v})\ \operatorname{div}(\boldsymbol{w}) \,\mathrm{d}\boldsymbol{x}$$
$$+ \int_{\partial\Omega} (\boldsymbol{\nabla} \boldsymbol{v} \cdot \boldsymbol{n}) \cdot \boldsymbol{w} \,\mathrm{d}\gamma + \int_{\partial\Omega} \operatorname{curl}(\boldsymbol{v}) \cdot (\boldsymbol{w} \wedge \boldsymbol{n}) \,\mathrm{d}\gamma - \int_{\partial\Omega} \operatorname{div}(\boldsymbol{v})\ (\boldsymbol{w} \cdot \boldsymbol{n}) \,\mathrm{d}\gamma,$$

where, for any vector $\boldsymbol{w}$ and $\boldsymbol{n}$, we denote by $\boldsymbol{w} \wedge \boldsymbol{n}$:

$$\text{if } d = 2,\ \boldsymbol{w} \wedge \boldsymbol{n} = \boldsymbol{w}_1\, \boldsymbol{n}_2 - \boldsymbol{w}_2\, \boldsymbol{n}_1,\ \text{ and, if } d = 3,\ \boldsymbol{w} \wedge \boldsymbol{n} = \begin{bmatrix} \boldsymbol{w}_2\, \boldsymbol{n}_3 - \boldsymbol{w}_3\, \boldsymbol{n}_2 \\ \boldsymbol{w}_3\, \boldsymbol{n}_1 - \boldsymbol{w}_1\, \boldsymbol{n}_3 \\ \boldsymbol{w}_1\, \boldsymbol{n}_2 - \boldsymbol{w}_2\, \boldsymbol{n}_1 \end{bmatrix}.$$

The identity (1.3) is easily obtained by supposing that the functions $\boldsymbol{v}$ and $\boldsymbol{w}$ are regular, and then using a density argument. If $\boldsymbol{v} \in \mathrm{H}^1(\Omega)^d$ and $\boldsymbol{w} \in \mathrm{H}^1_0(\Omega)^d$, it boils down to:

$$\int_\Omega \boldsymbol{\nabla} \boldsymbol{v} : \boldsymbol{\nabla} \boldsymbol{w} \,\mathrm{d}\boldsymbol{x} = \int_\Omega \mathrm{curl}(\boldsymbol{v}) \cdot \mathrm{curl}(\boldsymbol{w}) \,\mathrm{d}\boldsymbol{x} + \int_\Omega \mathrm{div}(\boldsymbol{v})\ \mathrm{div}(\boldsymbol{w}) \,\mathrm{d}\boldsymbol{x}. \tag{1.4}$$

2. Continuity with respect to the data

We begin this section by a preliminary lemma, which is a simplified version of a result of the theory of renormalized solution of the transport equation [11]. The proof of this lemma is given in section A.

Lemma 2.1. *Let Ω be a bounded open set of $\mathbb{R}^d$ and $\gamma > 1$. Let $\rho \in \mathrm{L}^{2\gamma}(\Omega)$, $\rho \geq 0$ a.e. in Ω and $\boldsymbol{u} \in \mathrm{H}^1_0(\Omega)^d$. Assume that $(\rho, \boldsymbol{u})$ satisfies* (1.2b) *(which is the weak form of* $\mathrm{div}(\rho \boldsymbol{u}) = 0$*). Then, for all $\beta \in [1, \gamma]$,*

$$\int_\Omega \rho^\beta \mathrm{div}(\boldsymbol{u}) \,\mathrm{d}\boldsymbol{x} = 0. \tag{2.1}$$

We are now in position to prove the following result (which gives the continuity, up to a subsequence, of the weak solution of (1.1) with respect to the data).

Theorem 2.2. *Let $\boldsymbol{f} \in \mathrm{L}^2(\Omega)^d$, $M > 0$ and $(\boldsymbol{f}_n)_{n\in\mathbb{N}} \subset \mathrm{L}^2(\Omega)^d$, $(M_n)_{n\in\mathbb{N}} \subset \mathbb{R}^\star_+$ be two sequences satisfying $\boldsymbol{f}_n \to \boldsymbol{f}$ weakly in $\mathrm{L}^2(\Omega)^d$ and $M_n \to M$. For $n \in \mathbb{N}$, let $(\boldsymbol{u}_n, p_n, \rho_n)$ be a weak solution of* (1.1) *with $\boldsymbol{f}_n$ and M_n (instead of $\boldsymbol{f}$ and M). Then there exists $(\boldsymbol{u}, p, \rho)$ weak solution of* (1.1) *such that, up to a subsequence, as $n \to \infty$,*
- *$\boldsymbol{u}_n \to \boldsymbol{u}$ in $\mathrm{L}^2(\Omega)^d$ and weakly in $\mathrm{H}^1_0(\Omega)^d$,*
- *$p_n \to p$ in $\mathrm{L}^q(\Omega)$ for any $1 \leq q < 2$ and weakly in $\mathrm{L}^2(\Omega)$,*
- *$\rho_n \to \rho$ in $\mathrm{L}^q(\Omega)$ for any $1 \leq q < 2\gamma$ and weakly in $\mathrm{L}^{2\gamma}(\Omega)$.*

Remark 2.3. Theorem 2.2 is also true for $\gamma = 1$, at least with only a weak convergence of p_n and ρ_n. The proof of this result is simpler than the proof given below, since Step 4 is useless if $\gamma = 1$ for the passage to the limit in the equation of state.

Remark 2.4 (Forcing term involving the density). Theorem 2.2 is also true with $\boldsymbol{f}_n = \rho_n \boldsymbol{g}$ and $\boldsymbol{g} \in \mathrm{L}^\infty(\Omega)^d$ given (which correspond to gravity effect). Then, the limit of $\boldsymbol{f}_n$ is $\boldsymbol{f} = \rho \boldsymbol{g}$ and the fact that $\boldsymbol{f}_n$ converges to $\boldsymbol{f}$ in $\mathrm{L}^2(\Omega)^d$ is not a hypothesis but is proven. The difference in the proof is essentially in the derivation of the estimates, see Remark 2.5 below.

Proof. The proof of Theorem 2.2 is composed of four steps. In the first one, we obtain some estimates on $(\boldsymbol{u}_n, p_n, \rho_n)$. With these estimates we can assume the convergence, up to a subsequence, of $(\boldsymbol{u}_n, p_n, \rho_n)$ to some $(\boldsymbol{u}, p, \rho)$. Then, it is quite easy (Step 2) to prove that $(\boldsymbol{u}, p, \rho)$ satisfies (1.2a)-(1.2b) but it is not easy to prove that $p = \rho^\gamma$ (except for $\gamma = 1$) since, using the estimates of Step 1, the convergence of p_n and ρ_n is only weak. In Step 3, we prove the convergence of the integral of $p_n\rho_n$ to the integral of $p\rho$. This allows in Step 4 to obtain the "strong" convergence of ρ_n (or p_n) and to conclude the proof.

Step 1. Estimates.

The following equation is satisfied by $\boldsymbol{u}_n$ and p_n:

$$\int_\Omega \boldsymbol{\nabla}\boldsymbol{u}_n : \boldsymbol{\nabla}\boldsymbol{v} \,\mathrm{d}\boldsymbol{x} - \int_\Omega p_n \mathrm{div}(\boldsymbol{v}) \,\mathrm{d}\boldsymbol{x} = \int_\Omega \boldsymbol{f}_n \cdot \boldsymbol{v} \,\mathrm{d}\boldsymbol{x} \text{ for all } \boldsymbol{v} \in (\mathrm{H}^1_0(\Omega))^d. \tag{2.2}$$

Taking $\boldsymbol{v} = \boldsymbol{u}_n$ in this relation and noting that $p_n = \rho_n^\gamma$ a.e., we may use Lemma 2.1 (with ρ_n, $\boldsymbol{u}_n$ and $\beta = \gamma$) to obtain an estimate on $\boldsymbol{u}_n$ in $(\mathrm{H}^1_0(\Omega))^d$. Precisely speaking, there exists c_1, only depending on the L^2-bound of $(\boldsymbol{f}_n)_{n\in\mathbb{N}}$ and on Ω, such that:

$$\|\boldsymbol{u}_n\|_{\mathrm{H}^1(\Omega)^d} \le c_1. \tag{2.3}$$

In order to obtain a bound for p_n in $\mathrm{L}^2(\Omega)$, we now choose $\boldsymbol{v}_n$ given by Lemma B.7 with $q = p_n - m(p_n)$, where $m(p_n)$ is the mean value of p_n. Taking $\boldsymbol{v} = \boldsymbol{v}_n$ in (2.2) and using $\int_\Omega \mathrm{div}(\boldsymbol{v}_n) \,\mathrm{d}\boldsymbol{x} = 0$ gives:

$$\int_\Omega \big(p_n - m(p_n)\big)^2 \,\mathrm{d}\boldsymbol{x} = \int_\Omega (\boldsymbol{f}_n \cdot \boldsymbol{v}_n - \boldsymbol{\nabla}\boldsymbol{u}_n : \boldsymbol{\nabla}\boldsymbol{v}_n) \,\mathrm{d}\boldsymbol{x}.$$

Since $\|\boldsymbol{v}_n\|_{\mathrm{H}^1(\Omega)^d} \le c_2 \|p_n - m(p_n)\|_{\mathrm{L}^2(\Omega)}$ and $\|\boldsymbol{u}_n\|_{\mathrm{H}^1(\Omega)^d} \le c_1$, the preceding inequality leads to an estimate on $\|p_n - m(p_n)\|_{\mathrm{L}^2(\Omega)}$, *i.e.* the existence of c_3, only depending on Ω and the L^2-bound of $(\boldsymbol{f}_n)_{n\in\mathbb{N}}$, such that $\|p_n - m(p_n)\|_{\mathrm{L}^2(\Omega)} \le c_3$. We now remark that:

$$\int_\Omega p_n^{1/\gamma} \,\mathrm{d}\boldsymbol{x} = \int_\Omega \rho_n \,\mathrm{d}\boldsymbol{x} \le \sup\{M_k, k \in \mathbb{N}\}.$$

Then, using Lemma B.6 (with $q = 2$ and $r = 1/\gamma$), there exists c_4, only depending on the L^2-bound of $(\boldsymbol{f}_n)_{n\in\mathbb{N}}$, the bound of $(M_n)_{n\in\mathbb{N}}$, γ and Ω such that:

$$\|p_n\|_{\mathrm{L}^2(\Omega)} \le c_4. \tag{2.4}$$

Finally, thanks to $p_n = \rho_n^\gamma$ a.e. in Ω, we also have an estimate on ρ_n in $L^{2\gamma}$, namely:

$$\|\rho_n\|_{\mathrm{L}^{2\gamma}(\Omega)} \le c_5 = c_4^{1/\gamma}. \tag{2.5}$$

This concludes Step 1.

Remark 2.5 (Forcing term involving the density). In the case where $\boldsymbol{f}_n = \rho_n\, \boldsymbol{g}$, with $\boldsymbol{g} \in \mathrm{L}^\infty(\Omega)^d$, the estimates on $\boldsymbol{u}_n, p_n$ and ρ_n are not obtained with the preceding proof, since the hypotheses of the theorem do not give a direct bound in $\mathrm{L}^2(\Omega)^d$ of the sequence $(\boldsymbol{f}_n)_{n\in\mathbb{N}}$. In order to obtain the estimates on $\boldsymbol{u}_n, p_n$ and ρ_n, we proceed as follows. We first remark that:

$$\|\boldsymbol{u}_n\|^2_{\mathrm{H}^1(\Omega)^d} = \int_\Omega \rho_n\, \boldsymbol{g} \cdot \boldsymbol{u}_n \,\mathrm{d}\boldsymbol{x} \le \|\boldsymbol{u}_n\|_{\mathrm{L}^2(\Omega)^d} \, \|\boldsymbol{g}\|_{\mathrm{L}^\infty(\Omega)} \, \|\rho_n\|_{\mathrm{L}^2(\Omega)}.$$

Hölder's Inequality gives:

$$\|\rho_n\|_{\mathrm{L}^2(\Omega)} \le \|\rho_n\|^{1-\alpha}_{\mathrm{L}^1(\Omega)} \, \|\rho_n\|^{\alpha}_{\mathrm{L}^{2\gamma}(\Omega)} \le M_n^{1-\alpha} \, \|\rho_n\|^{\alpha}_{\mathrm{L}^{2\gamma}(\Omega)}$$

with $\alpha = \gamma/(2\gamma - 1)$. Then, there exists c_6 only depending on $\boldsymbol{g}$, γ, Ω and on the bound of $(M_n)_{n\in\mathbb{N}}$ such that:

$$\|\boldsymbol{u}_n\|_{\mathrm{H}^1(\Omega)^d} \le c_6 \left(\int_\Omega \rho_n^{2\gamma} \,\mathrm{d}\boldsymbol{x} \right)^{1/(2(2\gamma-1))} \le c_6 \, \|p_n\|_{\mathrm{L}^2(\Omega)}^{1/(2\gamma-1)} . \tag{2.6}$$

Taking $\boldsymbol{v} = \boldsymbol{v}_n$ in (2.2) with $\boldsymbol{v}_n$ given by Lemma B.7 with $q = p_n - m(p_n)$ and using $\int_\Omega \mathrm{div}(\boldsymbol{v}_n) \,\mathrm{d}\boldsymbol{x} = 0$ gives:

$$\int_\Omega \big(p_n - m(p_n)\big)^2 \,\mathrm{d}\boldsymbol{x} = \int_\Omega (\rho_n \boldsymbol{g} \cdot \boldsymbol{v}_n - \boldsymbol{\nabla} \boldsymbol{u}_n : \boldsymbol{\nabla} \boldsymbol{v}_n) \,\mathrm{d}\boldsymbol{x}.$$

Then, we deduce, using (2.6):

$$\int_\Omega \big(p_n - m(p_n)\big)^2 \,\mathrm{d}\boldsymbol{x} \le c_7 \, \|p_n\|_{\mathrm{L}^2(\Omega)}^{1+1/(2\gamma-1)} ,$$

with c_7 only depending on $\boldsymbol{g}$, γ, Ω and on the bound of $(M_n)_{n\in\mathbb{N}}$. This inequality, together with the fact that $(2\gamma - 1) > 1$ and that $\int_\Omega p_n^{1/\gamma} \,\mathrm{d}\boldsymbol{x} = M_n$, leads to an L^2-bound on p_n and therefore to an $\mathrm{L}^{2\gamma}$-bound on ρ_n and, with (2.6), to an H^1_0-bound on $\boldsymbol{u}_n$.

Step 2. Passing to the limit on the equations (1.1a) **and** (1.1b)**.**

Thanks to the estimates obtained in Step 1, it is possible to assume (up to a subsequence) that, as $n \to \infty$:

$$\begin{aligned} &\boldsymbol{u}_n \to \boldsymbol{u} \text{ in } \mathrm{L}^2(\Omega)^d \text{ and weakly in } \mathrm{H}^1_0(\Omega)^d, \\ &p_n \to p \text{ weakly in } \mathrm{L}^2(\Omega), \\ &\rho_n \to \rho \text{ weakly in } \mathrm{L}^{2\gamma}(\Omega). \end{aligned}$$

Passing to the limit in the first equation satisfied by $(\boldsymbol{u}_n, p_n)$, we obtain that $(\boldsymbol{u}, p)$ is a solution to (1.2a).
Since $\boldsymbol{u}_n \to \boldsymbol{u}$ in $\mathrm{L}^2(\Omega)^d$ and $\rho_n \to \rho$ weakly in $\mathrm{L}^2(\Omega)$, we have $\rho_n u_n \to \rho u$ weakly in $\mathrm{L}^1(\Omega)^d$. Then (ρ, u) is solution of (1.2b).
The weak convergence of ρ_n to ρ and the fact that $\rho_n \ge 0$ a.e. in Ω gives that $\rho \ge 0$ a.e. in Ω (indeed, taking $\psi = 1_{\rho<0}$ as test function gives $\int_\Omega \rho\psi \,\mathrm{d}\boldsymbol{x} = \lim_{n\to\infty} \int_\Omega \rho_n \psi \,\mathrm{d}\boldsymbol{x} \ge 0$, which proves that $\rho\psi = 0$ a.e.). The weak convergence of ρ_n to ρ also gives (taking now $\psi = 1$ as test function) that $\int_\Omega \rho \,\mathrm{d}\boldsymbol{x} = M$.
Then (1.2c) is proven except for the fact that $p = \rho^\gamma$ a.e. in Ω. This is the objective of the last two steps, where we also prove the strong convergence of ρ_n and p_n.

Step 3. Proving $\displaystyle\int_\Omega \rho_n p_n \,\mathrm{d}\boldsymbol{x} \to \int_\Omega \rho p \,\mathrm{d}\boldsymbol{x}$**.**

Since the sequence $(\rho_n)_{n\in\mathbb{N}}$ is bounded in $\mathrm{L}^2(\Omega)$, Lemma B.8 gives the existence of a bounded sequence $(\boldsymbol{v}_n)_{n\in\mathbb{N}}$ in $\mathrm{H}^1(\Omega)^d$ such that $\mathrm{div}(\boldsymbol{v}_n) = \rho_n$ and $\mathrm{curl}(\boldsymbol{v}_n) = 0$. It is possible to assume (up to a subsequence) that $\boldsymbol{v}_n \to \boldsymbol{v}$ in $\mathrm{L}^2(\Omega)^d$ and weakly in $\mathrm{H}^1(\Omega)^d$. Passing to the limit gives $\mathrm{div}(\boldsymbol{v}) = \rho$ and $\mathrm{curl}(\boldsymbol{v}) = 0$.

Let $\varphi \in C_c^\infty(\Omega)$ (so that $\boldsymbol{v}_n \varphi \in \mathrm{H}^1_0(\Omega)^d$). Taking $\boldsymbol{v} = \boldsymbol{v}_n \varphi$ in (2.2) leads to:

$$\int_\Omega \boldsymbol{\nabla} \boldsymbol{u}_n : \boldsymbol{\nabla}(\boldsymbol{v}_n \varphi) \,\mathrm{d}\boldsymbol{x} - \int_\Omega p_n \,\mathrm{div}(\boldsymbol{v}_n \varphi) \,\mathrm{d}\boldsymbol{x} = \int_\Omega \boldsymbol{f}_n \cdot (\boldsymbol{v}_n \varphi) \,\mathrm{d}\boldsymbol{x}.$$

From identity (1.4), we thus get:

$$\int_\Omega \operatorname{div}(\boldsymbol{u}_n)\ \operatorname{div}(\boldsymbol{v}_n\varphi)\,\mathrm{d}\boldsymbol{x} + \int_\Omega \operatorname{curl}(\boldsymbol{u}_n)\cdot\operatorname{curl}(\boldsymbol{v}_n\varphi)\,\mathrm{d}\boldsymbol{x} \\ - \int_\Omega p_n \operatorname{div}(\boldsymbol{v}_n\varphi)\,\mathrm{d}\boldsymbol{x} = \int_\Omega \boldsymbol{f}_n\cdot(\boldsymbol{v}_n\varphi)\,\mathrm{d}\boldsymbol{x}.$$

The choice of $\boldsymbol{v}_n$ gives $\operatorname{div}(\boldsymbol{v}_n\varphi) = \rho_n\varphi + \boldsymbol{v}_n\cdot\boldsymbol{\nabla}\varphi$ and $\operatorname{curl}(\boldsymbol{v}_n\varphi) = L(\varphi)\,\boldsymbol{v}_n$, where $L(\varphi)$ is a matrix with entries involving the first order derivatives of φ. Then, the preceding equality yields:

$$\int_\Omega \big(\operatorname{div}(\boldsymbol{u}_n) - p_n\big)\,\rho_n\,\varphi\,\mathrm{d}\boldsymbol{x} + \int_\Omega \operatorname{div}(\boldsymbol{u}_n)\,\boldsymbol{v}_n\cdot\boldsymbol{\nabla}\varphi\,\mathrm{d}\boldsymbol{x} \\ + \int \operatorname{curl}(\boldsymbol{u}_n)\cdot L(\varphi)\,\boldsymbol{v}_n\,\mathrm{d}\boldsymbol{x} - \int_\Omega p_n\boldsymbol{v}_n\cdot\boldsymbol{\nabla}\varphi\,\mathrm{d}\boldsymbol{x} = \int_\Omega \boldsymbol{f}_n\cdot(\boldsymbol{v}_n\varphi)\,\mathrm{d}\boldsymbol{x}.$$

Thanks to the weak convergence of $\boldsymbol{u}_n$ in $\mathrm{H}^1_0(\Omega)^d$ to $\boldsymbol{u}$, the weak convergence of p_n in $\mathrm{L}^2(\Omega)$ to p, the weak convergence of $\boldsymbol{f}_n$ in $\mathrm{L}^2(\Omega)^d$ to $\boldsymbol{f}$ and the convergence of $\boldsymbol{v}_n$ in $\mathrm{L}^2(\Omega)^d$ to $\boldsymbol{v}$, we obtain:

$$\lim_{n\to\infty}\int_\Omega \big(\operatorname{div}(\boldsymbol{u}_n) - p_n\big)\,\rho_n\,\varphi\,\mathrm{d}\boldsymbol{x} = \int_\Omega \boldsymbol{f}\cdot(\boldsymbol{v}\varphi)\,\mathrm{d}\boldsymbol{x} - \int_\Omega \operatorname{div}(\boldsymbol{u})\,\boldsymbol{v}\cdot\boldsymbol{\nabla}\varphi\,\mathrm{d}\boldsymbol{x} \\ - \int_\Omega \operatorname{curl}(\boldsymbol{u})\cdot L(\varphi)\,\boldsymbol{v}\,\mathrm{d}\boldsymbol{x} + \int_\Omega p\,\boldsymbol{v}\cdot\boldsymbol{\nabla}\varphi\,\mathrm{d}\boldsymbol{x}. \tag{2.7}$$

But, thanks to Step 2, $(\boldsymbol{u},p)$ satisfies (1.2a), and thus:

$$\int_\Omega \boldsymbol{\nabla}\boldsymbol{u} : \boldsymbol{\nabla}(\boldsymbol{v}\varphi)\,\mathrm{d}\boldsymbol{x} - \int_\Omega p\ \operatorname{div}(\boldsymbol{v}\varphi)\,\mathrm{d}\boldsymbol{x} = \int_\Omega \boldsymbol{f}\cdot(\boldsymbol{v}\varphi)\,\mathrm{d}\boldsymbol{x},$$

or equivalently:

$$\int_\Omega \operatorname{div}(\boldsymbol{u})\ \operatorname{div}(\boldsymbol{v}\varphi)\,\mathrm{d}\boldsymbol{x} + \int_\Omega \operatorname{curl}(\boldsymbol{u})\cdot\operatorname{curl}(\boldsymbol{v}\varphi)\,\mathrm{d}\boldsymbol{x} - \int_\Omega p\ \operatorname{div}(\boldsymbol{v}\varphi)\,\mathrm{d}\boldsymbol{x} = \int_\Omega \boldsymbol{f}\cdot(\boldsymbol{v}\varphi)\,\mathrm{d}\boldsymbol{x},$$

which gives (using $\operatorname{div}(\boldsymbol{v}) = \rho$ and $\operatorname{curl}(\boldsymbol{v}) = 0$):

$$\int_\Omega (\operatorname{div}(\boldsymbol{u}) - p)\,\rho\,\varphi\,\mathrm{d}\boldsymbol{x} + \int_\Omega \operatorname{div}(\boldsymbol{u})\,\boldsymbol{v}\cdot\boldsymbol{\nabla}\varphi\,\mathrm{d}\boldsymbol{x} + \int_\Omega \operatorname{curl}(\boldsymbol{u})\cdot L(\varphi)\,\boldsymbol{v}\,\mathrm{d}\boldsymbol{x} \\ - \int_\Omega p\,\boldsymbol{v}\cdot\boldsymbol{\nabla}\varphi\,\mathrm{d}\boldsymbol{x} = \int_\Omega \boldsymbol{f}\cdot(\boldsymbol{v}\varphi)\,\mathrm{d}\boldsymbol{x}.$$

Then, with (2.7), we obtain:

$$\lim_{n\to\infty}\int_\Omega \big(p_n - \operatorname{div}(\boldsymbol{u}_n)\big)\,\rho_n\,\varphi\,\mathrm{d}\boldsymbol{x} = \int_\Omega \big(p - \operatorname{div}(\boldsymbol{u})\big)\,\rho\,\varphi\,\mathrm{d}\boldsymbol{x}. \tag{2.8}$$

In (2.8), the function φ is an arbitrary element of $\mathrm{C}^\infty_c(\Omega)$. We are going to prove now that it is possible to take $\varphi = 1$ in this relation. To this goal, we first remark that, thanks to $\gamma > 1$, the sequence $((p_n - \operatorname{div}(\boldsymbol{u}_n))\,\rho_n)_{n\in\mathbb{N}}$ is equi-integrable (see Definition B.1). A simple proof of this assertion is obtained using the fact that $(p_n - \operatorname{div}(\boldsymbol{u}_n))_{n\in\mathbb{N}}$ is bounded in $\mathrm{L}^2(\Omega)$, $(\rho_n)_{n\in\mathbb{N}}$ is bounded in $\mathrm{L}^{2\gamma}(\Omega)$ and the following inequality holds for any Borelian subset A of Ω:

$$\int_A |\big(p_n - \operatorname{div}(\boldsymbol{u}_n)\big)\,\rho_n|\,\mathrm{d}\boldsymbol{x} \le\ \|p_n - \operatorname{div}(\boldsymbol{u}_n)\|_{\mathrm{L}^2(\Omega)}\ \|\rho_n\|_{\mathrm{L}^{2\gamma}(\Omega)}\ |A|^{1/r},$$

with $\dfrac{1}{2}+\dfrac{1}{2\gamma}+\dfrac{1}{r}=1$ and where $|A|$ stands for the measure of A.
Then, using Lemma B.2, we get:

$$\lim_{n\to\infty}\int_\Omega \big(p_n-\operatorname{div}(\boldsymbol{u}_n)\big)\,\rho_n\,\mathrm{d}\boldsymbol{x}=\int_\Omega \big(p-\operatorname{div}(\boldsymbol{u})\big)\,\rho\,\mathrm{d}\boldsymbol{x}. \tag{2.9}$$

In order to conclude Step 3, it remains to use Lemma 2.1 (with $\beta=1$), which is possible since $\operatorname{div}(\rho_n\boldsymbol{u}_n)=\operatorname{div}(\rho\boldsymbol{u})=0$. It gives:

$$\int_\Omega \rho_n\operatorname{div}(\boldsymbol{u}_n)\,\mathrm{d}\boldsymbol{x}=\int_\Omega \rho\operatorname{div}(\boldsymbol{u})\,\mathrm{d}\boldsymbol{x}=0.$$

Then, (2.9) yields:

$$\lim_{n\to\infty}\int_\Omega p_n\,\rho_n\,\mathrm{d}\boldsymbol{x}=\int_\Omega p\,\rho\,\mathrm{d}\boldsymbol{x}. \tag{2.10}$$

Remark 2.6. For Step 4, the equality in (2.10) is not necessary. It would be sufficient to have:

$$\liminf_{n\to\infty}\int_\Omega p_n\,\rho_n\,\mathrm{d}\boldsymbol{x}\le\int_\Omega p\,\rho\,\mathrm{d}\boldsymbol{x},$$

and thus, instead of $\int_\Omega \rho_n\operatorname{div}(\boldsymbol{u}_n)\,\mathrm{d}\boldsymbol{x}=0$:

$$\liminf_{n\to\infty}\int_\Omega \rho_n\operatorname{div}(\boldsymbol{u}_n)\,\mathrm{d}\boldsymbol{x}\le 0.$$

Step 4. Passing to the limit on the EOS and strong convergence of ρ_n and p_n.

For $n\in\mathbb{N}$, let $G_n=(\rho_n^\gamma-\rho^\gamma)(\rho_n-\rho)$. For all $n\in\mathbb{N}$, the function G_n belongs to $L^1(\Omega)$ and $G_n\ge 0$ a.e. in Ω. Futhermore $G_n=(p_n-\rho^\gamma)(\rho_n-\rho)=p_n\rho_n-p_n\rho-\rho^\gamma\rho_n+\rho^\gamma\rho$ and:

$$\int_\Omega G_n\,\mathrm{d}\boldsymbol{x}=\int_\Omega p_n\rho_n\,\mathrm{d}\boldsymbol{x}-\int_\Omega p_n\rho\,\mathrm{d}\boldsymbol{x}-\int_\Omega \rho^\gamma\rho_n\,\mathrm{d}\boldsymbol{x}+\int_\Omega \rho^\gamma\rho\,\mathrm{d}\boldsymbol{x}.$$

Using the weak convergence in $L^2(\Omega)$ of p_n to p and of ρ_n to ρ, the fact that $\rho,\rho^\gamma\in L^2(\Omega)$ and (2.10) gives:

$$\lim_{n\to\infty}\int_\Omega G_n\,\mathrm{d}\boldsymbol{x}=0,$$

that is $G_n\to 0$ in $L^1(\Omega)$. Then, up to a subsequence, we have $G_n\to 0$ a.e. in Ω. Since $y\mapsto y^\gamma$ is an increasing function on $\mathbb{R}_+$, we then deduce that $\rho_n\to\rho$ a.e., as $n\to\infty$. Then, we also have $p_n=\rho_n^\gamma\to\rho^\gamma$ a.e.. Since $(\rho_n)_{n\in\mathbb{N}}$ is bounded in $\mathrm{L}^{2\gamma}(\Omega)$ and $(p_n)_{n\in\mathbb{N}}$ is bounded in $\mathrm{L}^2(\Omega)$, Lemma B.9 (which is classical) gives, as $n\to\infty$:

$$\rho_n\to\rho \text{ in } \mathrm{L}^q(\Omega) \text{ for all } 1\le q<2\gamma,$$
$$p_n\to\rho^\gamma \text{ in } L^q(\Omega) \text{ for all } 1\le q<2.$$

Since we already know that $p_n\to p$ weakly in $\mathrm{L}^2(\Omega)$, we necessarily have (by uniqueness of the weak limit in $\mathrm{L}^q(\Omega)$) that $p=\rho^\gamma$ a.e. in Ω. The proof of Theorem 2.2 is now complete. □

3. Discrete spaces and relevant lemmata

From now on, we suppose that the computational domain Ω is polygonal ($d = 2$) or polyhedral ($d = 3$). Let $\mathcal{T}$ be a decomposition of the domain Ω in simplices, which we call herafter a triangulation of Ω, regardless of the space dimension. By $\mathcal{E}(K)$, we denote the set of the edges ($d = 2$) or faces ($d = 3$) σ of the element $K \in \mathcal{T}$; for short, each edge or face will be called an edge hereafter. The set of all edges of the mesh is denoted by $\mathcal{E}$; the set of edges included in the boundary of Ω is denoted by $\mathcal{E}_{\rm ext}$ and the set of internal edges (*i.e.* $\mathcal{E} \setminus \mathcal{E}_{\rm ext}$) is denoted by $\mathcal{E}_{\rm int}$. The decomposition $\mathcal{T}$ is assumed to be regular in the usual sense of the finite element literature (*e.g.* [2]), and, in particular, $\mathcal{T}$ satisfies the following properties: $\bar\Omega = \bigcup_{K\in\mathcal{T}} \bar K$; if $K, L \in \mathcal{T}$, then $\bar K \cap \bar L = \emptyset$, $\bar K \cap \bar L$ is a vertex or $\bar K \cap \bar L$ is a common edge of K and L, which is denoted by $K|L$. For each internal edge of the mesh $\sigma = K|L$, $\boldsymbol{n}_{KL}$ stands for the normal vector of σ, oriented from K to L (so that $\boldsymbol{n}_{KL} = -\boldsymbol{n}_{LK}$). By $|K|$ and $|\sigma|$ we denote the (d and $d-1$ dimensional) measure, respectively, of an element K and of an edge σ, and h_K and h_σ stand for the diameter of K and σ, respectively. We measure the regularity of the mesh through the parameter θ defined by:

$$\theta = \inf \{ \frac{\xi_K}{h_K},\ K \in \mathcal{T} \} \tag{3.1}$$

where ξ_K stands for the diameter of the largest ball included in K. Note that, $\forall \sigma \in \mathcal{E}_{\rm int}$, $\sigma = K|L$, we have $h_\sigma \geq \xi_K \geq \theta\, h_K$ and $h_\sigma \leq h_L$ and so $\theta\, h_K \leq h_L \leq \theta^{-1} h_K$. Note also that, $\forall K \in \mathcal{T}$, $\forall \sigma \in \mathcal{E}(K)$, the inequality $h_\sigma\, |\sigma| \leq 2\ \theta^{-d}\, |K|$ holds [7, relation (2.2)]. These relations will be used throughout this paper. Finally, as usual, we denote by h the quantity $\max_{K\in\mathcal{T}} h_K$.

The space discretization relies on the Crouzeix-Raviart element (see [3] for the seminal paper and, for instance, [4, pp. 199–201] for a synthetic presentation). The reference element is the unit d-simplex and the discrete functional space is the space P_1 of affine polynomials. The degrees of freedom are determined by the following set of edge functionals:

$$\{F_\sigma,\ \sigma \in \mathcal{E}(K)\}, \qquad F_\sigma(v) = |\sigma|^{-1} \int_\sigma v \,{\rm d}\gamma. \tag{3.2}$$

The mapping from the reference element to the actual one is the standard affine mapping. Finally, the continuity of the average value of a discrete functions v across each edge of the mesh, $F_\sigma(v)$, is required, thus the discrete space V_h is defined as follows:

$$\begin{aligned} V_h = \{\, v \in L^2(\Omega)\ :\ & \forall K \in \mathcal{T},\ v|_K \in P_1(K)\,; \\ & \forall \sigma \in \mathcal{E}_{\rm int},\ \sigma = K|L,\ \ F_\sigma(v|_K) = F_\sigma(v|_L);\ \forall \sigma \in \mathcal{E}_{\rm ext},\ F_\sigma(v) = 0\}. \end{aligned} \tag{3.3}$$

The space of approximation for the velocity is the space $\boldsymbol{W}_h$ of vector-valued functions each component of which belongs to V_h: $\boldsymbol{W}_h = (V_h)^d$. The pressure is approximated by the space L_h of piecewise constant functions:

$$L_h = \left\{ q \in L^2(\Omega)\ :\ q|_K = \text{ constant}, \forall K \in \mathcal{T} \right\}.$$

Since only the continuity of the integral over each edge of the mesh is imposed, the functions of V_h are discontinuous through each edge; the discretization is thus nonconforming in $H^1(\Omega)^d$. We then define, for $1 \leq i \leq d$ and $u \in V_h$, $\partial_{h,i}\, u$ as the function of $\mathrm{L}^2(\Omega)$ which is equal to the derivative of u with respect to the

i^{th} space variable almost everywhere. This notation allows to define the discrete gradient, denoted by $\boldsymbol{\nabla}_h$, for both scalar and vector-valued discrete functions and the discrete divergence of vector-valued discrete functions, denoted by div_h.

The Crouzeix-Raviart pair of approximation spaces for the velocity and the pressure is *inf-sup* stable, in the usual sense for "piecewise H^1" discrete velocities, *i.e.* there exists $c_{\mathrm{i}} > 0$ only depending on Ω and, in a non-increasing way, on θ, such that:

$$\forall p \in L_h, \qquad \sup_{\boldsymbol{v} \in \boldsymbol{W}_h} \frac{\displaystyle\int_\Omega p \, \mathrm{div}_h(\boldsymbol{v}) \, \mathrm{d}\boldsymbol{x}}{\|\boldsymbol{v}\|_{1,b}} \geq c_{\mathrm{i}} \, \|p - m(p)\|_{\mathrm{L}^2(\Omega)} \, ,$$

where $m(p)$ is the mean value of p over Ω and $\|\cdot\|_{1,b}$ stands for the broken Sobolev H^1 semi-norm, which is defined for scalar as well as for vector-valued functions by:

$$\|v\|_{1,b}^2 = \sum_{K \in \mathcal{T}} \int_K |\boldsymbol{\nabla} v|^2 \, \mathrm{d}\boldsymbol{x} = \int_\Omega |\boldsymbol{\nabla}_h v|^2 \, \mathrm{d}\boldsymbol{x}.$$

This norm is known to control the L^2 norm by a Poincaré inequality (*e.g.* [4, lemma 3.31]). We also define a discrete semi-norm on L_h, similar to the usual H^1 semi-norm used in the finite volume context:

$$\forall p \in L_h, \qquad |p|_{\mathcal{T}}^2 = \sum_{\substack{\sigma \in \mathcal{E}_{\mathrm{int}},\\ \sigma = K|L}} \frac{|\sigma|}{h_\sigma} \, (p_K - p_L)^2.$$

From the definition (3.2), each velocity degree of freedom may be indexed by the number of the component and the associated edge, thus the set of velocity degrees of freedom reads:

$$\{v_{\sigma,i}, \ \sigma \in \mathcal{E}_{\mathrm{int}}, \ 1 \leq i \leq d\}$$

We denote by φ_σ the usual Crouzeix-Raviart shape function associated to σ, *i.e.* the scalar function of V_h such that $F_\sigma(\varphi_\sigma) = 1$ and $F_{\sigma'}(\varphi_\sigma) = 0, \ \forall \sigma' \in \mathcal{E} \setminus \{\sigma\}$.

Similarly, each degree of freedom for the pressure is associated to a cell K, and the set of pressure degrees of freedom is denoted by $\{p_K, \ K \in \mathcal{T}\}$.

We define by r_h the following interpolation operator:

$$r_h : \left| \begin{array}{lll} \mathrm{H}_0^1(\Omega) & \longrightarrow & V_h \\ u & \longmapsto & r_h u = \displaystyle\sum_{\sigma \in \mathcal{E}} F_\sigma(u) \, \varphi_\sigma = \sum_{\sigma \in \mathcal{E}} |\sigma|^{-1} \left(\int_\sigma v \, \mathrm{d}\gamma \right) \varphi_\sigma . \end{array} \right. \tag{3.4}$$

This operator naturally extends to vector-valued functions (*i.e.* to perform the interpolation from $\mathrm{H}_0^1(\Omega)^d$ to $\boldsymbol{W}_h$), and we keep the same notation r_h for both the scalar and vector case. The properties of r_h are gathered in the following lemma. They are proven in [3].

Theorem 3.1. *Let $\theta_0 > 0$ and let $\mathcal{T}$ be a triangulation of the computational domain Ω such that $\theta \geq \theta_0$, where θ is defined by* (3.1)*. The interpolation operator r_h enjoys the following properties:*

(1) *preservation of the divergence:*

$$\forall \boldsymbol{v} \in \mathrm{H}_0^1(\Omega)^d, \ \forall q \in L_h, \qquad \int_\Omega q \, \mathrm{div}_h(r_h \boldsymbol{v}) \, \mathrm{d}\boldsymbol{x} = \int_\Omega q \, \mathrm{div}(\boldsymbol{v}) \, \mathrm{d}\boldsymbol{x},$$

(2) *stability:*

$$\forall v \in \mathrm{H}^1_0(\Omega), \qquad \|r_h v\|_{1,b} \le c_1(\theta_0)\, |v|_{\mathrm{H}^1(\Omega)} ,$$

(3) *approximation properties:*

$$\forall v \in \mathrm{H}^2(\Omega) \cap \mathrm{H}^1_0(\Omega),\ \forall K \in \mathcal{T},$$
$$\|v - r_h v\|_{\mathrm{L}^2(K)} + h_K \, \|\boldsymbol{\nabla}_h (v - r_h v)\|_{\mathrm{L}^2(K)} \le c_2(\theta_0)\, h_K^2 \, |v|_{\mathrm{H}^2(K)} .$$

In both above inequalities, the notation $c_i(\theta_0)$ means that the real number c_i only depends on θ_0 and Ω, and, in particular, does not depend on the parameter h characterizing the size of the cells; this notation will be kept throughout the paper.

The following compactness result was proven in [7, Theorem 3.3].

Theorem 3.2. *Let $(v^{(n)})_{n\in\mathbb{N}}$ be a sequence of functions satisfying the following assumptions:*

1. *$\forall n \in \mathbb{N}$, there exists a triangulation of the domain $\mathcal{T}^{(n)}$ such that $v^{(n)} \in V_h^{(n)}$, where $V_h^{(n)}$ is the space of Crouzeix-Raviart discrete functions associated to $\mathcal{T}^{(n)}$, as defined by (3.3), and the parameter $\theta^{(n)}$ characterizing the regularity of $\mathcal{T}^{(n)}$ is bounded away from zero independently of n,*
2. *the sequence $(v^{(n)})_{n\in\mathbb{N}}$ is uniformly bounded with respect to the broken Sobolev H^1 semi-norm,* i.e.*:*

$$\forall n \in \mathbb{N}, \qquad \|v^{(n)}\|_{1,b} \le C,$$

where C is a constant real number and $\|\cdot\|_{1,b}$ stands for the broken Sobolev H^1 semi-norm associated to $\mathcal{T}^{(n)}$ (with a slight abuse of notation, namely dropping, for short, the index $^{(n)}$ pointing the dependence of the norm with respect to the mesh).

Then, when $n \to \infty$, possibly up to the extraction of a subsequence, the sequence $(v^{(n)})_{n\in\mathbb{N}}$ converges (strongly) in $\mathrm{L}^2(\Omega)$ to a limit $\bar v$ such that $\bar v \in \mathrm{H}^1_0(\Omega)$.

Finally, the following technical lemma, together with its proof, can be found in [7, lemma 2.4].

Lemma 3.3. *Let $\theta_0 > 0$ and let $\mathcal{T}$ be a triangulation of the computational domain Ω such that $\theta \ge \theta_0$, where θ is defined by (3.1); let $(a_\sigma)_{\sigma\in\mathcal{E}_{\rm int}}$ be a family of real numbers such that $\forall \sigma \in \mathcal{E}_{\rm int}$, $|a_\sigma| \le 1$ and let v be a function of the Crouzeix-Raviart space V_h associated to $\mathcal{T}$. Then the following bound holds:*

$$\sum_{\sigma\in\mathcal{E}_{\rm int}} \left| \int_\sigma a_\sigma \ [v] \, f \, \mathrm{d}\gamma \right| \le c(\theta_0)\, h\, \|v\|_{1,b} \ |f|_{\mathrm{H}^1(\Omega)} ,\ \forall f \in \mathrm{H}^1_0(\Omega).$$

where $[v]$ stands for the jump of the function v across the edge.

4. The numerical scheme

Let ρ^* be the mean density, *i.e.* $\rho^* = M/|\Omega|$ where $|\Omega|$ stands for the measure of the domain Ω. We consider the following numerical scheme for the discretization of Problem (1.1):

$$\forall \boldsymbol{v} \in \boldsymbol{W}_h, \qquad \int_\Omega \boldsymbol{\nabla}_h \boldsymbol{u} : \boldsymbol{\nabla}_h \boldsymbol{v} \,\mathrm{d}\boldsymbol{x} - \int_\Omega p \,\mathrm{div}_h(\boldsymbol{v}) \,\mathrm{d}\boldsymbol{x} = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} \,\mathrm{d}\boldsymbol{x}, \tag{4.1a}$$

$$\forall K \in \mathcal{T}, \qquad \sum_{\sigma=K|L} \mathrm{v}^+_{\sigma,K}\, \rho_K - \mathrm{v}^-_{\sigma,K}\, \rho_L + (T_{\mathrm{stab},1})_K + (T_{\mathrm{stab},2})_K = 0, \tag{4.1b}$$

$$\forall K \in \mathcal{T}, \qquad p_K = \rho_K^\gamma, \tag{4.1c}$$

where:

- $\mathrm{v}_{\sigma,K}$, $\mathrm{v}^+_{\sigma,K}$ and $\mathrm{v}^-_{\sigma,K}$ stands respectively for:

$$\mathrm{v}_{\sigma,K} = |\sigma|\, \boldsymbol{u}_\sigma \cdot \boldsymbol{n}_{KL},\ \ \mathrm{v}^+_{\sigma,K} = \max(\mathrm{v}_{\sigma,K},\ 0),\ \ \mathrm{v}^-_{\sigma,K} = -\min(\mathrm{v}_{\sigma,K},\ 0), \tag{4.2}$$

  so that $\mathrm{v}_{\sigma,K} = \mathrm{v}^+_{\sigma,K} - \mathrm{v}^-_{\sigma,K}$;

- the stabilization terms read, $\forall K \in \mathcal{T}$:

$$(T_{\mathrm{stab},1})_K = h^\alpha\, |K|\, (\rho_K - \rho^*)\,, \tag{4.3a}$$

$$(T_{\mathrm{stab},2})_K = \sum_{\sigma=K|L} (h_K + h_L)^\xi\, \frac{|\sigma|}{h_\sigma}\, (|\rho_K| + |\rho_L|)^\zeta\, (\rho_K - \rho_L)\,, \tag{4.3b}$$

  with $\zeta = \max(0, 2-\gamma)$.

Equation (4.1a) may be considered as the standard finite element discretization of Equation (1.2a). Since the pressure is piecewise constant, the finite element discretization of Relation (1.2b), *i.e.* the mass balance, is similar to a finite volume formulation, in which we introduce in (4.1b) the standard first-order upwinding and two stabilizing terms. The first one, *i.e.* $T_{\mathrm{stab},1}$, guarantees that the integral of the density over the computational domain is always M (this can easily be seen by summing (4.1b) for $K \in \mathcal{T}$). The second one, *i.e.* $T_{\mathrm{stab},2}$, is useful in the convergence analysis. It may be seen as a finite volume analogue of a continuous term of the form $\mathrm{div}(|\rho|^\zeta \boldsymbol{\nabla}\rho)$ weighted by a mesh-dependent coefficient tending to zero as h^ξ. Note, however, that h_σ is not the distance which is encountered when diffusive terms are approximated by the two-points finite volume method; consequently, the usual restriction for the mesh (namely, the Delaunay condition) is not required here. We take $\alpha \geq 1$ and the convergence analysis uses $0 < \xi < 2$. Finally, any solution to (4.1) satisfies $\rho > 0$ (see Theorem 5.3), which gives sense to the equation of state (4.1c).

Remark 4.1 (Forcing term involving the density). To deal with a right hand side in the first equation of the continuous problem (Equation (1.1a)) reading $\rho\, \boldsymbol{g}$ with $\boldsymbol{g} \in \mathrm{L}^\infty(\Omega)^d$, Equation (4.1a) must simply be changed to:

$$\forall \boldsymbol{v} \in \boldsymbol{W}_h, \qquad \int_\Omega \boldsymbol{\nabla}_h \boldsymbol{u} : \boldsymbol{\nabla}_h \boldsymbol{v} \,\mathrm{d}\boldsymbol{x} - \int_\Omega p \,\mathrm{div}_h(\boldsymbol{v}) \,\mathrm{d}\boldsymbol{x} = \int_\Omega \rho\, \boldsymbol{g} \cdot \boldsymbol{v} \,\mathrm{d}\boldsymbol{x}. \tag{4.4}$$

Remark 4.2 (Solution of the nonlinear algebraic system (4.1)). System (4.1) is a (potentially large) nonlinear algebraic system, and the design of an algorithm for its solution is not an easy task. To this purpose, one of the following iterative processes may perhaps be a starting point. First, one can remark that the fixed point iteration which consists in (*i*) solving (4.1a) for $\boldsymbol{u}$ with a fixed pressure, (*ii*) solving (4.1b) for ρ with the obtained velocity and (*iii*) finally computing the pressure from the equation of state is well-posed and stable; this result is used, for a slightly different (and essentially more complex) problem in [8, proof of Lemma 3.2] to prove, by a Brouwer fixed point argument, the existence of a solution. Second, one will find in [10, section 4.1] an extension to variable density flows (but with a pressure-independent density) of the augmented Lagrangian technique classical for incompressible flow problems.

5. Existence and convergence of approximate solutions

5.1. **Existence of a solution and a priori estimates.** We begin this section by a preliminary lemma, the proof of which can be found in [6, section 2]. This result may be seen as a discrete version of the identity introduced in the theory of renormalized solution of the transport equation [11, 12].

Lemma 5.1. *Let ρ be a non-negative function of L_h, $\boldsymbol{u}$ be a function of $\boldsymbol{W}_h$ and $\mathrm{v}_{\sigma,K}$, $\mathrm{v}^+_{\sigma,K}$ and $\mathrm{v}^-_{\sigma,K}$ be given by Equation* (4.2)*. Let $\phi\,:\,[0,+\infty)\longrightarrow\mathbb{R}$ be a once continuously differentiable convex function. Then the following estimate holds:*

$$\sum_{K\in\mathcal{T}}\phi'(\rho_K)\left(\sum_{\sigma=K|L}\mathrm{v}^+_{\sigma,K}\,\rho_K-\mathrm{v}^-_{\sigma,K}\,\rho_L\right)\geq\int_\Omega\big(\rho\phi'(\rho)-\phi(\rho)\big)\ \mathrm{div}_h(\boldsymbol{u})\,\mathrm{d}\boldsymbol{x}.$$

We are now in position to prove the following result, which is a discrete analog of Lemma 2.1.

Lemma 5.2. *Let $\mathcal{T}$ be a triangulation of the computational domain Ω and $(\boldsymbol{u},\rho)\in\boldsymbol{W}_h\times L_h$ satisfy the second equation of the scheme,* i.e. *Equation* (4.1b)*. We assume $\rho\geq 0$. Then, for all $\beta\geq 1$:*

$$\int_\Omega\rho^\beta\mathrm{div}_h(\boldsymbol{u})\,\mathrm{d}\boldsymbol{x}\leq c h^\alpha,$$

where the real number c only depends on β, the domain Ω and M.

Proof. Let us first consider the case $\beta>1$. Let the function ϕ be defined by $\phi(s)=s^\beta$ (ϕ is indeed a continuously differentiable convex function over $[0,+\infty)$). Multiplying (4.1b) by $\phi'(\rho_K)$, summing over $K\in\mathcal{T}$ and applying Lemma 5.1 yields $T_1+T_2+T_3\leq 0$ with:

$$\begin{aligned}
&T_1=(\beta-1)\int_\Omega\rho^\beta\mathrm{div}_h(\boldsymbol{u})\,\mathrm{d}\boldsymbol{x},\\
&T_2=\beta\sum_{K\in\mathcal{T}}h^\alpha\,|K|\,\rho_K^{\beta-1}\,(\rho_K-\rho^*)\,,\\
&T_3=\beta\sum_{K\in\mathcal{T}}\rho_K^{\beta-1}\sum_{\sigma=K|L}(h_K+h_L)^\xi\,\frac{|\sigma|}{h_\sigma}\,(\rho_K+\rho_L)^\zeta\,(\rho_K-\rho_L)\,.
\end{aligned}$$

By convexity of the function ϕ, we have:

$$T_2\geq h^\alpha\sum_{K\in\mathcal{T}}|K|\,\left(\rho_K^\beta-(\rho^*)^\beta\right)\geq-|\Omega|\,(\rho^*)^\beta\,h^\alpha.$$

Reordering the sums in T_3, we get:

$$T_3=\beta\sum_{\mathcal{E}\in\mathcal{E}_{\rm int},\ \sigma=K|L}(h_K+h_L)^\xi\,\frac{|\sigma|}{h_\sigma}\,(\rho_K+\rho_L)^\zeta\,(\rho_K-\rho_L)\,\left(\rho_K^{\beta-1}-\rho_L^{\beta-1}\right),$$

and so $T_3 \geq 0$. Finally, we thus get:

$$\int_\Omega \rho^\beta \mathrm{div}_h(\boldsymbol{u}) \,\mathrm{d}\boldsymbol{x} \leq \frac{1}{\beta - 1} \; |\Omega| \; (\rho^*)^\beta \; h^\alpha,$$

which concludes the proof for $\beta > 1$.

We now turn to the case $\beta = 1$. Let ϵ be a positive real number. Choosing now $\phi(s) = s \, \log(s + \epsilon)$ (so $s\phi'(s) - \phi(s) = s^2/(s+\epsilon)$) yields by the same arguments:

$$\int_\Omega \frac{\rho^2}{\rho+\epsilon} \; \mathrm{div}_h(\boldsymbol{u}) \,\mathrm{d}\boldsymbol{x} \leq \rho^* \;\; \log(\rho^* + \epsilon) \; |\Omega| \; h^\alpha,$$

thus, since for any non-negative real number s, $|s - \dfrac{s^2}{s+\epsilon}| = \epsilon \dfrac{s}{s+\epsilon} \leq \epsilon$:

$$\int_\Omega \rho \, \mathrm{div}_h(\boldsymbol{u}) \,\mathrm{d}\boldsymbol{x} \leq \rho^* \;\; \log(\rho^* + \epsilon) \; |\Omega| \; h^\alpha + \epsilon \int_\Omega |\mathrm{div}_h(\boldsymbol{u})| \,\mathrm{d}\boldsymbol{x}.$$

Since $\mathrm{div}_h(\boldsymbol{u})$ is bounded in $\mathrm{L}^1(\Omega)$ ($\boldsymbol{u}$ is a discrete function) and $\rho^* > 0$, letting ϵ tend to zero yields the conclusion. □

The existence of a solution to (4.1) follows, with minor changes to cope with the diffusion stabilization term $T_{\mathrm{stab},2}$, from the theory developed in [6, section 2]. In this latter paper, it is obtained for fairly general equations of state by a topological degree argument. We only give here the obtained result, together with a proof of the *a priori* estimates verified by the solution, and we refer to [6] for the proof of existence. The following estimate may be seen as an equivalent for the discrete case of Step 1 of the proof of Theorem 2.2.

Theorem 5.3. *Let $\theta_0 > 0$ and let $\mathcal{T}$ be a triangulation of the computational domain Ω such that $\theta \geq \theta_0$, where θ is defined by* (3.1)*. The problem* (4.1) *admits at least one solution $(\boldsymbol{u}, p, \rho) \in \boldsymbol{W}_h \times L_h \times L_h$, and any possible solution is such that $p_K > 0$ and $\rho_K > 0$, $\forall K \in \mathcal{T}$, and satisfies:*

$$\|\boldsymbol{u}\|_{1,b} \; + \; \|p\|_{\mathrm{L}^2(\Omega)} \; + \; \|\rho\|_{\mathrm{L}^{2\gamma}(\Omega)} \; + h^{\xi/2} \, |\rho|_{\mathcal{T}} \; \leq C, \tag{5.1}$$

where the real number C only depends on the data of the problem Ω, $\boldsymbol{f}$, M and on θ_0.

Proof. From [6, section 2], we know that (4.1) admits at least a solution and that, for any solution, the pressure and the density are positive. Let $(\boldsymbol{u}, p, \rho) \in \boldsymbol{W}_h \times L_h \times L_h$ be such a solution. On one hand, taking $\boldsymbol{u}$ as test function in (4.1a) yields:

$$\|\boldsymbol{u}\|_{1,b}^2 \; - \int_\Omega p \; \mathrm{div}_h(\boldsymbol{u}) \,\mathrm{d}\boldsymbol{x} = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{u} \,\mathrm{d}\boldsymbol{x}.$$

On the other hand, multiplying (4.1b) by $\gamma \, \rho_K^{\gamma-1}/(\gamma-1)$, summing over $K \in \mathcal{T}$ and invoking Lemma 5.2 yields:

$$\int_\Omega p \; \mathrm{div}_h(\boldsymbol{u}) \,\mathrm{d}\boldsymbol{x} + \frac{h^\alpha}{\gamma - 1} \sum_{K \in \mathcal{T}} |K| \; \big(\rho_K^\gamma - (\rho^*)^\gamma\big) + \frac{\gamma}{\gamma - 1} \sum_{K \in \mathcal{T}} \rho_K^{\gamma-1} \; (T_{\mathrm{stab},2})_K \leq 0.$$

Summing these two relations yields:

$$\|\boldsymbol{u}\|_{1,b}^2 \; + \frac{\gamma}{\gamma - 1} \sum_{K \in \mathcal{T}} \rho_K^{\gamma-1} \; (T_{\mathrm{stab},2})_K \leq \int_\Omega \boldsymbol{f} \cdot \boldsymbol{u} \,\mathrm{d}\boldsymbol{x} + \frac{1}{\gamma - 1} \; |\Omega| \; (\rho^*)^\gamma \; h^\alpha. \tag{5.2}$$

As in the proof of Lemma 5.2, the last term in the left hand side is easily seen to be non-negative. By Poincaré and Young's inequalities, we thus obtain a control on

$\|\boldsymbol{u}\|_{1,b}$. Using the *inf-sup* stability of the discretization, we hence get from (4.1a) a control of $\|p - m(p)\|_{\mathrm{L}^2(\Omega)}$ (where $m(p)$ stands for the mean value of p over Ω). Finally, by summing equation (4.1b) for $K \in \mathcal{T}$, we obtain that the integral of ρ over Ω is M, and, as in the continuous case, this yields an estimate for $\|p\|_{\mathrm{L}^2(\Omega)}$ (or $\|\rho\|_{\mathrm{L}^{2\gamma}(\Omega)}$) by lemma B.6.

To conclude the proof, we now need to estimate $|\rho|_{\mathcal{T}}$. Let us first suppose that $\gamma < 2$. By reordering the summations, we get in this case:

$$\sum_{K\in\mathcal{T}} \rho_K^{\gamma-1}\ (T_{\mathrm{stab},2})_K = \sum_{\substack{\sigma\in\mathcal{E}_{\mathrm{int}},\\ \sigma=K|L}} (h_K+h_L)^\xi\ \frac{|\sigma|}{h_\sigma}\,(\rho_K+\rho_L)^{2-\gamma}\ (\rho_K-\rho_L)\ \left(\rho_K^{\gamma-1}-\rho_L^{\gamma-1}\right),$$

where, without loss of generality, we can assume that $\rho_K \geq \rho_L$. Since, for $\gamma < 2$, the function $s \mapsto s^{\gamma-1}$ is concave, we have:

$$\left(\rho_K^{\gamma-1}-\rho_L^{\gamma-1}\right) \geq (\gamma-1)\ \rho_K^{\gamma-2}\ (\rho_K-\rho_L),$$

and thus:

$$\sum_{K\in\mathcal{T}} \rho_K^{\gamma-1}\ (T_{\mathrm{stab},2})_K \geq (\gamma-1) \sum_{\substack{\sigma\in\mathcal{E}_{\mathrm{int}},\\ \sigma=K|L}} (h_K+h_L)^\xi\ \frac{|\sigma|}{h_\sigma}\ (\rho_K-\rho_L)^2\,.$$

Using this inequality in (5.2) concludes the proof for $\gamma < 2$. If $\gamma \geq 2$, multiplying Equation (4.1b) by ρ_K, sumning over $K \in \mathcal{T}$ and using Lemma 5.2 yields:

$$\frac{1}{2}\int_\Omega \rho^2\ \mathrm{div}_h(\boldsymbol{u})\,\mathrm{d}\boldsymbol{x} + \frac{1}{2}\,h^\alpha \sum_{K\in\mathcal{T}} |K|\ \left(\rho_K^2-(\rho^*)^2\right) + \sum_{\substack{\sigma\in\mathcal{E}_{\mathrm{int}},\\ \sigma=K|L}} (h_K+h_L)^\xi\ \frac{|\sigma|}{h_\sigma}\ (\rho_K-\rho_L)^2 \leq 0,$$

which concludes the proof since, for $\gamma \geq 2$, ρ is bounded in $\mathrm{L}^4(\Omega)$. □

Remark 5.4 (Forcing term involving the density). This estimate also holds in the case where $\boldsymbol{f} = \rho\,\boldsymbol{g}$, replacing Equation (4.1a) by Equation (4.4). It is proved by modifying the above aguments as in the continuous case (see remark 2.5).

5.2. **Passing to the limit in the mass and momentum balance equations.** The following result may be seen as the equivalent for the discrete case of Step 2 of the proof of Theorem 2.2.

Proposition 5.5. *Let $(\mathcal{T}^{(n)})_{n\in\mathbb{N}}$ be a given a sequence of triangulations of Ω. We assume that h_n tends to zero when $n \to \infty$. In addition, we assume that the sequence of discretizations is regular, in the sense that there exists $\theta_0 > 0$ such that $\theta_n \geq \theta_0$, $\forall n \in \mathbb{N}$. For $n \in \mathbb{N}$, we denote by $\boldsymbol{W}_h^{(n)}$ and $L_h^{(n)}$ respectively the discrete spaces for the velocity and the pressure or the density associated to $\mathcal{T}^{(n)}$ and by $(\boldsymbol{u}_n, p_n, \rho_n) \in \boldsymbol{W}_h^{(n)} \times L_h^{(n)} \times L_h^{(n)}$ a corresponding solution to the discrete problem* (4.1)*, with $\alpha \geq 1$ and $0 < \xi < 2$. Then:*

(1) *up to the extraction of a subsequence, the sequence $(\boldsymbol{u}_n)_{n\in\mathbb{N}}$ (strongly) converges in $\mathrm{L}^2(\Omega)^d$ to a limit $\boldsymbol{u} \in \mathrm{H}^1_0(\Omega)^d$ when $n \to \infty$, $(p_n)_{n\in\mathbb{N}}$ converges to p weakly in $\mathrm{L}^2(\Omega)$ and ρ_n converges to ρ weakly in $\mathrm{L}^{2\gamma}(\Omega)$;*

(2) $(\boldsymbol{u}, p, \rho)$ *satisfies the equations* (1.2a) *and* (1.2b) *of the weak continuous problem,* $\rho \geq 0$ *a.e. in* Ω *and* $\int_\Omega \rho \, \mathrm{d}\boldsymbol{x} = M$.

Remark 5.6 (Forcing term involving the density). This convergence result also holds, without any additional difficulty, in the case where $\boldsymbol{f} = \rho\, \boldsymbol{g}$, replacing Equation (4.1a) by Equation (4.4).

Proof. The convergence (up to the extraction of a subsequence) of the sequence $(\boldsymbol{u}_n, p_n, \rho_n)$ is a consequence of the uniform (with respect to n) estimate of Theorem 5.3, applying Theorem 3.2 to each component $\boldsymbol{u}_i^{(m)}$, $1 \leq i \leq d$. The proof that the limit satisfies $\rho \geq 0$ a.e. in Ω, $\int_\Omega \rho \, \mathrm{d}\boldsymbol{x} = M$ and Equation (1.2a) is strictly the same than the proof of the same result for a linear equation of state, *i.e.* Theorem 6.1 in [7]. For Equation (1.2b), it only differs from this latter by the treatment of the (slightly) different second stabilization term $T_{\mathrm{stab},2}$, but, once again, following the same arguments shows the convergence of the corresponding term to zero. This computation is not reproduced here. □

5.3. **Passing to the limit in the equation of state.** This section gathers the analogues for the discrete case of Step 3 and Step 4 of the proof of Theorem 2.2. It begins with some technical preliminaries.

Definition 5.7. Let $\mathcal{T}$ be a triangulation of the domain Ω and $\mathcal{A}$ be the set of the vertices of the mesh. For $a \in \mathcal{A}$, we denote by $\mathcal{N}_a \subset \mathcal{T}$ the set of the elements $K \in \mathcal{T}$ of which a is a vertex. Let q be a function of L_h. We denote by $i_h q$ the function defined as follows:

$$
\left|
\begin{array}{l}
i_h q \in \mathrm{C}^0(\Omega), \\
\forall K \in \mathcal{T}, \ \text{the restriction to } K \text{ of } i_h q \text{ is affine}, \\
\forall a \in \mathcal{A}, \ i_h q(a) = \dfrac{1}{\mathrm{card}(\mathcal{N}_a)} \displaystyle\sum_{K \in \mathcal{N}_a} q_K.
\end{array}
\right.
\tag{5.3}
$$

In other words, the function $i_h q$ belongs to the usual space of piecewise affine continuous finite element functions (often called the P_1 finite element space), and corresponds to some regularization of q. The operator i_h satisfies the following stability and approximation results.

Lemma 5.8. *Let* $\theta_0 > 0$ *and let* $\mathcal{T}$ *be a triangulation of the computational domain* Ω *such that* $\theta \geq \theta_0$*, where* θ *is defined by* (3.1)*. Let* $q \in L_h$ *be given. Then:*

$$
|i_h q|_{\mathrm{H}^1(\Omega)} \leq c_{\mathrm{s}} \, |q|_{\mathcal{T}} \tag{5.4}
$$

$$
\|i_h q - q\|_{\mathrm{L}^2(\Omega)} \leq c_{\mathrm{a}} \ h \, |q|_{\mathcal{T}} \tag{5.5}
$$

where the real numbers c_{s} *and* c_{a} *only depend on* θ_0.

Proof. Let q be a function of L_h. For $K \in \mathcal{T}$, we denote by $\mathcal{A}_K$ the set of the vertices of K and, for $a \in \mathcal{A}$, we denote by ψ_a the P_1 shape function associated to a. By definition of the operator i_h, we get:

$$
|i_h q|^2_{\mathrm{H}^1(\Omega)} = \sum_{K \in \mathcal{T}} \int_K \Bigl| \sum_{a \in \mathcal{A}_K} i_h q(a) \ \boldsymbol{\nabla} \psi_a(\boldsymbol{x}) \Bigr|^2 \, \mathrm{d}\boldsymbol{x}.
$$

Since, over each cell K, we have $\sum_{a\in\mathcal{A}_K} \psi_a(\boldsymbol{x}) = 1$, the preceding relation yields:

$$\begin{aligned} |i_h q|^2_{\mathrm{H}^1(\Omega)} &= \sum_{K\in\mathcal{T}} \int_K \Big| \sum_{a\in\mathcal{A}_K} \big(i_h q(a) - q_K\big)\ \boldsymbol{\nabla}\psi_a(\boldsymbol{x})\Big|^2 \,\mathrm{d}\boldsymbol{x} \\ &= \sum_{K\in\mathcal{T}} \int_K \Big| \sum_{a\in\mathcal{A}_K} \frac{1}{\mathrm{card}(\mathcal{N}_a)} \sum_{L\in\mathcal{N}_a} (q_L - q_K)\ \boldsymbol{\nabla}\psi_a(\boldsymbol{x})\Big|^2 \,\mathrm{d}\boldsymbol{x} \end{aligned}$$

and so:

$$|i_h q|^2_{\mathrm{H}^1(\Omega)} \leq c_1 \sum_{K\in\mathcal{T}} \frac{|K|}{h_K^2} \sum_{a\in\mathcal{A}_K} \sum_{L\in\mathcal{N}_a} (q_L - q_K)^2,$$

where the real number c_1 only depends on the regularity of the mesh (*i.e.* the parameter θ_0). Since, still by regularity of the mesh, a vertex belongs to a bounded number of cells, the quantity $q_L - q_K$, for $L \in \mathcal{N}_a$ and $a \in \mathcal{A}_K$, can be developped as a sum of differences $q_M - q_N$ involving a bounded number of terms and such that M and N are two neighbouring control volumes. Invoking once again the regularity of the mesh (and, in particular, the fact that, for two neighbouring control volumes K and L, the ratio h_K/h_L is bounded), we obtain, for each edge σ separating two cells involved in one of the terms of this sum, the inequality $|K|/h_K^2 \leq c_2\,|\sigma|/h_\sigma$. This yields the estimate (5.4) of this lemma.

We now turn to the bound (5.5). We have:

$$\|i_h q - q\|^2_{\mathrm{L}^2(\Omega)} = \sum_{K\in\mathcal{T}} \int_K \Big(q_K - \sum_{a\in\mathcal{A}_K} i_h q(a)\ \psi_a(\boldsymbol{x})\Big)^2 \,\mathrm{d}\boldsymbol{x},$$

and so, once again since $\forall \boldsymbol{x} \in K,\ \sum_{a\in\mathcal{A}_K} \psi_a(\boldsymbol{x}) = 1$:

$$\begin{aligned} \|i_h q - q\|^2_{\mathrm{L}^2(\Omega)} &= \sum_{K\in\mathcal{T}} \int_K \Big(\sum_{a\in\mathcal{A}_K} (q_K - i_h q(a))\ \psi_a(\boldsymbol{x})\Big)^2 \,\mathrm{d}\boldsymbol{x} \\ &= \sum_{K\in\mathcal{T}} \int_K \Big(\sum_{a\in\mathcal{A}_K} \frac{1}{\mathrm{card}(\mathcal{N}_a)} \sum_{L\in\mathcal{N}_a} (q_K - q_L)\ \psi_a(\boldsymbol{x})\Big)^2 \,\mathrm{d}\boldsymbol{x} \\ &\leq c_3 \sum_{K\in\mathcal{T}} |K| \sum_{a\in\mathcal{A}_K} \sum_{L\in\mathcal{N}_a} (q_K - q_L)^2, \end{aligned}$$

where c_3 only depends on the regularity of the mesh. The result then follows by the same arguments as previously (developping the difference $q_K - q_L$ as a a sum of differences between the values of q over two neighbouring cells and invoking the regularity of the mesh). □

Proposition 5.9. *Let $(\mathcal{T}^{(n)})_{n\in\mathbb{N}}$ be a sequence of discretizations satisfying the same assumptions that in Proposition 5.5, and $(\boldsymbol{u}_n, p_n, \rho_n)_{n\in\mathbb{N}}$ be a corresponding sequence of discrete solutions. Let $(\boldsymbol{u}, p, \rho) \in \mathrm{H}^1_0(\Omega)^d \times \mathrm{L}^2(\Omega) \times \mathrm{L}^{2\gamma}(\Omega)$ be the limit (up to the extraction of a subsequence and in the sense of Proposition 5.5) of this sequence of solutions when $n \to \infty$. Let $(q_n)_{n\in\mathbb{N}}$ be a sequence of functions such that, $\forall n \in \mathbb{N}$, $q_n \in L_h^{(n)}$ and:*

$$|q_n|_{\mathcal{T}} \leq c\ h_n^{-\eta},$$

where c is a positive real number and η is such that $\eta < 1$. We assume in addition that the sequence $(q_n)_{n\in\mathbb{N}}$ is bounded in $\mathrm{L}^2(\Omega)$ and weakly converges in $\mathrm{L}^2(\Omega)$ to

$q \in \mathrm{L}^2(\Omega)$. *Then:*

$$\forall \varphi \in \mathrm{C}_c^\infty(\Omega), \quad \lim_{n\to\infty} \int_\Omega \big(\mathrm{div}_h(\boldsymbol{u}_n) - p_n\big)\, q_n \varphi \,\mathrm{d}\boldsymbol{x} = \int_\Omega \big(\mathrm{div}(\boldsymbol{u}) - p\big)\, q\varphi \,\mathrm{d}\boldsymbol{x}.$$

Proof. Throughout this proof, we denote by c_i, $i \in \mathbb{N}$, a positive real number possibly depending on θ_0, Ω and the data of the problem, but independent on n.

Let us consider the sequence $(i_h q_n)_{n\in\mathbb{N}}$ of functions of $\mathrm{H}^1(\Omega)$, where i_h is the operator given by Definition 5.7. Since q_n is bounded in $\mathrm{L}^2(\Omega)$ and $|q_n|_{\mathcal{T}} \leq C h^{-\eta}$, thanks to properties (5.4)–(5.5) and the regularity of the sequence of discretizations ($\forall n \in \mathbb{N},\ \theta_n \geq \theta$), we have:

$$\|i_h q_n\|_{\mathrm{L}^2(\Omega)} \leq c_1, \ \ \|i_h q_n\|_{\mathrm{H}^1(\Omega)} \leq c_2\ h_n^{-\eta} \text{ and } \|i_h q_n - q_n\|_{\mathrm{L}^2(\Omega)} \leq c_3\ h_n^{1-\eta}. \tag{5.6}$$

Let $(\boldsymbol{w}_n)_{n\in\mathbb{N}}$ be the sequence of functions of $\mathrm{H}^1(\Omega)^d$ defined from $(i_h q_n)_{n\in\mathbb{N}}$ by Lemma B.8. We thus have:

$$\boldsymbol{w}_n \in \mathrm{H}^2(\Omega)^d,\ \mathrm{curl}(\boldsymbol{w}_n) = 0 \text{ and } \mathrm{div}(\boldsymbol{w}_n) = i_h q_n \text{ a.e. in } \Omega.$$

In addition, Inequalities (5.6) yield:

$$\|\boldsymbol{w}_n\|_{\mathrm{H}^1(\Omega)^d} \leq c_4 \text{ and } \|\boldsymbol{w}_n\|_{\mathrm{H}^2(\Omega)^d} \leq c_5\ h_n^{-\eta},$$

and thus, up to the extraction of a subsequence, the sequence $(\boldsymbol{w}_n)_{n\in\mathbb{N}}$ converges to $\boldsymbol{w} \in \mathrm{H}^1(\Omega)^d$ (strongly) in $\mathrm{L}^2(\Omega)^d$ and weakly in $\mathrm{H}^1(\Omega)^d$; in addition, $\boldsymbol{w}$ satisfies $\mathrm{curl}(\boldsymbol{w}) = 0$ and $\mathrm{div}(\boldsymbol{w}) = q$ a.e. in Ω.

Let φ be a function of $\mathrm{C}_c^\infty(\Omega)$, and let us take $r_h^{(n)}(\varphi\boldsymbol{w}_n)$ as test function in the first relation of the discrete problem (4.1a) associated to $\mathcal{T}^{(n)}$:

$$\int_\Omega \boldsymbol{\nabla}_h \boldsymbol{u}_n : \boldsymbol{\nabla}_h (r_h^{(n)}(\varphi\boldsymbol{w}_n)) \,\mathrm{d}\boldsymbol{x} - \int_\Omega p_n \,\mathrm{div}_h(r_h^{(n)}(\varphi\boldsymbol{w}_n)) \,\mathrm{d}\boldsymbol{x} = \int_\Omega \boldsymbol{f} \cdot (r_h^{(n)}(\varphi\boldsymbol{w}_n)) \,\mathrm{d}\boldsymbol{x}.$$

We thus get:

$$\int_\Omega \boldsymbol{\nabla}_h \boldsymbol{u}_n : \boldsymbol{\nabla}(\varphi\boldsymbol{w}_n) \,\mathrm{d}\boldsymbol{x} - \int_\Omega p_n \,\mathrm{div}(\varphi\boldsymbol{w}_n) \,\mathrm{d}\boldsymbol{x} = \int_\Omega \boldsymbol{f} \cdot (\varphi\boldsymbol{w}_n) \,\mathrm{d}\boldsymbol{x} + R_{\Omega,n}, \tag{5.7}$$

where:

$$R_{\Omega,n} = -\int_\Omega \boldsymbol{\nabla}_h \boldsymbol{u}_n : \boldsymbol{\nabla}_h \delta_n \,\mathrm{d}\boldsymbol{x} + \int_\Omega p_n \,\mathrm{div}_h(\delta_n) \,\mathrm{d}\boldsymbol{x} + \int_\Omega \boldsymbol{f} \cdot \delta_n \,\mathrm{d}\boldsymbol{x},$$

with $\delta_n = r_h^{(n)}(\varphi\boldsymbol{w}_n) - \varphi\boldsymbol{w}_n$. Thanks to the approximation properties of the operator $r_h^{(n)}$ (Theorem 3.1), we get:

$$\|\delta_n\|_{\mathrm{H}^1(\Omega)^d} \leq c_6\ h_n \,\|\varphi\boldsymbol{w}_n\|_{\mathrm{H}^2(\Omega)^d} \leq c_7\ h_n^{1-\eta},$$

where c_7 and, from now on in this proof, the other real numbers denoted by c_i, $i \in \mathbb{N}$, also depends on φ. Thus, since the sequences $(\boldsymbol{\nabla}_h \boldsymbol{u}_n)_{n\in\mathbb{N}}$ and $(p_n)_{n\in\mathbb{N}}$ are bounded in $\mathrm{L}^2(\Omega)^{d\times d}$ and $\mathrm{L}^2(\Omega)$ respectively, we have:

$$|R_{\Omega,n}| \leq c_8\ h_n^{1-\eta}. \tag{5.8}$$

Returning now to Equation (5.7) and applying the identity (1.3) over each control volume, we get:

$$\int_\Omega \mathrm{curl}_h(\boldsymbol{u}_n)\cdot \mathrm{curl}(\varphi \boldsymbol{w}_n)\,\mathrm{d}\boldsymbol{x} + \int_\Omega \mathrm{div}_h(\boldsymbol{u}_n)\ \mathrm{div}(\varphi \boldsymbol{w}_n)\,\mathrm{d}\boldsymbol{x} \\ - \int_\Omega p_n\ \mathrm{div}(\varphi \boldsymbol{w}_n)\,\mathrm{d}\boldsymbol{x} = \int_\Omega \boldsymbol{f}\cdot(\varphi \boldsymbol{w}_n)\,\mathrm{d}\boldsymbol{x} + R_{\Omega,n} + R_{\partial\mathcal{T},n},$$

where the term $R_{\partial\mathcal{T},n}$ gathering the boundary terms appearing in (1.3) has the following structure:

$$R_{\partial\mathcal{T},n} = \sum_{\sigma\in\mathcal{E}_{\rm int}} \int_\sigma \sum_{1\leq i,j,k\leq d} c_{\sigma,i,j,k}\ [\boldsymbol{u}_n]_i\ (\boldsymbol{\nabla}(\varphi \boldsymbol{w}_n))_{j,k}\,\mathrm{d}\gamma,$$

and each real numbers $c_{\sigma,i,j,k}$ is a component of the unit normal vector to σ. Applying lemma 3.3, we thus get:

$$|R_{\partial\mathcal{T},n}| \leq c_9\ h_n\, \|\boldsymbol{u}_n\|_{1,b}\ \|\varphi \boldsymbol{w}_n\|_{\mathrm{H}^2(\Omega)^d}\ \leq c_{10}\ h_n^{1-\eta}. \tag{5.9}$$

The choice of $\boldsymbol{w}_n$ gives $\mathrm{div}(\varphi \boldsymbol{w}_n) = i_h q_n\,\varphi + \boldsymbol{w}_n\cdot\boldsymbol{\nabla}\varphi$ and $\mathrm{curl}(\varphi \boldsymbol{w}_n) = L(\varphi)\,\boldsymbol{w}_n$, where $L(\varphi)$ is a matrix with entries involving some first order derivatives of φ. Thus, we get:

$$\int_\Omega \mathrm{curl}_h(\boldsymbol{u}_n)\cdot L(\varphi)\,\boldsymbol{w}_n\,\mathrm{d}\boldsymbol{x} + \int_\Omega \mathrm{div}_h(\boldsymbol{u}_n)\ (q_n\varphi + \boldsymbol{w}_n\cdot\boldsymbol{\nabla}\varphi)\,\mathrm{d}\boldsymbol{x} \\ - \int_\Omega p_n\ (q_n\varphi + \boldsymbol{w}_n\cdot\boldsymbol{\nabla}\varphi)\,\mathrm{d}\boldsymbol{x} = \int_\Omega \boldsymbol{f}\cdot(\varphi \boldsymbol{w}_n)\,\mathrm{d}\boldsymbol{x} + R_{\Omega,n} + R_{\partial\mathcal{T},n} + R_{i_h,n},$$

where $R_{i_h,n}$ reads:

$$R_{i_h,n} = -\int_\Omega \big(\mathrm{div}_h(\boldsymbol{u}_n) - p_n\big)\ (i_h q_n - q_n)\ \varphi\,\mathrm{d}\boldsymbol{x}.$$

Thanks to the fact that both $\mathrm{div}_h(\boldsymbol{u}_n)$ and p_n are bounded in $\mathrm{L}^2(\Omega)$, the third inequality in (5.6) yields:

$$|R_{i_h,n}| \leq c_{11}\ h_n^{1-\eta}. \tag{5.10}$$

Using the estimates (5.8), (5.9) and (5.10), together with the convergence properties of the sequences $(\boldsymbol{u}_n)_{n\in\mathbb{N}}$, $(\boldsymbol{w}_n)_{n\in\mathbb{N}}$ and $(p_n)_{n\in\mathbb{N}}$ to pass to the limit when $n\to\infty$ in the preceding equation, we get:

$$\lim_{n\to\infty} \int_\Omega \big(\mathrm{div}_h(\boldsymbol{u}_n) - p_n\big)\ q_n\varphi\,\mathrm{d}\boldsymbol{x} = -\int_\Omega \mathrm{curl}(\boldsymbol{u})\cdot L(\varphi)\,\boldsymbol{w}\,\mathrm{d}\boldsymbol{x} \\ - \int_\Omega \mathrm{div}(\boldsymbol{u})\ (\boldsymbol{w}\cdot\boldsymbol{\nabla}\varphi)\,\mathrm{d}\boldsymbol{x} + \int_\Omega p\ (\boldsymbol{w}\cdot\boldsymbol{\nabla}\varphi)\,\mathrm{d}\boldsymbol{x} + \int_\Omega \boldsymbol{f}\cdot(\varphi\boldsymbol{w})\,\mathrm{d}\boldsymbol{x}.$$

But, from Proposition 5.5, we know that $\boldsymbol{u}$ and p satisfies Equation (1.2a). Taking $\boldsymbol{v} = \varphi\boldsymbol{w}$ in this last relation and using $\mathrm{curl}(\boldsymbol{w}) = 0$ and $\mathrm{div}(\boldsymbol{w}) = q$ yields:

$$\int_\Omega \big(\mathrm{div}(\boldsymbol{u}) - p\big)\ q\varphi\,\mathrm{d}\boldsymbol{x} = -\int_\Omega \mathrm{curl}(\boldsymbol{u})\cdot L(\varphi)\,\boldsymbol{w}\,\mathrm{d}\boldsymbol{x} \\ - \int_\Omega \mathrm{div}(\boldsymbol{u})\ (\boldsymbol{w}\cdot\boldsymbol{\nabla}\varphi)\,\mathrm{d}\boldsymbol{x} + \int_\Omega p\ (\boldsymbol{w}\cdot\boldsymbol{\nabla}\varphi)\,\mathrm{d}\boldsymbol{x} + \int_\Omega \boldsymbol{f}\cdot(\varphi\boldsymbol{w})\,\mathrm{d}\boldsymbol{x},$$

which concludes the proof. □

Proposition 5.10. *Let $(\mathcal{T}^{(n)})_{n\in\mathbb{N}}$ be a sequence of discretizations satisfying the same assumptions that in Proposition 5.5, and $(\boldsymbol{u}_n, p_n, \rho_n)_{n\in\mathbb{N}}$ be a corresponding sequence of discrete solutions. Let $(\boldsymbol{u}, p, \rho) \in \mathrm{H}_0^1(\Omega)^d \times \mathrm{L}^2(\Omega) \times \mathrm{L}^{2\gamma}(\Omega)$ be the limit (up to the extraction of a subsequence and in the sense of Proposition 5.5) of this sequence of solutions when $n \to \infty$. Then we have:*

$$\lim_{n\to\infty} \int_\Omega \big(\mathrm{div}_h(\boldsymbol{u}_n) - p_n\big)\, \rho_n \,\mathrm{d}\boldsymbol{x} = \int_\Omega (\mathrm{div}(\boldsymbol{u}) - p)\, \rho \,\mathrm{d}\boldsymbol{x}$$

Proof. By the stability result of Theorem 5.3, the sequence $(\rho_n)_{n\in\mathbb{N}}$ satisfies:

$$\forall n \in \mathbb{N}, \quad \|\rho_n\|_{\mathrm{L}^{2\gamma}(\Omega)} + h^{\xi/2}\, |\rho_n|_{\mathcal{T}} \le C.$$

Hence Proposition 5.9 with $q_n = \rho_n$, $q = \rho$ and $\eta = \xi/2$ applies and thus:

$$\forall \varphi \in \mathrm{C}_c^\infty(\Omega), \quad \lim_{n\to\infty} \int_\Omega \big(\mathrm{div}_h(\boldsymbol{u}_n) - p_n\big)\, \rho_n \varphi \,\mathrm{d}\boldsymbol{x} = \int_\Omega \big(\mathrm{div}(\boldsymbol{u}) - p\big)\, \rho \varphi \,\mathrm{d}\boldsymbol{x}.$$

As in Step 3 of the proof of Theorem 2.2, since the sequence $(\mathrm{div}_h(\boldsymbol{u}_n) - p_n)_{n\in\mathbb{N}}$ is bounded in $\mathrm{L}^2(\Omega)$ and $(\rho_n)_{n\in\mathbb{N}}$ is bounded in $\mathrm{L}^{2\gamma}(\Omega)$ with $\gamma > 1$, the sequence $((\mathrm{div}_h(\boldsymbol{u}_n) - p_n)\, \rho_n)_{n\in\mathbb{N}}$ is equi-integrable, and thus, Lemma B.2 yields the conclusion. □

This result concludes the analogue at the discrete level of Step 3 in the proof of theorem 2.2. Let us now state the final convergence result, the proof of which closely follows Step 4.

Theorem 5.11. *Let a sequence of triangulations $(\mathcal{T}^{(n)})_{n\in\mathbb{N}}$ of Ω be given. We assume that h_n tends to zero when $n \to \infty$. In addition, we assume that the sequence of discretizations is regular, in the sense that there exists $\theta_0 > 0$ such that $\theta_n \ge \theta_0$, $\forall n \in \mathbb{N}$. For $n \in \mathbb{N}$, we denote by $\boldsymbol{W}_h^{(n)}$ and $L_h^{(n)}$ respectively the discrete spaces for the velocity and the pressure associated to $\mathcal{T}^{(n)}$ and by $(\boldsymbol{u}_n, p_n, \rho_n) \in \boldsymbol{W}_h^{(n)} \times L_h^{(n)} \times L_h^{(n)}$ a corresponding solution to the discrete problem* (4.1)*, with $\alpha \ge 1$ and $0 < \xi < 2$. Then, up to the extraction of a subsequence, when $n \to \infty$:*

(1) *the sequence $(\boldsymbol{u}_n)_{n\in\mathbb{N}}$ (strongly) converges in $\mathrm{L}^2(\Omega)^d$ to a limit $\boldsymbol{u} \in \mathrm{H}_0^1(\Omega)^d$,*
(2) *the sequence $(p_n)_{n\in\mathbb{N}}$ converges weakly in $\mathrm{L}^2(\Omega)$ and (strongly) in $\mathrm{L}^p(\Omega)$, for $1 \le p < 2$, to $p \in \mathrm{L}^2(\Omega)$,*
(3) *the sequence $(\rho_n)_{n\in\mathbb{N}}$ converges weakly in $\mathrm{L}^{2\gamma}(\Omega)$ and (strongly) in $\mathrm{L}^p(\Omega)$, for $1 \le p < 2\gamma$, to $\rho \in \mathrm{L}^{2\gamma}(\Omega)$,*
(4) *$(\boldsymbol{u}, p, \rho)$ are solution to Problem* (1.2a)–(1.2c)*.*

Remark 5.12 (Forcing term involving the density)*.* This convergence result also holds in the case where $\boldsymbol{f} = \rho\, \boldsymbol{g}$, replacing for the scheme Equation (4.1a) by Equation (4.4). The only additional difficulty lies in the solution estimates, and the modification of their proof is explained in remark 2.5.

Proof. Since, by Proposition 5.5, we know that the weak limit $(\boldsymbol{u}, p, \rho)$ exists and satisfies the weak momentum and mass balance equations (1.2a)–(1.2b), together with $\rho \ge 0$ a.e. in Ω and $\int_\Omega \rho \,\mathrm{d}\boldsymbol{x} = M$, we only need here to prove the strong convergence of $(p_n)_{n\in\mathbb{N}}$ and $(\rho_n)_{n\in\mathbb{N}}$, together with the fact that the equation of state is satisfied:

$$p = \rho^\gamma \quad \text{a.e. in } \Omega.$$

To this purpose, we mimic in this proof Step 4 of theorem 2.2.

Thanks to the fact that $\rho \in \mathrm{L}^{2\gamma}(\Omega)$, $\rho \geq 0$ a.e. in Ω, $\boldsymbol{u} \in (\mathrm{H}^1_0(\Omega))^d$ and that $(\rho, \boldsymbol{u})$ satisfies (1.2b) (which is the weak form of $\mathrm{div}(\rho \boldsymbol{u}) = 0$), we may apply Lemma 2.1 to obtain:

$$\int_\Omega \rho \, \mathrm{div}(\boldsymbol{u}) \, \mathrm{d}\boldsymbol{x} = 0. \tag{5.11}$$

As in the proof of Theorem 2.2, we now consider the sequence $(G_n)_{n\in\mathbb{N}}$ with $G_n = (p_n - \rho^\gamma)(\rho_n - \rho)$. For all $n \in \mathbb{N}$, the function G_n belongs to $L^1(\Omega)$ and $G_n \geq 0$ a.e. in Ω. Futhermore, expanding the product in G_n, we get:

$$\int_\Omega G_n \, \mathrm{d}\boldsymbol{x} = \int_\Omega p_n \rho_n \, \mathrm{d}\boldsymbol{x} - \int_\Omega p_n \rho \, \mathrm{d}\boldsymbol{x} - \int_\Omega \rho^\gamma \rho_n \, \mathrm{d}\boldsymbol{x} + \int_\Omega \rho^\gamma \rho \, \mathrm{d}\boldsymbol{x}.$$

From Proposition 5.10 and using (5.11), we have:

$$\lim_{n\to\infty} \int_\Omega \big(p_n - \mathrm{div}_h(\boldsymbol{u}_n)\big) \, \rho_n \, \mathrm{d}\boldsymbol{x} - \int_\Omega p \, \rho \, \mathrm{d}\boldsymbol{x} = 0.$$

Lemma 5.2 yields:

$$\limsup_{n\to\infty} \int_\Omega \mathrm{div}_h(\boldsymbol{u}_n) \, \rho_n \, \mathrm{d}\boldsymbol{x} \leq 0.$$

Hence:

$$\limsup_{n\to\infty} \int_\Omega p_n \, \rho_n \, \mathrm{d}\boldsymbol{x} \leq \int_\Omega p \, \rho \, \mathrm{d}\boldsymbol{x}.$$

Using the weak convergence in $L^2(\Omega)$ of p_n to p and of ρ_n to ρ, we see that $\limsup_{n\to\infty} \int_\Omega G_n \, \mathrm{d}\boldsymbol{x} \leq 0$ and, since $G_n \geq 0$ a.e., $G_n \to 0$ in $L^1(\Omega)$. By strictly the same arguments as in the end of the proof of Theorem 2.2, we thus obtain that $p = \rho^\gamma$ a.e. in Ω and then the desired strong convergence of $(p_n)_{n\in\mathbb{N}}$ and $(\rho_n)_{n\in\mathbb{N}}$. □

6. Discussion

This paper extends to the case where the equation of state reads $p = \rho^\gamma$ the convergence results proven in [7] for the linear law $p = \rho$. To our knowledge, these convergence analyses seem to be the first results of this kind for the compressible Stokes problem. Beside the convergence of the scheme, they also provide an existence result for solutions of the continuous problem (which could also be derived from the continuous existence theory ingredients for the steady Navier-Stokes equations, as stated in [11, p. 162]).

A puzzling fact is that the present theory relies on two ingredients which are usually not present in actual implementations. Firstly, the stabilization term $T_{\mathrm{stab},2}$ is used in two of our proofs: first to ensure the convergence of the discretization of the mass convection flux $\mathrm{div}(\rho \boldsymbol{u})$ and, second, for the proof of the discrete "effective viscous pressure" lemma (Proposition 5.9). To our knowledge, this term has never been introduced elsewhere, and in particular never appears in the schemes implemented in "real-life" computer codes. Secondly, the control of the pressure in some $\mathrm{L}^q(\Omega)$ space with $q > 1$ (here in $\mathrm{L}^2(\Omega)$) relies on the stability of the discrete gradient (*i.e.* the satisfaction of the so-called discrete *inf-sup* condition), which is not verified by collocated discretizations; note that this argument is not needed to obtain some control on the solution (see the proof of *a priori* estimates here and [6, 5] for stability studies of schemes for the Navier-Stokes equations). Assessing the effective relevance of these requirements for the discretization should deserve more work in the future.

An easy extension of this work consists in replacing the diffusive term $-\Delta \boldsymbol{u}$ in (1.1) by its complete expression $-\mu\,\Delta \boldsymbol{u} - \mu/3\,\boldsymbol{\nabla}(\mathrm{div}\boldsymbol{u})$ with $\mu > 0$ (*i.e.* the usual form of the divergence of the shear stress tensor in a constant viscosity compressible flow). Concerning higher order approximation issues, let us note that the fact that the pressure is approximated by a piecewise constant function appears crucial in both stability and convergence proofs: extending this study to higher degree finite element discretizations thus certainly represents a difficult task. Finally, let us mention that an extension of the present work to the MAC scheme is underway.

Appendix A. Proof of lemma 2.1

Proof. We handle the proof of this lemma in two steps.

Step 1. We assume in this step that there exists some $\alpha > 0$ such that $\rho \geq \alpha$ a.e. in Ω.

Let us first consider the case of a regular function ρ, say $\rho \in \mathrm{C}^1(\bar\Omega)$. In this case, if $1 < \beta \leq \gamma$, we take $\varphi = \rho^{\beta-1}$ in (1.2b) which yields:

$$\int_\Omega \rho^{\beta-1} \boldsymbol{u} \cdot \boldsymbol{\nabla} \rho \,\mathrm{d}\boldsymbol{x} = 0,$$

or, equivalently:

$$\int_\Omega \boldsymbol{u} \cdot \boldsymbol{\nabla} \rho^\beta \,\mathrm{d}\boldsymbol{x} = 0$$

and thus (2.1), using the boundary condition on $\boldsymbol{u}$. Assume now that $\beta = 1$. A first way to prove (2.1) in this case is to pass to the limit, as $n \to \infty$, on (2.1) with $\beta = 1 + 1/n$ (and $n \geq n_0$, $1/n_0 \leq \gamma - 1$). A second way (which is also valid if $\gamma = 1$) is to take $\varphi = \ln(\rho)$ in (1.2b). We obtain:

$$0 = \int_\Omega \boldsymbol{u} \cdot \boldsymbol{\nabla} \rho \,\mathrm{d}\boldsymbol{x} = \int_\Omega \rho\, \mathrm{div}(\boldsymbol{u}) \,\mathrm{d}\boldsymbol{x}.$$

This simple proof, which works for a regular function ρ, is adapted in Section 5 in a discrete setting, the discretization playing a similar role to the regularization.

We now have to prove (2.1) without assuming regularity of ρ. Let B be an open ball containing $\bar\Omega$. Taking $\boldsymbol{u} = 0$ in $\mathbb{R}^d \setminus \Omega$, $\rho = \alpha$ in $B \setminus \Omega$ and $\rho = 0$ in $\mathbb{R}^d \setminus B$, we have $\rho \in \mathrm{L}^{2\gamma}(\mathbb{R}^d)$, $\rho \geq 0$ a.e. in $\mathbb{R}^d$, $\rho \geq \alpha$ a.e. in B and $\boldsymbol{u} \in \mathrm{H}^1(\mathbb{R}^d)^d$. We also deduce from (1.2b):

$$\int_{\mathbb{R}^d} \rho \boldsymbol{u} \cdot \boldsymbol{\nabla} \varphi \,\mathrm{d}\boldsymbol{x} = 0 \text{ for all } \varphi \in \mathrm{C}^1(\mathbb{R}^d). \tag{A.1}$$

Let $(r_n)_{n \in \mathbb{N}^\star}$ be a sequence of mollifiers, that is:

$$r \in \mathrm{C}_c^\infty(\mathbb{R}^d, \mathbb{R}),\ \int_{\mathbb{R}^d} r \,\mathrm{d}\boldsymbol{x} = 1,\ r \geq 0 \text{ in } \mathbb{R}^d \\ \text{and, for } n \in \mathbb{N}^\star,\ \boldsymbol{x} \in \mathbb{R}^d,\ r_n(\boldsymbol{x}) = n^d r(n\boldsymbol{x}). \tag{A.2}$$

For $n \in \mathbb{N}^\star$, we set $\rho_n = \rho \star r_n$ and $(\rho\boldsymbol{u})_n = (\rho\boldsymbol{u}) \star r_n$. Thanks to (A.1), we have $\mathrm{div}((\rho\boldsymbol{u})_n) = 0$ in $\mathbb{R}^d$. Since $\boldsymbol{u} \in \mathrm{H}^1(\mathbb{R}^d)^d$, and $\rho \in \mathrm{L}^{2\gamma}(\mathbb{R}^d)$, we will prove in

Lemma B.4 that $\mathrm{div}((\rho\boldsymbol{u})_n - \rho_n \boldsymbol{u}) \to 0$ weakly in $\mathrm{L}^{(2\gamma)/(\gamma+1)}(\mathbb{R}^d)$ as $n \to \infty$. Then, if $(q_n)_{n\in\mathbb{N}^\star}$ is a converging sequence in $\mathrm{L}^{(2\gamma)/(\gamma-1)}(\mathbb{R}^d)$, we have:

$$-\int_{\mathbb{R}^d} \mathrm{div}(\rho_n \boldsymbol{u})\, q_n \,\mathrm{d}\boldsymbol{x} = \int_{\mathbb{R}^d} \mathrm{div}((\rho\boldsymbol{u})_n - \rho_n \boldsymbol{u})\, q_n \,\mathrm{d}\boldsymbol{x} \to 0 \text{ as } n \to \infty.$$

Let $1 < \beta \leq \gamma$. Taking $q_n = \rho_n^{\beta-1}$ (which is actually a converging sequence in $\mathrm{L}^{(2\gamma)/(\gamma-1)}(\mathbb{R}^d)$, which converges towards $\rho^{\beta-1}$) yields:

$$-\int_{\Omega} \mathrm{div}(\rho_n \boldsymbol{u})\, \rho_n^{\beta-1} \,\mathrm{d}\boldsymbol{x} = -\int_{\mathbb{R}^d} \mathrm{div}(\rho_n \boldsymbol{u})\, \rho_n^{\beta-1} \,\mathrm{d}\boldsymbol{x} \to 0 \text{ as } n \to \infty. \tag{A.3}$$

The function $\rho_n^{\beta-1}$ belongs to $\mathrm{C}^1(\bar{\Omega})$, at least for n large enough, since $\rho \geq \alpha$ in $B \supset \bar{\Omega}$. Then, (A.3) reads:

$$\int_{\Omega} \rho_n^{\beta-1} \boldsymbol{u} \cdot \boldsymbol{\nabla} \rho_n \,\mathrm{d}\boldsymbol{x} \to 0 \text{ as } n \to \infty,$$

or, equivalently:

$$\int_{\Omega} \rho_n^{\beta} \mathrm{div}(\boldsymbol{u}) \,\mathrm{d}\boldsymbol{x} = -\int_{\Omega} \boldsymbol{u} \cdot \boldsymbol{\nabla} \rho_n^{\beta} \,\mathrm{d}\boldsymbol{x} \to 0 \text{ as } n \to \infty.$$

As $n \to \infty$, we have $\rho_n \to \rho$ in $\mathrm{L}^{2\gamma}(\Omega)$ and therefore $\rho_n^\beta \to \rho^\beta$ in $\mathrm{L}^2(\Omega)$. Then:

$$\int_{\Omega} \rho_n^{\beta}\, \mathrm{div}(\boldsymbol{u}) \,\mathrm{d}\boldsymbol{x} \to \int_{\Omega} \rho^{\beta}\, \mathrm{div}(\boldsymbol{u}) \,\mathrm{d}\boldsymbol{x},$$

and we obtain (2.1).

As in the case of a regular ρ, it remains to prove (2.1) if $\beta = 1$. A simple way to do so is to pass to the limit, as $n \to \infty$, on (2.1) with $\beta = 1 + 1/n$ (and $n \geq n_0$, $1/n_0 \leq \gamma - 1$).

Step 2, General Case. Let $\alpha > 0$ and set $\rho_\alpha = \rho + \alpha$, so that $\rho_\alpha \geq \alpha$ a.e.. We begin by using, for a fixed $\alpha > 0$, the same proof as in Step 1, except that $\mathrm{div}(\rho_\alpha \boldsymbol{u})$ is not equal to 0 since $\mathrm{div}(\rho_\alpha \boldsymbol{u}) = \alpha\, \mathrm{div}(\boldsymbol{u})$. Setting, for simplicity, $\bar{\rho} = \rho_\alpha$, we proceed as in Step 1. Taking $\boldsymbol{u} = 0$ in $\mathbb{R}^d \setminus \Omega$, $\bar{\rho} = \alpha$ in $B \setminus \Omega$ and $\bar{\rho} = 0$ in $\mathbb{R}^d \setminus B$, we have $\bar{\rho} \in \mathrm{L}^{2\gamma}(\mathbb{R}^d)$, $\bar{\rho} \geq 0$ a.e. in $\mathbb{R}^d$, $\bar{\rho} \geq \alpha$ a.e. in B and $\boldsymbol{u} \in \mathrm{H}^1(\mathbb{R}^d)^d$. We also deduce from (1.2b):

$$\int_{\mathbb{R}^d} \bar{\rho} \boldsymbol{u} \cdot \boldsymbol{\nabla} \varphi \,\mathrm{d}\boldsymbol{x} = \int_{\mathbb{R}^d} \alpha \boldsymbol{u} \cdot \boldsymbol{\nabla} \varphi \,\mathrm{d}\boldsymbol{x} \text{ for all } \varphi \in \mathrm{C}^1(\mathbb{R}^d), \tag{A.4}$$

which gives $\mathrm{div}(\bar{\rho}\boldsymbol{u}) = h$ (in the sense of distributions) with $h = \alpha\, \mathrm{div}(\boldsymbol{u})$ (which is a function belonging to $L^2(\mathbb{R}^d)$).

For $n \in \mathbb{N}^\star$, we set $\bar{\rho}_n = \bar{\rho} \star r_n$ and $(\bar{\rho}\boldsymbol{u})_n = (\bar{\rho}\boldsymbol{u}) \star r_n$. Thanks to (A.4), we have $\mathrm{div}((\bar{\rho}\boldsymbol{u})_n) = h \star r_n$ in $\mathbb{R}^d$ (note that $h \star r_n$ tends to h in $\mathrm{L}^2(\mathbb{R}^d)$ as $n \to \infty$). Since $\boldsymbol{u} \in \mathrm{H}^1(\mathbb{R}^d)^d$, and $\bar{\rho} \in \mathrm{L}^{2\gamma}(\mathbb{R}^d)$, Lemma B.4 gives that $\mathrm{div}((\bar{\rho}\boldsymbol{u})_n - \bar{\rho}_n \boldsymbol{u}) \to 0$ weakly in $\mathrm{L}^{2\gamma/(\gamma+1)}(\mathbb{R}^d)$ as $n \to \infty$. Then, if $(q_n)_{n\in\mathbb{N}^\star}$ is a converging sequence in $\mathrm{L}^{2\gamma/(\gamma-1)}(\mathbb{R}^d)$ and in $\mathrm{L}^2(\mathbb{R}^d)$ to some q, we have:

$$\begin{aligned} -\int_{\mathbb{R}^d} \mathrm{div}(\bar{\rho}_n \boldsymbol{u})\, q_n \,\mathrm{d}\boldsymbol{x} \quad &= \int_{\mathbb{R}^d} \mathrm{div}((\bar{\rho}\boldsymbol{u})_n - \bar{\rho}_n \boldsymbol{u})\, q_n \,\mathrm{d}\boldsymbol{x} - \int_{\mathbb{R}^d} (h \star r_n)\, q_n \,\mathrm{d}\boldsymbol{x} \\ &\to -\int_{\mathbb{R}^d} h\, q \,\mathrm{d}\boldsymbol{x} \text{ as } n \to \infty. \end{aligned}$$

Let $1 < \beta \leq \gamma$. Taking $q_n = \bar\rho_n^{\beta-1}$, which actually converges in $\mathrm{L}^{2\gamma/(\gamma-1)}(\mathbb{R}^d)$ and in $\mathrm{L}^2(\mathbb{R}^d)$ towards $\bar\rho^{\beta-1}$, yields:

$$\begin{aligned} -\int_\Omega \mathrm{div}(\bar\rho_n \boldsymbol{u})\bar\rho_n^{\beta-1}\,\mathrm{d}\boldsymbol{x} &= -\int_{\mathbb{R}^d} \mathrm{div}(\bar\rho_n \boldsymbol{u})\bar\rho_n^{\beta-1}\,\mathrm{d}\boldsymbol{x} \\ &\to -\int_{\mathbb{R}^d} h\,\bar\rho^{\beta-1}\,\mathrm{d}\boldsymbol{x} \text{ as } n\to\infty. \end{aligned} \tag{A.5}$$

The function $\bar\rho_n^{\beta-1}$ belongs to $\mathrm{C}^1(\bar\Omega)$, at least for n large enough, since $\bar\rho \geq \alpha$ in $B \supset \bar\Omega$. Then, (A.5) reads:

$$(\beta-1)\int_\Omega \bar\rho_n^{\beta-1}\,\boldsymbol{u}\cdot\boldsymbol{\nabla}\bar\rho_n\,\mathrm{d}\boldsymbol{x} \to \int_{\mathbb{R}^d} h\,\bar\rho^{\beta-1}\,\mathrm{d}\boldsymbol{x} \text{ as } n\to\infty,$$

or, equivalently:

$$\frac{\beta-1}{\beta}\int_\Omega \bar\rho_n^{\beta}\,\mathrm{div}(\boldsymbol{u})\,\mathrm{d}\boldsymbol{x} \to -\int_{\mathbb{R}^d} h\,\bar\rho^{\beta-1}\,\mathrm{d}\boldsymbol{x} \text{ as } n\to\infty.$$

As $n\to\infty$, we have $\bar\rho_n \to \bar\rho$ in $\mathrm{L}^{2\gamma}(\Omega)$ and therefore $\bar\rho_n^\beta \to \bar\rho^\beta$ in $\mathrm{L}^2(\Omega)$. Then:

$$\int_\Omega \bar\rho_n^{\beta}\,\mathrm{div}(\boldsymbol{u})\,\mathrm{d}\boldsymbol{x} \to \int_\Omega \bar\rho^{\beta}\,\mathrm{div}(\boldsymbol{u})\,\mathrm{d}\boldsymbol{x}$$

and we obtain:

$$\frac{\beta-1}{\beta}\int_\Omega \bar\rho^{\beta}\,\mathrm{div}(\boldsymbol{u})\,\mathrm{d}\boldsymbol{x} = -\int_{\mathbb{R}^d} h\,\bar\rho^{\beta-1}\,\mathrm{d}\boldsymbol{x}.$$

Replacing $\bar\rho$ by $\rho+\alpha$ and h by $\alpha\,\mathrm{div}(\boldsymbol{u})$ gives, for all $\alpha > 0$:

$$\frac{\beta-1}{\beta}\int_\Omega (\rho+\alpha)^{\beta}\,\mathrm{div}(\boldsymbol{u})\,\mathrm{d}\boldsymbol{x} = -\int_\Omega \alpha\,\mathrm{div}(\boldsymbol{u})\,(\rho+\alpha)^{\beta-1}\,\mathrm{d}\boldsymbol{x}.$$

As $\alpha\to 0$, the $\mathrm{L}^2(\Omega)-$convergence of $(\rho+\alpha)^\beta$ and $(\rho+\alpha)^{\beta-1}$ towards ρ^β and $\rho^{\beta-1}$ leads to (2.1).

Once again, it remains to prove (2.1) if $\beta = 1$, which can be done passing to the limit, as $n\to\infty$, on (2.1) with $\beta = 1+1/n$ (and $n \geq n_0$, $1/n_0 \leq \gamma-1$). ☐

Appendix B. Lemmata

Definition B.1. A sequence $(F_n)_{n\in\mathbb{N}} \subset \mathrm{L}^1(\Omega)$ is said equi-integrable if:

$$\lim_{\lambda_d(A)\to 0}\int_A |F_n|\,\mathrm{d}\boldsymbol{x} = 0, \text{ uniformly with respect to } n\in\mathbb{N},$$

where $\lambda_d(A)$ denotes the $d-$dimensional Lebesgue measure of the Borelian subset $A\subset\Omega$.

Lemma B.2. *Let $(F_n)_{n\in\mathbb{N}} \subset \mathrm{L}^1(\Omega)$ be an equi-integrable sequence, and F be a function of $\mathrm{L}^1(\Omega)$. We assume that:*

$$\lim_{n\to\infty}\int_\Omega F_n\varphi\,\mathrm{d}\boldsymbol{x} = \int_\Omega F\varphi\,\mathrm{d}\boldsymbol{x} \textit{ for all } \varphi\in\mathrm{C}_c^\infty(\Omega). \tag{B.1}$$

Then:

$$\lim_{n\to\infty}\int_\Omega F_n\,\mathrm{d}\boldsymbol{x} = \int_\Omega F\,\mathrm{d}\boldsymbol{x}.$$

Proof. Let ϵ be a positive real number. Using the equi-integrability of $(F_n)_{n\in\mathbb{N}}$ and $F \in L^1(\Omega)$, there exists $\eta > 0$ such that, for any Borelian subset A of Ω:

$$\lambda_d(A) \leq \eta \Rightarrow \int_A |F_n| \,\mathrm{d}\boldsymbol{x} \leq \epsilon,\ \forall n \in \mathbb{N},\ \text{and} \int_A |F| \,\mathrm{d}\boldsymbol{x} \leq \epsilon. \tag{B.2}$$

Let K be a compact subset of Ω. Then, there exists $\varphi \in \mathrm{C}_c^\infty(\Omega)$ such that $0 \leq \varphi \leq 1$ in Ω, $\varphi = 1$ in K and $\lambda_d(\Omega \setminus K) \leq \eta$ and we obtain, for all $n \in \mathbb{N}$,

$$\int_\Omega (F_n - F) \,\mathrm{d}\boldsymbol{x} = \int_\Omega (F_n - F)(1-\varphi) \,\mathrm{d}\boldsymbol{x} + \int_\Omega (F_n - F)\, \varphi \,\mathrm{d}\boldsymbol{x},$$

which gives, thanks to (B.2) with $A = \Omega \setminus K$,

$$|\int_\Omega (F_n - F) \,\mathrm{d}\boldsymbol{x}| \leq 2\epsilon + |\int_\Omega (F_n - F)\, \varphi \,\mathrm{d}\boldsymbol{x}|.$$

Finally, (B.1) gives the existence of $n_0 \in N$ such that:

$$n \geq n_0 \Rightarrow |\int_\Omega (F_n - F) \,\mathrm{d}\boldsymbol{x}| \leq 3\epsilon.$$

which concludes the proof. □

Remark B.3. Indeed, with the assumptions of lemma B.2, we may prove that $F_n \to F$ weakly in $L^1(\Omega)$, but we do not need this finer result here.

Lemma B.4. *Let $\gamma > 1$, $\rho \in \mathrm{L}^{2\gamma}(\mathbb{R}^d)$, and $\boldsymbol{u} \in \mathrm{H}^1(\mathbb{R}^d)^d$. Let $(r_n)_{n\in\mathbb{N}^\star}$ be a sequence of mollifiers as given by* (A.2) *and, for $n \in \mathbb{N}^\star$, $\rho_n = \rho \star r_n$ and $(\rho\boldsymbol{u})_n = (\rho\boldsymbol{u}) \star r_n$. Then, $[(\rho\boldsymbol{u})_n - \rho_n\boldsymbol{u}] \to 0$ weakly in $\mathrm{W}^{1,(2\gamma)/(\gamma+1)}(\mathbb{R}^d)^d$ (which gives, in particular, that $\mathrm{div}((\rho\boldsymbol{u})_n - \rho_n\boldsymbol{u}) \to 0$ weakly in $\mathrm{L}^{(2\gamma)/(\gamma+1)}(\mathbb{R}^d)$).*

Proof. First, we remark that $\rho_n \to \rho$ in $\mathrm{L}^{2\gamma}(\mathbb{R}^d)$ as $n \to \infty$. Let $\delta = 2\gamma/(\gamma+1)$. Since $\boldsymbol{u} \in \mathrm{L}^2(\mathbb{R}^d)^d$, we deduce that $\rho_n\boldsymbol{u} \to \rho\boldsymbol{u}$ in $\mathrm{L}^\delta(\mathbb{R}^d)^d$ as $n \to \infty$. Since $\rho\boldsymbol{u} \in \mathrm{L}^\delta(\mathbb{R}^d)^d$, $(\rho\boldsymbol{u})_n \to \rho\boldsymbol{u}$ in $\mathrm{L}^\delta(\mathbb{R}^d)^d$ as $n \to \infty$. This gives $[(\rho\boldsymbol{u})_n - \rho_n\boldsymbol{u}] \to 0$ in $\mathrm{L}^\delta(\mathbb{R}^d)^d$. Then, in order to prove the lemma, thanks to the reflexivity of $\mathrm{W}^{1,\delta}(\mathbb{R}^d)$, we only have to prove that the sequence $((\rho\boldsymbol{u})_n - \rho_n\boldsymbol{u})_{n\in\mathbb{N}^\star}$ is bounded in $\mathrm{W}^{1,\delta}(\mathbb{R}^d)^d$. Then, we only have to prove that, for any $i, j \in \{1, \ldots, d\}$ (denoting by $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_d$ the components of $\boldsymbol{u}$), the sequence:

$$(\partial_i[(\rho\boldsymbol{u}_j)_n - \rho_n\boldsymbol{u}_j])_{n\in\mathbb{N}^\star}$$

is bounded in $\mathrm{L}^\delta(\mathbb{R}^d)$ (where ∂_i denotes the derivative with respect to $\boldsymbol{x}_i$).

Let $i, j \in \{1, \ldots, d\}$ and $n \in \mathbb{N}^\star$. We have:

$$\partial_i[(\rho\boldsymbol{u}_j)_n - \rho_n\boldsymbol{u}_j] = (\rho\boldsymbol{u}_j) \star \partial_i r_n - (\rho \star \partial_i r_n)\ \boldsymbol{u}_j - \rho_n \star \partial_i \boldsymbol{u}_j = F_n - G_n,$$

with

$$F_n = (\rho\boldsymbol{u}_j) \star \partial_i r_n - (\rho \star \partial_i r_n)\ \boldsymbol{u}_j \text{ and } G_n = \rho_n \star \partial_i \boldsymbol{u}_j.$$

The sequence $(G_n)_{n\in\mathbb{N}^\star}$ is bounded in $\mathrm{L}^\delta(\mathbb{R}^d)$ since:

$$\|G_n\|_{\mathrm{L}^\delta(\mathbb{R}^d)} \ \leq\ \|\rho_n\|_{\mathrm{L}^{2\gamma}(\mathbb{R}^d)}\ \|\partial_i\boldsymbol{u}_j\|_{\mathrm{L}^2(\mathbb{R}^d)} \ \leq\ \|\rho\|_{\mathrm{L}^{2\gamma}(\mathbb{R}^d)}\ \|\boldsymbol{u}\|_{\mathrm{H}^1(\mathbb{R}^d)^d}\ .$$

We have now to prove that the sequence $(F_n)_{n\in\mathbb{N}^\star}$ is bounded in $\mathrm{L}^\delta(\mathbb{R}^d)$. For a.e. $\boldsymbol{x} \in \mathbb{R}^d$, we have:

$$\begin{aligned} F_n(\boldsymbol{x}) &= \int_{\mathbb{R}^d} \rho(\boldsymbol{x}-\boldsymbol{y})\ \big(\boldsymbol{u}_j(\boldsymbol{x}-\boldsymbol{y}) - \boldsymbol{u}_j(\boldsymbol{x})\big)\ \partial_i r_n(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y} \\ &= \int_B \rho(\boldsymbol{x}-\frac{\boldsymbol{z}}{n})\ \big(\boldsymbol{u}_j(\boldsymbol{x}-\frac{\boldsymbol{z}}{n}) - \boldsymbol{u}_j(\boldsymbol{x})\big)\ n\ \partial_i r(\boldsymbol{z})\,\mathrm{d}\boldsymbol{z}, \end{aligned}$$

where B is a ball of center 0 and radius R containing the support of r. Then, setting $\delta' = (2\gamma)/(\gamma-1)$ (so that $1/\delta + 1/\delta' = 1$) and using Hölder Inequality, we get:

$$|F_n(\boldsymbol{x})|^\delta \le n^\delta \int_B |\rho(\boldsymbol{x}-\frac{\boldsymbol{z}}{n})\ \big(\boldsymbol{u}_j(\boldsymbol{x}-\frac{\boldsymbol{z}}{n}) - \boldsymbol{u}_j(\boldsymbol{x})\big)|^\delta\ |\partial_i r(\boldsymbol{z})|\,\mathrm{d}\boldsymbol{z} \\ \left[\int_B |\partial_i r(\boldsymbol{z})|\,\mathrm{d}\boldsymbol{z}\right]^{\delta/\delta'}.$$

We set:

$$C = \left[\int_B |\partial_i r(\boldsymbol{z})|\,\mathrm{d}\boldsymbol{z}\right]^{\delta/\delta'},$$

we integrate over $\mathbb{R}^d$ the preceding inequality and we use Fubini-Tonelli Theorem:

$$\begin{aligned} &\int_{\mathbb{R}^d} |F_n(\boldsymbol{x})|^\delta\,\mathrm{d}\boldsymbol{x} \le \\ &\qquad C n^\delta \int_B \left[\int_{\mathbb{R}^d} |\rho(\boldsymbol{x}-\frac{\boldsymbol{z}}{n})(\boldsymbol{u}_j(\boldsymbol{x}-\frac{\boldsymbol{z}}{n}) - \boldsymbol{u}_j(\boldsymbol{x}))|^\delta\,\mathrm{d}\boldsymbol{x}\right]\ |\partial_i r(\boldsymbol{z})|\,\mathrm{d}\boldsymbol{z}. \end{aligned} \tag{B.3}$$

Using again Hölder Inequality, we have for $\boldsymbol{z} \in B$:

$$\int_{\mathbb{R}^d} |\rho(\boldsymbol{x}-\frac{\boldsymbol{z}}{n})(\boldsymbol{u}_j(\boldsymbol{x}-\frac{\boldsymbol{z}}{n}) - \boldsymbol{u}_j(\boldsymbol{x}))|^\delta\,\mathrm{d}\boldsymbol{x} \le \left[\int_{\mathbb{R}^d} |\rho(\boldsymbol{x}-\frac{\boldsymbol{z}}{n})|^{2\gamma}\,\mathrm{d}\boldsymbol{x}\right]^{1/(\gamma+1)} \\ \left[\int_{\mathbb{R}^d} |\boldsymbol{u}_j(\boldsymbol{x}-\frac{\boldsymbol{z}}{n}) - \boldsymbol{u}_j(\boldsymbol{x})|^2\,\mathrm{d}\boldsymbol{x}\right]^{\gamma/(\gamma+1)}.$$

Since $\displaystyle\int_{\mathbb{R}^d} |\rho(\boldsymbol{x}-\frac{\boldsymbol{z}}{n})|^{2\gamma}\,\mathrm{d}\boldsymbol{x} = \|\rho\|_{\mathrm{L}^{2\gamma}}^{2\gamma}$ and, for all $\boldsymbol{z} \in B$ (see Lemma B.5):

$$\int_{\mathbb{R}^d} |\boldsymbol{u}_j(\boldsymbol{x}-\frac{\boldsymbol{z}}{n}) - \boldsymbol{u}_j(\boldsymbol{x})|^2\,\mathrm{d}\boldsymbol{x} \le (\frac{R}{n})^2\ \|\boldsymbol{u}\|_{\mathrm{H}^1(\mathbb{R}^d)^d}^2,$$

we obtain:

$$\int_{\mathbb{R}^d} |\rho(\boldsymbol{x}-\frac{\boldsymbol{z}}{n})(\boldsymbol{u}_j(\boldsymbol{x}-\frac{\boldsymbol{z}}{n}) - \boldsymbol{u}_j(\boldsymbol{x}))|^\delta\,\mathrm{d}\boldsymbol{x} \le (\frac{R}{n})^\delta\ \|\rho\|_{\mathrm{L}^{2\gamma}}^\delta\ \|\boldsymbol{u}\|_{\mathrm{H}^1(\mathbb{R}^d)^d}^\delta.$$

Then, (B.3) yields:

$$\int_{\mathbb{R}^d} |F_n(\boldsymbol{x})|^\delta\,\mathrm{d}\boldsymbol{x} \le C n^\delta (\frac{R}{n})^\delta\ \|\rho\|_{\mathrm{L}^{2\gamma}}^\delta\ \|\boldsymbol{u}\|_{\mathrm{H}^1(\mathbb{R}^d)^d}^\delta\ C^{\frac{\delta'}{\delta}}.$$

This proves that the sequence $(F_n)_{n\in\mathbb{N}}$ is bounded in $\mathrm{L}^\delta(\mathbb{R}^d)$ and concludes the proof of the lemma since $\delta = (2\gamma)/(\gamma+1)$. □

Lemma B.5. *Let $w \in \mathrm{H}^1(\mathbb{R}^d)$ and $\boldsymbol{h} \in \mathbb{R}^d$. Then:*

$$\|w(\cdot+\boldsymbol{h}) - w\|_{\mathrm{L}^2(\mathbb{R}^d)} \le |\boldsymbol{h}|\ \|w\|_{\mathrm{H}^1(\mathbb{R}^d)}, \tag{B.4}$$

where $|\boldsymbol{h}|$ is the Euclidean norm of $\boldsymbol{h}$.

Proof. Let $\boldsymbol{h} \in \mathbb{R}^d$ and $\varphi \in \mathrm{C}_c^\infty(\mathbb{R}^d)$. For all $\boldsymbol{x} \in \mathbb{R}^d$, we have:

$$\varphi(\boldsymbol{x}+\boldsymbol{h}) - \varphi(\boldsymbol{x}) = \int_0^1 \boldsymbol{\nabla}\varphi(\boldsymbol{x}+t\boldsymbol{h}) \cdot \boldsymbol{h} \,\mathrm{d}t.$$

Then:

$$|\varphi(\boldsymbol{x}+\boldsymbol{h}) - \varphi(\boldsymbol{x})|^2 \leq |\boldsymbol{h}|^2 \int_0^1 |\boldsymbol{\nabla}\varphi(\boldsymbol{x}+t\boldsymbol{h})|^2 \,\mathrm{d}t.$$

Integrating the preceding inequality over $\mathbb{R}^d$ and using Fubini-Tonelli Theorem gives:

$$\|\varphi(\cdot+\boldsymbol{h}) - \varphi\|^2_{\mathrm{L}^2(\mathbb{R}^d)} \leq |\boldsymbol{h}|^2 \, \|\varphi\|^2_{\mathrm{H}^1(\mathbb{R}^d)},$$

which leads to (B.4) with $w = \varphi$. Finally, the density of $\mathrm{C}_c^\infty(\mathbb{R}^d)$ in $\mathrm{H}^1(\mathbb{R}^d)$ gives (B.4) for all $w \in \mathrm{H}^1(\mathbb{R}^d)$. □

Lemma B.6. *Let Ω be a bounded open set of $\mathbb{R}^d$, $q \geq 1$ and $a,\ b,\ r > 0$. Then there exists C, only depending on $\Omega,\ q,\ a,\ b,\ r$, such that:*

$$p \in \mathrm{L}^q(\Omega), \quad \int_\Omega |p - m(p)|^q \,\mathrm{d}\boldsymbol{x} \leq a, \quad \int_\Omega |p|^r \,\mathrm{d}\boldsymbol{x} \leq b \Rightarrow \int_\Omega |p|^q \,\mathrm{d}\boldsymbol{x} \leq C,$$

where $m(p)$ denotes the mean value of p.

Proof. The proof is trivial if $r \geq 1$ since, in this case, Hölder inequality gives $\lambda_d(\Omega)|m(p)| \leq b^{1/r}\lambda_d(\Omega)^{1-1/r}$ (where $\lambda_d(\Omega)$ is the $d-$dimensional Lebesgue measure of Ω) and we conclude with $|p|^q \leq 2^q|p-m(p)|^q + 2^q m(p)^q$ a.e. in Ω.

We assume now that $r < 1$. Then $|m(p)|^r \leq 2^r|p-m(p)|^r + 2^r|p|^r \leq 2^r|p - m(p)|^q + 2^r + 2^r|p|^r$ a.e. in Ω. This yields $\lambda_d(\Omega)|m(p)|^r \leq 2^r a + 2^r\lambda_d(\Omega) + 2^r b$ and we conclude, once again, using $|p|^q \leq 2^q|p-m(p)|^q + 2^q m(p)^q$ a.e. in Ω. □

The following lemma is due to Necas, and a simple proof of this result is given in [1].

Lemma B.7. *Let $q \in \mathrm{L}^2(\Omega)$ such that $\int_\Omega q \,\mathrm{d}\boldsymbol{x} = 0$. Then, there exists $\boldsymbol{v} \in (\mathrm{H}^1_0(\Omega))^d$ such that $\mathrm{div}(\boldsymbol{v}) = q$ a.e. in Ω and $\|\boldsymbol{v}\|_{\mathrm{H}^1(\Omega)^d} \leq c_2 \|q\|_{\mathrm{L}^2(\Omega)}$ where c_2 only depends on Ω.*

Lemma B.8. *Let Ω be a bounded open set of $\mathbb{R}^d$ and $\rho \in \mathrm{L}^2(\Omega)$. Then, there exists $\boldsymbol{v} \in \mathrm{H}^1(\Omega)^d$ such that $\mathrm{div}(\boldsymbol{v}) = \rho$ a.e. in Ω, $\mathrm{curl}(\boldsymbol{v}) = 0$ a.e. in Ω and $\|\boldsymbol{v}\|_{\mathrm{H}^1(\Omega)^d} \leq C \|\rho\|_{\mathrm{L}^2(\Omega)}$ where C only depends on Ω.*
Furthermore, if the boundary of Ω is Lipschitz continuous and $\rho \in \mathrm{H}^1(\Omega)$, it is possible to have $\boldsymbol{v} \in \mathrm{H}^2(\Omega)^d$ and $\|\boldsymbol{v}\|_{\mathrm{H}^2(\Omega)^d} \leq C \|\rho\|_{\mathrm{H}^1(\Omega)}$ where C only depends on Ω.

Proof. Let B be a ball containing Ω, $\rho \in \mathrm{L}^2(\Omega)$. We extend ρ to B by zero outside Ω. Let w be the solution of the Dirichlet problem on B associated to ρ. A classical regularity result gives that $w \in \mathrm{H}^2(B)$ and gives an estimate on the H^2-norm of w in terms of the L^2-norm of ρ, only depending on the radius of B (and thus on Ω). Taking $\boldsymbol{v} = -\boldsymbol{\nabla} w$ concludes the proof of the first part of the lemma.
Let us now suppose that $\rho \in \mathrm{H}^1(\Omega)$. We now extend ρ to B in such a way that the extension, still denoted by ρ, belongs to $\mathrm{H}^1(B)$ and $\|\rho\|_{\mathrm{H}^1(B)} \leq c\,\|\rho\|_{\mathrm{H}^1(\Omega)}$, with c only depending on Ω (such an extension exists since Ω is supposed to have a Lipschitz continuous boundary). We then proceed as previously, taking for w the solution of the Dirichlet problem on B associated to ρ, $\boldsymbol{v} = -\boldsymbol{\nabla} w$, and invoking the regularity of the Laplace operator. □

Lemma B.9. *Let Ω be a bounded open set of $\mathbb{R}^d$, $p, q \in [1, \infty]$, $q < p$ and $(g_n)_{n\in\mathbb{N}}$ a bounded sequence of $\mathrm{L}^p(\Omega)$. Assume that $g_n \to g$ a.e. in Ω. Then, $g \in \mathrm{L}^p(\Omega)$ and $g_n \to g$ in $\mathrm{L}^q(\Omega)$, as $n \to \infty$.*

Proof. The fact that $g \in \mathrm{L}^p(\Omega)$ is an easy consequence of Fatou's Lemma applied to the sequence $(|g_n|^p)_{n\in\mathbb{N}}$. The fact that $g_n \to g$ in $\mathrm{L}^q(\Omega)$, as $n \to \infty$ is an easy consequence of the Egorov Theorem (and uses the a.e. convergence of g_n to g) and of the equi-integrability of the sequence $(|g_n|^q)_{n\in\mathbb{N}}$ (which follows from the boundedness in $\mathrm{L}^p(\Omega)$ of $(g_n)_{n\in\mathbb{N}}$). □

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