Nonlinear methods for linear equations

and numerical schemes

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ABSTRACT. This paper presents some methods for the study of linear elliptic equations. These methods, now classical, are essentially due to G. Stampacchia. The second part of the paper is devoted to the possible use of these methods for the study of numerical schemes for these linear elliptic equations.

RÉSUMÉ. Cet article présente tout d'abord des méthodes non linéaires pour l'étude d'équations elliptiques non linéaires. Ces méthodes, maintenant assez classiques, sont essentiellement dues à G. Stampacchia. La seconde partie est consacrée aux possibilités d'utlilisation de ces méthodes pour l'étude des schémas numériques pour ces équations elliptiques linéaires.

KEYWORDS : elliptic equations, nonlinear methods, numerical schemes

MOTS-CLÉS : équations elliptiques, methodes non linéaires, schémas numériques

1. Introduction

The first objective of this paper is to present some (now classical) results on linear elliptic equations with Stampacchia's methods (initiated in [1], 1965). The common feature of these methods is that there are nonlinear. Indeed, they use some test functions, in the weak formulation of the equation, which are nonlinear functions of the unknown. Then, the second objective is to see if it is possible to use these methods for the study of numerical schemes, in order to prove, for instance, some properties on the approximate solutions or to prove the convergence of the approximate solution towards the exact solution as the mesh size goes to zero. We will see that the use of these nonlinear methods requires some restrictions on the numerical schemes which are not needed when using linear methods. A possible consequence of this study is to give some guideline for the construction of new numerical schemes allowing the use of these nonlinear methods.

In all this paper, Ω is a bounded open set of $\mathbb{R}^d$ ($d \geq 1$) with a Lipschitz continuous boundary. The set $M_d(\mathbb{R})$ is the set of $d \times d$ positive definite symmetric matrices. The function A is a function from Ω to $M_d(\mathbb{R})$ with coefficients in $L^\infty(\Omega)$ and uniformly coercive, that is, for some $\alpha > 0$, $A(x)\xi.\xi \geq \alpha\xi.\xi$ for all $\xi \in \mathbb{R}^d$ and for a.e. $x \in \Omega$. The principal part of the considered linear elliptic operator is $u \mapsto -\mathrm{div}(A\nabla u)$.

In Section 2 are presented four examples of application of these Stampacchia's methods. In Section 3 is presented the study of numerical schemes.

2. Stampacchia's methods

One fisrt considers the Dirichlet problem with the function A and a second member f in $L^2(\Omega)$, whose the classical weak formulation is:

$$\begin{aligned} & u \in H_0^1(\Omega), \\ & \int_\Omega A\nabla u \cdot \nabla v dx = \int_\Omega f v dx, \text{ for all } v \in H_0^1(\Omega). \end{aligned} \tag{1}$$

2.1. positivity

As it is well known, (1) has a unique solution u and $f \geq 0$ a.e. implies $u \geq 0$ a.e. (or, equivalently, $f \leq 0$ a.e. implies $u \leq 0$ a.e.). Assuming $f \leq 0$ a.e., a simple way to prove this result is to take $v = u^+$ in (1), this is possible since $u^+ \in H_0^1(\Omega)$ and leads to:

$$\alpha \| |\nabla u^+| \|_2 \leq \int_\Omega A\nabla u^+ \cdot \nabla u^+ = \int_\Omega A\nabla u \cdot \nabla u^+ = \int_\Omega f u^+ \leq 0.$$

Then, $\nabla u^+ = 0$ a.e. and $u^+ = 0$ a.e.., $u \le 0$ a.e... A property used in this proof is that, for $u \in H_0^1(\Omega)$, one has $\nabla u^+ = 1_{u>0}\nabla u = 1_{u\ge 0}\nabla u$ a.e.. More generally (cf. [1]), if $\varphi : \mathbb{R} \to \mathbb{R}$ is a Lipschitz continuous function such that $\varphi(0) = 0$ and if $u \in H_0^1(\Omega)$, the function $\varphi(u)$ belongs to $H_0^1(\Omega)$ and $\nabla\varphi(u) = \varphi'(u)\nabla u$ a.e.. (Actually, for all the results given in this paper, it is possible to use only C^1 functions φ.)

2.2. Bounded solutions (Stampacchia)

Let $f \in H^{-1}(\Omega)$ and u be the unique of (1) where the second member is replaced by $\langle f, v\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}$. The aim is to find a condition on the "regularity" of f in order to obtain $u \in L^\infty(\Omega)$ ($d \ge 2$, since there is no condition on f if $d = 1$). A result of G. Stampacchia ([1]) gives that $u \in L^\infty(\Omega)$ if $f \in W^{-1,p}(\Omega)$ for some $p > d$. In the case of (1), this gives that a sufficient conditiion in order to have $u \in L^\infty(\Omega)$ is that $f \in L^p(\Omega)$ for some $p > d/2$. In order to prove this result, let $p > d$ such that $f \in W^{-1,p}(\Omega)$. Then, it exists $F \in (L^p(\Omega))^d$ such that $f =\text{div}F$ and one has:

$$u \in H_0^1(\Omega), \int_\Omega A\nabla u \cdot \nabla v dx = \int_\Omega F \cdot \nabla v dx \text{ for all } v \in H_0^1(\Omega).$$

Let $k \in \mathbb{R}_+^\star$. Taking $v = \psi(u) = (u-k)^+ - (u+k)^-$ (ψ is nondecreasing). One has $\nabla\psi(u) = 1_{A_k}\nabla u$ a.e., with $A_k = \{|u| \ge k\}$ and:

$$\int_{A_k} A\nabla u \cdot \nabla u dx = \int_{A_k} F \cdot \nabla u dx.$$

Then, with Cauchy-Schwarz and Hölder inequalites (with $p/2$ and its conjugate):

$$\alpha \| |\nabla u| \|_{L^2(A_k)} \le C_1 \|f\|_{W^{-1,p}} \lambda(A_k)^{\frac{1}{2}-\frac{1}{p}}.$$

Using Sobolev imbedding ($W_0^{1,1}(\Omega) \subset L^{d/(d-1)}(\Omega)$) and Cauchy-Schwarz again:

$$\lambda(A_h) \le \frac{C_2 \|f\|_{W^{-1,p}}^\gamma}{h-k} \lambda(A_k)^\beta, \text{ for } 0 \le k < h,$$

with $\gamma = d/(d-1)$ and $\beta = \frac{p-1}{p}\frac{d}{d-1} > 1$ (since $p > d$).

Since $\beta > 1$, this gives $\lambda(A_h) = 0$ if $h \ge C_3\|f\|_{W^{-1,p}}$. Then: $\|u\|_\infty \le C_3\|f\|_{W^{-1,p}}$.

A further developpement of this proof leads to $u \in C(\overline{\Omega})$ and finally to the Hölder continuity of u.

2.3. Existence of a solution for a measure as second member

Let f be a measure on Ω ($f \in (C(\overline{\Omega}))'$). The aim is now to prove existence (and possibly uniqueness) of the solution of the Dirichlet problem with f as second member (for $d > 2$, the solution is, in general, no longer in $H_0^1(\Omega)$ since a measure is, in general, not in $H^{-1}(\Omega)$). The First method to solve this problem is a duality method (Stampacchia, [1]). A second method is to pass to the limit on approximate solutions (cf. [2]). The main difficulty is to obtain estimates on u, solution of (1) with $f \in L^2$, only depending of the L^1-norm of f. For $\theta > 1$, let, for $s \in \mathbb{R}$, $\varphi(s) = \int_0^s \frac{1}{(1+|t|)^\theta} dt$. Taking $v = \varphi(u) \in H_0^1(\Omega)$ in (1) leads to:

$$\int_\Omega |\nabla \phi(u)|^2 dx = \int_\Omega \frac{|\nabla u|^2}{(1+|u|)^\theta} dx \le C_\theta \|f\|_1,$$

with $C_\theta = \int_0^\infty \frac{1}{(1+|t|)^\theta} dt < \infty$ and $\phi(s) = \int_0^s \sqrt{\varphi'(t)} dt$. Using Hölder Inequality, Sobolev imbedding and θ close to 1, one obtains, for $q < \frac{d}{d-1}$, a bound on u in $W_0^{1,q}$ only depending on the L^1-norm of f and on q. Passing to the limit on a sequence of approximate solutions (corresponding to regular second members converging towards f), one obtains existence of a solution (in the disctribution sense) if f is a measure. This solution belongs to $W_0^{1,q}(\Omega)$ for all $q < \frac{d}{d-1}$.

2.4. Convection-diffusion without coercivity

Let $w \in C(\overline{\Omega})^d$ and $f \in L^2(\Omega)$. One consider now the following problem:

$$\begin{aligned} & u \in H_0^1(\Omega), \\ & \int_\Omega A\nabla u \cdot \nabla v dx - \int_\Omega u w \cdot \nabla v dx = \int_\Omega f v dx, \text{ for all } v \in H_0^1(\Omega). \end{aligned} \tag{2}$$

Existence and uniqueness of a solution for this problem is known (J. Droniou, [3]). The main step is to obtain an *a priori* estimate on meas($\{|u| \ge k\}$) (this measure goes to 0 as $k \to \infty$). (Then, one obtains an $H_0^1(\Omega)-$estimate and existence follows with a topological degree argument. Uniqueness is a consequence of an existence result for the dual problem.) In order to obtain this *a priori* estimate, one takes takes $v = \varphi(u)$ with $\varphi(s) = \int_0^s \frac{1}{(1+|s|)^2}$ in (2), it gives:

$$\alpha \int_\Omega \frac{|\nabla u|^2}{(1+|u|)^2} dx \le \|f\|_1 + \int_\Omega \frac{|w||u||\nabla u|}{(1+|u|)^2} dx \le \|f\|_1 + \|w\|_\infty \int_\Omega \frac{|\nabla u|}{1+|u|} dx,$$

with $\|w\|_\infty = \sup_{x\in\Omega} |w(x)| < \infty$. and, with Young Inequality:

$$\int_\Omega |\nabla \phi(u)|^2 dx = \int_\Omega |\nabla \ln(1+|u|)|^2 dx = \int_\Omega \frac{|\nabla u|^2}{(1+|u|)^2} dx \le C,$$

where C only depends on α, $\|f\|_1$ and $\|w\|_\infty$, and where $\phi(s) = \int_0^s \sqrt{\varphi'(t)} dt$. Since $\ln(1+|u|) \in H_0^1(\Omega)$, one deduces an estimate on $\ln(1+|u|)$ in $L^2(\Omega)$ and then an estimate on meas($\{|u| \ge k\}$).

3. Numerical schemes

In this section, the objective is to use the nonlinear methods of the preceding section for proving some properties of the approximate solutions (of the same equations) given by a numerical scheme and for proving convergence results of the approximate solution. It is possible to use Finite Volumes schemes or Finite Element schemes. One present here Finite Element schemes (it is simpler to present, since Finite Element schemes are closer to the weak formulation of the continuous problem). Let $\mathcal{M}$ is a mesh of Ω, with triangles ($d = 2$) or tetrahedra ($d = 3$). Let $H = \{u \in C(\overline{\Omega});\, u_{|_K} \in P^1\}$ and $H_0 = \{u \in H;\, u = 0 \text{ on } \partial\Omega\}$. The approximate problem is:

$$u_{\mathcal{M}} \in H_0, \int_\Omega A\nabla u_{\mathcal{M}} \cdot \nabla v dx \;(- \int_\Omega u_{\mathcal{M}} w \cdot \nabla v dx) = T(v), \text{ for all } v \in H_0, \qquad (3)$$

where $T(v) = \int_\Omega f v dx$ or $\langle f, v\rangle$ of $\int_\Omega v df$ in the examples of Section 2. The first difficulty is that $u \in H_0$ does not implies $u^+, \psi(u), \varphi(u) \in H_0$ (with the notations of Section 2). But, it is possible to take, as test function in (3), the interpolate of the test function of the “continuous" case. Before to do this, we rewrite slightly differently (3) (taking, for simplicity, $w = 0$ and $T(v) = \int f v dx$).

Let us denote by $\mathcal{V}$ the set of vertices of the mesh and , for $K \in \mathcal{V}$, by ϕ_K the basis function associated to K. The unknown $u_{\mathcal{M}}$ may be writen as $u_{\mathcal{M}} = \sum_{K\in\mathcal{V}} u_K \phi_K$, and (3) may be written as, for all $v = \sum_{L\in\mathcal{V}} v_L \phi_L \in H_0$:

$$\sum_{(K,L)\in(\mathcal{V})^2} T_{K,L}(u_K - u_L)(v_K - v_L) = \int f v dx. \qquad (4)$$

where $T_{K,L} = -\int_\Omega A\nabla\phi_K \cdot \nabla\phi_L dx$. (Note that $\sum_{L\in\mathcal{V}} T_{K,L} = 0$.)

If $v \in C(\overline{\Omega})$, let us denote by $\Pi_{\mathcal{M}}(v)$ the element of H such $\Pi_{\mathcal{M}}(v) = v$ at the vertices of the mesh. Let $\varphi \in C(\mathbb{R}, \mathbb{R})$ Lipschitz continuous and nondecreasing and such that $\varphi(0) = 0$. Taking $v = \Pi_{\mathcal{M}}\varphi(u_{\mathcal{M}})$ (which belongs to H_0) in (4) leads to:

$$\sum_{(K,L)\in(\mathcal{V})^2} T_{K,L}(u_K - u_L)(\varphi(u_K) - \varphi(u_L)) = \int f v dx. \qquad (5)$$

Define ϕ by $\phi(s) = \int_0^s \sqrt{\varphi'(t)}dt$. For $a, b \in \mathbb{R}$, one has (thanks to Cauchy-Schwarz Inequality) $(\phi(a) - \phi(b))^2 \le (a - b)(\varphi(a) - \varphi(b))$. In order to be able to proceed as in Section 2, we will assume now that $T_{K,L} \ge 0$ for all $K, L \in \mathcal{V}$. Then (5) gives:

$$\sum_{(K,L)\in(\mathcal{V})^2} T_{K,L}(\phi(u_K) - \phi(u_L))^2 \le \sum_{(K,L)\in(\mathcal{V})^2} T_{K,L}(u_K - u_L)(\varphi(u_K) - \varphi(u_L)).$$

Since $\int_\Omega A\nabla\Pi_{\mathcal{M}}\phi(u) \cdot \nabla\Pi_{\mathcal{M}}\phi(u) = \sum_{(K,L)\in(\mathcal{V})^2} T_{K,L}(\phi(u_K) - \phi(u_L))^2$, it is now possible to continue as in Section 2. Let $u = u_{\mathcal{M}}$. If $f \le 0$ a.e., taking $\varphi(s) = s^+$ one obtains $\int_\Omega A\nabla\Pi_{\mathcal{M}}u^+ \cdot \nabla\Pi_{\mathcal{M}}u^+ dx = \sum_{(K,L)\in(\mathcal{V})^2} T_{K,L}(u_K^+ - u_L^+)^2 \leq 0$, from which one deduces $u^+ = 0$ and $u \leq 0$ a.e..

In the case of the third example of Section 2 (where f is a measure data), taking, for $\theta > 1$ and $s \in \mathbb{R}$, $\varphi(s) = \int_0^s \frac{1}{(1+|t|)^\theta} dt$, one obtains $\int_\Omega |\nabla\Pi_{\mathcal{M}}\phi(u)|^2 dx \leq C_\theta \|f\|_1$, with $\phi(s) = \int_0^s \sqrt{\varphi'(t)} dt$. We then deduce convenient bound on u to conclude to the convergence of the scheme (see, for instance [4], [5], [6]).

In the case of the fourth example (convection-diffusion without coercivity), one takes the same φ as before with $\theta = 2$. If the mesh size is small enough (or using an "upwinding" for the convection part), one obtains an $H_0^1(\Omega)-$estimate on $\Pi_{\mathcal{M}} \ln(1+|u|) \in H_0^1(\Omega)$, then, an estimate on $\ln(1+|u|)$ in $L^2(\Omega)$ and finally, as in the "continuous" case, an estimate on meas($\{|u| \geq k\}$) (see [5]).

In conclusion, If $T_{K,L} \geq 0$, for all K, L, the methods of Stampacchia can be used for the study of numerical schemes. Without the condition $T_{K,L} \geq 0$, it seems not easy to use the methods of Stampacchia... Without changing the mesh (Finite Element or Finite Volume with the so called "non admissible" meshes), a possible solution is perhaps to discretize this elliptic linear problem with a nonlinear scheme taking in (5) $T_{K,L}$ depending on the approximate solution and with $T_{K,L}(u) \geq 0$, for all K, L.

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