On the time continuity of entropy solutions

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Abstract

We show that any entropy solution u of a convection diffusion equation $\partial_t u + \mathrm{div} F(u) - \Delta\phi(u) = b$ in $\Omega \times (0,T)$ belongs to $C([0,T), L^1_{\mathrm{loc}}(\Omega))$. The proof does not use the uniqueness of the solution.

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1 The problem, and main result

Convection diffusion equations appear in a large class of problems, and have been widely studied. We consider in the sequel only equations under conservative form:

$$\partial_t u + \mathrm{div} F(u) - \Delta\phi(u) = b, \tag{1}$$

so that we can give some sense to (1) in the distributional sense. In this paper, we consider entropy solutions of (1) that do not take into account any boundary condition, or condition for $|x| \to +\infty$.

The proof does not use a L^1-contraction principle (see e.g. Alt & Luckaus [1] or Otto [11]), so that it can be applied in case where uniqueness is not ensured, like for example complex spatial coupling of different conservation laws as in [3], or for cases where uniqueness fails because of boundary conditions or conditions at $|x| = +\infty$, as it will be stressed in the sequel.

Let us now state the required assumptions on the data. Let Ω be an open subset of $\mathbb{R}^d$ ($d \geq 1$), and let T be a positive real value or $+\infty$.

$$F \text{ is a continuous function,} \tag{H1}$$

$$\phi \text{ is a nondecreasing Lipschitz function,} \tag{H2}$$

$$u_0 \in L^1_{\mathrm{loc}}(\Omega). \tag{H3}$$

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One has to make the following assumption on the source term:

$$b \in L^2_{\mathrm{loc}}([0,T); H^{-1}(\Omega)) \cap L^1_{\mathrm{loc}}(\Omega \times [0,T)). \tag{H4}$$

In the sequel, sign is the function defined by

$$\mathrm{sign}(s) = \begin{cases} 0 & \text{if} \quad s = 0, \\ 1 & \text{if} \quad s > 0, \\ -1 & \text{if} \quad s < 0. \end{cases}$$

We consider entropy weak solutions of (1), as in the famous work of Kružkov [9] for hyperbolic equations. This notion can be extended to degenerated parabolic equations, as noticed by Carrillo [4]. This leads to the following definition of entropy weak solution:

Definition 1 *A function u is said to be an entropy weak solution if:*

1. $u \in L^1_{\mathrm{loc}}(\Omega \times [0,T))$,
2. $F(u) \in \left(L^2_{\mathrm{loc}}(\Omega \times [0,T))\right)^d$,
3. $\phi(u) \in L^2_{\mathrm{loc}}([0,T); H^1_{\mathrm{loc}}(\Omega))$,
4. $\forall \psi \in \mathcal{D}^+(\Omega \times [0,T))$, $\forall \kappa \in \mathbb{R}$,

$$\int_0^T \int_\Omega |u-\kappa| \partial_t \psi dx dt + \int_\Omega |u_0 - \kappa| \psi(0) dx \\ + \int_0^T \int_\Omega sign(u-\kappa) \left(F(u) - F(k) - \nabla \phi(u)\right) \cdot \nabla \psi dx dt \\ + \int_0^T \int_\Omega sign(u-\kappa) b \psi dx dt \geq 0. \tag{2}$$

Proposition 1.1 *Any entropy weak solution is a weak solution, that is it fulfills the three first points in Definition 1, and:* $\forall \psi \in \mathcal{D}(\Omega \times [0,T))$,

$$\int_0^T \int_\Omega u \partial_t \psi dx dt + \int_\Omega u_0 \psi(0) dx \\ + \int_0^T \int_\Omega (F(u) - \nabla \phi(u)) \cdot \nabla \psi dx dt + \int_0^T \int_\Omega b \psi dx dt = 0. \tag{3}$$

Reciprocally, if ϕ is increasing, then any weak solution is an entropy solution.

Proof

Suppose first that ϕ is increasing, then the fact that any weak solution u is an entropy weak solution is just based on a convexity inequality, and on the fact that $\mathrm{sign}(\phi(a) - \phi(b)) = \mathrm{sign}(a-b)$ for all $(a,b) \in \mathbb{R}^2$. More details are available in [4] (see also [8]).

The fact that an entropy weak solution u is a weak solution is obvious if u belongs to $L^\infty_{\mathrm{loc}}(\Omega \times [0,T))$ (consider $\kappa = \pm \|u\|_{L^\infty(\mathrm{supp}(\psi))}$).

Suppose now that u only belongs to $L^1_{\mathrm{loc}}(\Omega \times [0,T))$. Let $\kappa \in \mathbb{R}$, then for all $\psi \in \mathcal{D}(\Omega \times [0,T))$, one has

$$\int_0^T \int_\Omega \kappa \partial_t \psi dx dt + \int_\Omega \kappa \psi(0) dx + \int_0^T \int_\Omega F(\kappa) \cdot \nabla \psi dx dt = 0, \tag{4}$$

which added to (2) yields: $\forall \psi \in \mathcal{D}^+(\Omega \times [0,T))$,

$$\begin{aligned} &\int_0^T \int_\Omega \left(|u-\kappa|+\kappa\right) \partial_t \psi dx dt + \int_\Omega \left(|u_0-\kappa|+\kappa\right) \psi(0) dx \\ &+ \iint_{\{u<\kappa\}} \left(2F(\kappa) - F(u)\right) \cdot \nabla \psi dx dt + \iint_{\{u\geq\kappa\}} F(u) \cdot \nabla \psi dx dt \\ &+ \int_0^T \int_\Omega \operatorname{sign}(u-\kappa) \left(-\nabla \phi(u) \cdot \nabla \psi + b\psi\right) dx dt \geq 0. \end{aligned} \tag{5}$$

One will now let κ tend to $-\infty$ in (5). Suppose that $\kappa < 0$, then

$$\big||u-\kappa|+\kappa\big| \leq |u| \text{ and } \big||u-\kappa|+\kappa\big| \to u \text{ a.e. in } \operatorname{supp}(\psi),$$

then, from the dominated convergence theorem, one has

$$\lim_{\kappa\to-\infty} \int_0^T \int_\Omega \left(|u-\kappa|+\kappa\right) \partial_t \psi dx dt = \int_0^T \int_\Omega u \partial_t \psi dx dt,$$

$$\lim_{\kappa\to-\infty} \int_\Omega \left(|u_0-\kappa|+\kappa\right) \psi(0) dx = \int_\Omega u_0 \psi(0) dx.$$

It follows also from the dominated convergence theorem that

$$\lim_{\kappa\to-\infty} \iint_{\{u\geq\kappa\}} F(u) \cdot \nabla \psi dx dt = \int_0^T \int_\Omega F(u) \cdot \nabla \psi dx dt,$$

and that

$$\begin{aligned} \lim_{\kappa\to-\infty} &\int_0^T \int_\Omega \operatorname{sign}(u-\kappa) \left(-\nabla \phi(u) \cdot \nabla \psi + b\psi\right) dx dt \\ &= \int_0^T \int_\Omega \left(-\nabla \phi(u) \cdot \nabla \psi + b\psi\right) dx dt. \end{aligned}$$

It appears clearly that

$$\left(2F(\kappa) - F(u)\right) \chi_{\{u<\kappa\}} \to 0 \text{ a.e. in } \operatorname{supp}(\psi) \text{ as } \kappa \to -\infty,$$

where $\chi_{\mathcal{E}}(x,t) = 1$ if $(x,t) \in \mathcal{E}$ and 0 otherwise. In order to obtain the domination, we will follow the method proposed in Remark 2.1 of [5]. Assume that there exists a sequence $(\kappa_n)_n$ with $\kappa_n \to -\infty$ such that $(|F(\kappa_n)|)_n$ is bounded by C, then

$$\left|\left(2F(\kappa_n) - F(u)\right) \chi_{\{u<\kappa_n\}}\right| \leq 2C + F(u) \in L^1(\operatorname{supp}(\psi)).$$

Otherwise, $\lim_{s\to-\infty} |F(s)| = \infty$. Let $n_0 \in \mathbb{N}$ such that $n_0 > \inf_{s\in\mathbb{R}} |F(s)|$, we set for $n \geq n_0$:

$$\kappa_n = \min\{s \in \mathbb{R} \mid |F(s)| \leq n\},$$

then one has $\lim_{n\to\infty} \kappa_n = -\infty$, and

$$s \leq \kappa_n \implies |F(s)| \geq |F(\kappa_n)|.$$

Hence, in this case, one has

$$\left|\left(2F(\kappa_n) - F(u)\right) \chi_{\{u<\kappa_n\}}\right| \leq 3|F(u)| \in L^1(\operatorname{supp}(\psi)),$$

and thus

$$\lim_{n\to\infty}\iint_{\{u>\kappa_n\}}(2F(\kappa_n)-F(u))\cdot\nabla\psi dxdt=0.$$

Therefore, letting $(\kappa_n)_n$ tend to $-\infty$ in (5) provides that, for all $\psi\in\mathcal{D}^+(\Omega\times[0,T))$,

$$\int_0^T\int_\Omega u\partial_t\psi dxdt+\int_\Omega u_0\psi(0)dx$$
$$+\int_0^T\int_\Omega(F(u)-\nabla\phi(u))\cdot\nabla\psi dxdt+\int_0^T\int_\Omega b\psi dxdt\geq 0.$$

The same way, subtracting (4) to (2) and considering a convenient sequence $(\kappa_n)_n$ tending to $+\infty$, one obtains

$$\int_0^T\int_\Omega u\partial_t\psi dxdt+\int_\Omega u_0\psi(0)dx$$
$$+\int_0^T\int_\Omega(F(u)-\nabla\phi(u))\cdot\nabla\psi dxdt+\int_0^T\int_\Omega b\psi dxdt\leq 0.$$

This ensures that: $\forall\psi\in\mathcal{D}^+(\Omega\times[0,T))$,

$$\int_0^T\int_\Omega u\partial_t\psi dxdt+\int_\Omega u_0\psi(0)dx$$
$$+\int_0^T\int_\Omega(F(u)-\nabla\phi(u))\cdot\nabla\psi dxdt+\int_0^T\int_\Omega b\psi dxdt=0. \qquad (6)$$

It is now easy to check that (6) still holds for $\psi\in\mathcal{D}(\Omega\times[0,T))$, and so this achieves the proof of Propostion 1.1 ■

Remark 1.1 *In the case where $\phi\equiv 0$, the point 2 of definition 1 can be replaced by*

$$F(u)\in\left(L^1_{\text{loc}}(\Omega\times[0,T))\right)^d,$$

and one can remove the assumption $b\in L^2_{\text{loc}}([0,T);H^{-1}(\Omega))$ in (H4). Actually, in such a case, Kružkov entropies $|\cdot-\kappa|$ are sufficient to obtain the time continuity. The assumptions $F(u)\in\left(L^2_{\text{loc}}(\Omega\times[0,T))\right)^d$ and $b\in L^2_{\text{loc}}([0,T);H^{-1}(\Omega))$ will only be useful to ensure $\partial_t u$ belongs to $L^2_{\text{loc}}([0,T);H^{-1}(\Omega))$ in order to recover the regular convex entropies, which are necessary to treat the parabolic case, as it was shown in the work of Carrillo [4].

The Definition 1 does not take into account any boundary condition, or condition at $|x|\to+\infty$. This lack of regularity can lead to non-uniqueness cases, as the one shown in the book of Friedman [7] (also available in the one of Smoller [14]): the very simple problem

$$\begin{cases}\partial_t u-\partial^2_{xx}u=0 & \text{in } \mathbb{R}\times\mathbb{R}_+,\\ u(\cdot,0)=0 & \text{in } \mathbb{R}\end{cases} \qquad (7)$$

admits multiple classical solutions if one does not ask some condition for large x like e.g. $u\in\mathcal{S}'(\mathbb{R}\times\mathbb{R}_+)$. Indeed, it is easy to check that

$$u(x,t)=\sum_{k=0}^{\infty}\frac{1}{2k!}x^{2k}\frac{d^k}{dt^k}e^{-1/t^2}$$

is a classical solution of (7). So u is a weak solution of (7), and thus an entropy weak solution thanks to Proposition 1.1. It also belongs to $C([0,T], L^1_{\text{loc}}(\mathbb{R}))$, thanks to its regularity.

Let us give another example, proposed by Michel Pierre [13]. We now consider the problem

$$\begin{cases} \partial_t u - \partial^2_{xx} u = 0 & \text{in } [0,1] \times \mathbb{R}_+, \\ u(\cdot,0) = 0 & \text{in } [0,1], \\ u(0,\cdot) = u(1,\cdot) = 0 & \text{in } \mathbb{R}_+ \end{cases} \tag{8}$$

which admits the constant function equal to 0 as unique smooth solution. A non-smooth solution to the problem (8) can be built as follows. Denote by u_f the fundamental solution of the heat equation in the one-dimensional case:

$$u_f(x,t) = \frac{1}{\sqrt{4\pi t}} \exp\left(-\frac{x^2}{4t}\right),$$

then $v := \partial_x u_f$ also satisfies the heat equation in the distributional sense. The function v, given by

$$v(x,t) = -\frac{2x}{t\sqrt{4\pi t}} \exp\left(-\frac{x^2}{4t}\right),$$

satisfies $v(0,t) = 0$ for all $t > 0$, belongs to $C^\infty([0,1] \times [0,T] \setminus \{(0,0)\})$ but is not continuous in $(x,t) = (0,0)$. Indeed, one has

$$\lim_{s \to 0^+} v(\sqrt{s}, s) = -\infty.$$

The function $t \mapsto v(1,t)$ belongs to $C^\infty(\mathbb{R}_+)$, then there exists a unique $w \in C^\infty([0,1] \times \mathbb{R}_+)$ solution to the problem

$$\begin{cases} \partial_t w - \partial^2_{xx} w = 0 & \text{in } [0,1] \times \mathbb{R}_+, \\ w(\cdot,0) = 0 & \text{in } [0,1], \\ w(0,\cdot) = 0 & \text{in } \mathbb{R}_+, \\ w(1,t) = v(1,t) & \text{in } \mathbb{R}_+. \end{cases}$$

Defining $u := v - w$, then u is a solution to the problem (8) which is not the trivial solution since it is not regular. Nevertheless, u is a weak solution to the problem and thus a entropy weak solution thanks to Proposition 1.1. Thanks to its regularity, it clearly appears that u belongs to $C(\mathbb{R}_+; L^1_{\text{loc}}((0,1))$.

In the following theorem, we claim that any entropy solution is time continuous with respect with the time variable, at least locally with respect to the space variable.

Theorem 1.2 *Let u be a entropy solution in the sense of Definition 1, then there exists $\overline{u}$ such that $u = \overline{u}$ a.e. on $\Omega \times [0,T)$ and fulfilling*

$$\overline{u} \in C([0,T); L^1_{\text{loc}}(\Omega)).$$

Furthermore, if there exists $p > 1$ and a neighborhood $\mathcal{U}$ of $\partial\Omega$ in Ω such that

$$u_0 \in L^p_{\text{loc}}(\mathcal{U}), \qquad u \in L^\infty_{\text{loc}}([0,T); L^p_{\text{loc}}(\mathcal{U})),$$

then we have:

$$\overline{u} \in C([0,T); L^1_{\text{loc}}(\overline{\Omega})).$$

2 Essential continuity for $t = 0$

In this section, we give a simple way to prove the classical result stated in Proposition 2.1.

Definition 2 *One says that $t \in [0,T)$ is a right-Lebesgue point if there exists $\overline{u}(t)$ in $L^1_{\text{loc}}(\Omega)$ such that for all compact subset K of Ω,*

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_t^{t+\varepsilon} \|u(s) - \overline{u}(t)\|_{L^1(K)} ds = 0.$$

We denote by $\mathcal{L}$ the set of right-Lebesgue points.

It is well known that $meas\,((0,T) \setminus \mathcal{L}) = 0$ and that $u = \overline{u}$ (in the $L^1_{\text{loc}}(\Omega)$-sense) a.e. in $(0,T)$. In the sequel, we will prove that $\mathcal{L} = [0,T)$, and that $\overline{u}$ belongs to $C([0,T); L^1_{\text{loc}}(\Omega))$. We begin by considering the essential continuity for the initial time $t = 0$.

Proposition 2.1 *For all $\zeta \in \mathcal{D}^+(\Omega)$, one has:*

$$\lim_{\substack{t \to 0 \\ t \in \mathcal{L}}} \int_\Omega |\overline{u}(x,t) - u_0(x)|\zeta(x)dx = 0.$$

Particularly, this ensures that $0 \in \mathcal{L}$.

The limit as t tends to 0, $t \in \mathcal{L}$ can be seen as an essential limit, as it is done in Lemma 7.41 in the book of Màlek et al. [10] in the case of a purely hyperbolic problem, or by Otto [11] in the case of a non strongly degenerated parabolic equation. See also the paper of Blanchard and Porretta [2] for the case of renormalized solutions for degenerate parabolic equations.

Proof

First, notice that for all $t \in \mathcal{L}$, and for all $\kappa \in \mathbb{R}$, t is also a right-hand side Lebesgue point of $|u - \kappa|$. Indeed, if K denotes a compact subset of $\overline{\Omega}$, one has for a.e $(x,s) \in \Omega \cap K \times (0,T)$

$$\big||u(x,s) - \kappa| - |u(x,t) - \kappa|\big| \leq |u(x,s) - u(x,t)|,$$

and so, for all $t \in \mathcal{L}$,

$$\lim_{\alpha \to 0} \frac{1}{\alpha} \int_t^{t+\alpha} \int_{\Omega \cap K} \big||u(x,s) - \kappa| - |u(x,t) - \kappa|\big| dxds = 0. \tag{9}$$

Let $\alpha > 0$, and $t^\star \in \mathcal{L}$, one denotes

$$\chi^\alpha_{[0,t^\star[}(t) = \begin{cases} 1 & \text{if } t \leq t^\star \\ 0 & \text{if } t \geq t^\star + \alpha \\ \frac{t^\star + \alpha - t}{\alpha} & \text{if } t^\star < t < t^\star + \alpha. \end{cases}$$

Let $\zeta \in \mathcal{D}^+(\Omega)$, and let $\varepsilon > 0$ be such that $d(supp(\zeta), \partial\Omega) > \varepsilon$. Let $\rho \in \mathcal{D}^+(\mathbb{R}^d)$, with $supp(\rho) \subset B(0,1)$ and $\int_{\mathbb{R}^d} \rho(z)dz = 1$. One denotes $\rho_\varepsilon(z) = \frac{1}{\varepsilon^d}\rho(\frac{z}{\varepsilon})$. The function $y \mapsto \zeta(x)\rho_\varepsilon(x-y)$ belongs to $\mathcal{D}^+(\Omega)$.

Taking $\kappa = u_0(y)$ and $\psi(x,y,t) = \zeta(x)\rho_\varepsilon(x-y)\chi^\alpha_{[0,t^\star[}(t)$ in (2), an integrating with respect to $y \in \Omega$ yields:

$$
\begin{aligned}
&\int_0^T \int_\Omega \int_\Omega |u(x,t) - u_0(y)| \zeta(x)\rho_\varepsilon(x-y)\partial_t \chi^\alpha_{[0,t^\star[}(t) dxdydt \\
&\quad + \int_\Omega \int_\Omega |u_0(x) - u_0(y)|\zeta(x)\rho_\varepsilon(x-y)dxdy \\
&\quad + \int_0^T \chi^\alpha_{[0,t^\star[}(t) \int_\Omega \int_\Omega \left[\begin{array}{c} \mathrm{sign}(u(x,t) - u_0(y)) \\ (F(u(x,t)) - F(u_0(y))) \\ \cdot\nabla\left(\zeta(x)\rho_\varepsilon(x-y)\right) \end{array} \right] dxdydt \\
&\quad - \int_0^T \chi^\alpha_{[0,t^\star[}(t) \int_\Omega \int_\Omega \left[\begin{array}{c} \mathrm{sign}(u(x,t) - u_0(y))\nabla\phi(u(x,t)) \\ \cdot\nabla\left(\zeta(x)\rho_\varepsilon(x-y)\right) \end{array} \right] dxdydt \\
&\quad + \int_0^T \chi^\alpha_{[0,t^\star[}(t) \int_\Omega \int_\Omega \left[\begin{array}{c} \mathrm{sign}(u(x,t) - u_0(y))b(x,t) \\ \zeta(x)\rho_\varepsilon(x-y) \end{array} \right] dxdydt \geq 0, \quad (10)
\end{aligned}
$$

where all the gradient are considered with respect to x, and not y.

One has

$$
|u(x,t) - u_0(y)| = |u(x,t) - u_0(x)| + |u(x,t) - u_0(y)| - |u(x,t) - u_0(x)|,
$$

then, since $\int_{\mathbb{R}^d} \rho_\varepsilon(x-y)dy = 1$ for all x in $\mathrm{supp}(\zeta)$, using

$$
|u_0(x) - u_0(y)| \geq \big||u(x,t) - u_0(y)| - |u(x,t) - u_0(x)|\big|,
$$

we obtain

$$
\begin{aligned}
&\int_0^T \partial_t \chi^\alpha_{[0,t^\star[}(t) \int_\Omega \int_\Omega |u(x,t) - u_0(y)|\zeta(x)\rho_\varepsilon(x-y)dxdydt \\
&\quad \leq \int_0^T \partial_t \chi^\alpha_{[0,t^\star[}(t) \int_\Omega |u(x,t) - u_0(x)|\zeta(x)dxdt \\
&\quad + \|\partial_t \chi^\alpha_{[0,t^\star[}\|_{L^1(0,T)} \int_\Omega \int_\Omega |u_0(x) - u_0(y)|\zeta(x)\rho_\varepsilon(x-y)dxdy. \quad (11)
\end{aligned}
$$

For all $\alpha \in]0, T - t^\star]$,

$$
\|\partial_t \chi^\alpha_{[0,t^\star[}\|_{L^1(0,T)} = 1,
$$

and then, one can let α tend to 0 in (11), so that (10) implies:

$$
\begin{aligned}
&- \int_\Omega \int_\Omega |\overline{u}(x,t^\star) - u_0(x)|\zeta(x)dxdy \\
&+2\int_\Omega \int_\Omega |u_0(x) - u_0(y)|\zeta(x)\rho_\varepsilon(x-y)dxdy + \int_0^{t^\star} \mathcal{R}_\varepsilon(t)dt \geq 0, \quad (12)
\end{aligned}
$$

where $\mathcal{R}_\varepsilon$ belongs to $L^1(0,T)$ for all $\varepsilon > 0$. Since $\mathcal{L}$ is dense in $[0,T]$, one can let in a first step $t^\star$ tend to 0, so that $\int_0^{t^\star} \mathcal{R}_\varepsilon(t)dt$ vanishes:

$$
\begin{aligned}
&\limsup_{\substack{t^\star \to 0 \\ t^\star \in \mathcal{L}}} \int_\Omega \int_\Omega |\overline{u}(x,t^\star) - u_0(x)|\zeta(x)dxdy \\
&\leq 2\int_\Omega \int_\Omega |u_0(x) - u_0(y)|\zeta(x)\rho_\varepsilon(x-y)dxdy. \quad (13)
\end{aligned}
$$

One can now let ε tend to 0, and using the fact that u_0 belongs to $L^1_{\text{loc}}(\Omega)$, and that ζ is compactly supported in Ω, one gets:

$$\lim_{\substack{t^\star \to 0 \\ t^\star \in \mathcal{L}}} \int_\Omega \int_\Omega |\overline{u}(x,t^\star) - u_0(x)|\zeta(x)dxdy = 0.$$

This achieves the proof of Proposition 2.1. ■

3 Time continuity for any $t \geq 0$

In this section, we want to prove the following proposition:

Proposition 3.1 *Let u be a entropy solution in the sense of Definition 1, then there exists $\overline{u}$ such that $u = \overline{u}$ a.e. on $\Omega \times (0,T)$ and fulfilling*

$$\overline{u} \in C([0,T); L^1_{\text{loc}}(\Omega)).$$

In the sequel, we still denote by $\overline{u}$ the representative defined using the right Lebesgue points introduced in Definition 2. Proving the essential continuity for every $t^\star \in \mathcal{L}$ is easy. Indeed, if one replaces $\psi(x,t)$ by $(1 - \chi^\alpha_{[0,t^\star[})(t)\psi(x,t)$ in (2), and then if one lets α tend to 0, one gets:

$$\begin{aligned}\int_{t^\star}^T \int_\Omega |u-\kappa|\partial_t\psi dxdt + \int_\Omega |\overline{u}(t^\star) - \kappa|\psi(t^\star)dx \\ + \int_{t^\star}^T \int_\Omega \text{sign}(u-\kappa)\left(F(u) - F(\kappa) - \nabla\phi(u)\right)\cdot\nabla\psi dxdt \\ + \int_{t^\star}^T \int_\Omega \text{sign}(u-\kappa) b\psi dxdt \geq 0. \qquad (14)\end{aligned}$$

One can thus apply the Proposition 2.1 with $t^\star$ instead of 0, and $\overline{u}(t^\star)$ instead of u_0: $\forall \zeta \in \mathcal{D}^+(\Omega)$,

$$\lim_{\substack{s^\star \to t^\star \\ s^\star \in \mathcal{L}}} \int_\Omega \int_\Omega |\overline{u}(x,s^\star) - \overline{u}(x,t^\star)|\zeta(x)dxdy = 0.$$

We will prove the uniform continuity of $t \mapsto \overline{u}(t)$ from $\mathcal{L} \cap [0, T-\gamma]$ to $L^1_{\text{loc}}(\Omega)$ for all $\gamma \in (0,T)$. This will give as a direct consequence that $\mathcal{L} = [0,T)$ and $\overline{u} \in C([0,T); L^1_{\text{loc}}(\Omega))$. This uniform continuity will come from Theorem 13 in the paper of Carrillo [4], which, adapted to our case, can be stated as follow:

Theorem 3.2 *Suppose that (H1), (H2) hold. Let u_0, v_0 belong to $L^1_{\text{loc}}(\Omega)$, let b_u, b_v belong to $L^2((0,T); H^{-1}(\Omega)) \cap L^1((0,T); L^1_{\text{loc}}(\Omega))$, and let u, v be two entropy solutions associated to the choice of $b = b_u$ and initial data u_0 for u and $b = b_v$ and initial data v_0 for v in Definition 1. Then*

$\forall \psi \in \mathcal{D}^+(\Omega \times [0, T[)$,

$$\int_0^T \int_\Omega |u - v| \partial_t \psi dx dt + \int_\Omega |u_0 - v_0| \psi(0) dx$$
$$\int_0^T \int_\Omega (sign(u - v)(F(u) - F(v)) - \nabla |\phi(u) - \phi(v)|) \cdot \nabla \psi dx dt$$
$$+ \int_0^T \int_\Omega sign(u - v)(b_u - b_v) \psi dx dt \geq 0. \tag{15}$$

We now have all the tools for the proof of Proposition 3.1.

Proof of Proposition 3.1

Let $\gamma > 0$, let $t^\star \in \mathcal{L}_\gamma = \mathcal{L} \cap [0, T - \gamma]$, and $h \in \mathcal{L}_\gamma$ such that $t^\star + h \in \mathcal{L}_\gamma$ (this is the case of almost every $h \in (0, T - t^\star - \gamma)$). Let $\zeta \in \mathcal{D}^+(\Omega)$, let $\alpha \in]0, T - t^\star - \gamma - h[$.

Taking $\psi(x,t) = \zeta(x) \chi^\alpha_{[0,t^\star[}(t)$, $v_0(x) = u(x, h)$, $v(x,t) = u(x, t+h)$ in (15), and letting α tend to 0 yields:

$$- \int_\Omega |\overline{u}(x, t^\star) - \overline{u}(x, t^\star + h)| \zeta(x) dx + \int_\Omega |u_0(x) - \overline{u}(x, h)| \zeta(x) dx$$
$$\int_0^{t^\star} \int_\Omega \left[\begin{array}{c} \mathrm{sign}(u(x,t) - u(x, t+h)) \\ (F(u(x,t)) - F(u(x, t+h))) \\ -\nabla |\phi(u(x,t)) - \phi(u(x, t+h))| \end{array} \right] \cdot \nabla \zeta(x) dx dt$$
$$+ \int_0^{t^\star} \int_\Omega \left[\begin{array}{c} \mathrm{sign}(u(x,t) - u(x, t+h)) \\ (b(x,t) - b(x, t+h)) \end{array} \right] \zeta(x) dx dt \geq 0. \tag{16}$$

We deduce from (16) that

$$\int_\Omega |\overline{u}(x, t^\star) - \overline{u}(x, t^\star + h)| \zeta(x) dx \leq \int_\Omega |u_0(x) - \overline{u}(x, h)| \zeta(x) dx$$
$$+ \int_0^{T - \gamma - h} \int_\Omega |F(u(x,t)) - F(u(x, t+h))| \, |\nabla \zeta(x)| dx dt$$
$$+ \int_0^{T - \gamma - h} \int_\Omega |\nabla \phi(u)(x, t+h) - \nabla \phi(u)(x,t)| \, |\nabla \zeta(x)| dx dt$$
$$+ \int_0^{T - \gamma - h} \int_\Omega |b(x, t+h) - b(x,t)| \, \zeta(x) dx dt,$$

and since $F(u), \nabla \phi(u)$ and b belong to $L^1_{\mathrm{loc}}(\Omega \times (0,T))$, one can claim that:

$$\forall \varepsilon > 0, \forall t^\star \in \mathcal{L}_\gamma, \exists \eta > 0 \ s.t. \ \forall h \in \mathcal{L} \cap [0, T - \gamma - t^\star], h \leq \eta \Rightarrow$$
$$\int_\Omega |\overline{u}(x, t^\star) - \overline{u}(x, t^\star + h)| \zeta(x) dx \leq \int_\Omega |u_0(x) - \overline{u}(x, h)| \zeta(x) dx + \varepsilon. \tag{17}$$

One can now use Proposition 2.1 in (17), so that we get that

$$t \mapsto \overline{u}(x,t) \text{ is uniformly continuous from } \mathcal{L} \text{ to } L^1(\Omega, \zeta),$$

which is the L^1-space for measure of density ζ w.r.t. Lebesgue measure. We deduce that, for all $\gamma \in (0,T)$, $t \mapsto \overline{u}$ is uniformly continuous from

$\mathcal{L}_\gamma$ to $L^1_{\rm loc}(\Omega)$, and this ensures that $\mathcal{L}_\gamma = [0, T-\gamma]$. This holds for any $\gamma \in (0,T)$, and so we can claim that $\overline{u} \in C([0,T); L^1_{\rm loc}(\Omega))$. ■

It remains to prove the last part of Theorem 1.2 by considering some test functions $\zeta \in \mathcal{D}^+(\overline{\Omega})$ instead of $\zeta \in \mathcal{D}^+(\Omega)$. We will need some additional regularity on the solution:

$$\left\{ \begin{array}{c} \text{There exists an open neighborhood } \mathcal{U} \text{ of } \partial\Omega \text{ in } \overline{\Omega} \text{ s.t.} \\ u_0 \in L^p_{\rm loc}(\mathcal{U}), \qquad u \in L^\infty_{\rm loc}([0,T); L^p_{\rm loc}(\mathcal{U})). \end{array} \right\} \tag{H5}$$

(H5) gives the uniform (w.r.t. t) local equiintegrability of u (and so of $\overline{u}$) on a neighborhood of $\mathcal{U}$. We deduce, using $\overline{u} \in C([0,T); L^1_{\rm loc}(\Omega))$ that $\overline{u} \in C([0,T); L^1_{\rm loc}(\overline{\Omega}))$.

End of the proof of Theorem 1.2

Suppose that (H1),(H2),(H3),(H4) hold, then thanks to Proposition 3.1, there exists a weak solution $\overline{u} \in C([0,T), L^1_{\rm loc}(\Omega))$.

For $\varepsilon > 0$, $\gamma \in (0,T)$, $\zeta \in \mathcal{D}^+(\Omega)$, there exists $\eta > 0$ such that: $\forall t \in [0, T-\gamma]$, $\forall h \in [0, \min(\eta, T-t-\gamma)]$,

$$\int_\Omega |\overline{u}(x,t+h) - \overline{u}(x,t)|\zeta(x)dx \le \varepsilon.$$

Let K be a compact subset of $\overline{\Omega}$. Then there exists $\zeta \in \mathcal{D}^+(\overline{\Omega})$ such that $0 \le \zeta(x) \le 1$ for all $x \in \mathbb{R}^d$, and $\zeta(x) = 1$ if $x \in K$. Let $\alpha > 0$ and let $\beta_\alpha \in C^\infty(\mathbb{R}^d; \mathbb{R})$ such that:

$$\begin{array}{ll} 0 \le \beta_\alpha(x) \le 1 & \text{for all } x \in \mathbb{R}^d, \\ \beta_\alpha(x) = 1 & \text{if } d(x, \partial\Omega) \le \alpha/2, \\ \beta_\alpha(x) = 0 & \text{if } d(x, \partial\Omega) \ge \alpha. \end{array}$$

Suppose that (H5) holds. For α small enough, one has $supp(\zeta\beta_\alpha) \subset \mathcal{U}$ and then, for all $t \in [0, T-\gamma]$, for all $h \in [0, T-t-\gamma]$,

$$\int_\Omega |\overline{u}(x,t+h) - \overline{u}(x,t)|\zeta(x)\beta_\alpha dx \le 2\|u\|_{L^\infty((0,T-\gamma); L^p(\mathcal{U}_\zeta))}\|\beta_\alpha\|_{L^{p'}(\mathcal{U}_\zeta)},$$

where $\mathcal{U}_\zeta$ denotes $\mathcal{U} \cap supp(\zeta)$, and $p' = \frac{p}{p-1} < +\infty$. Since $\|\beta_\alpha\|_{L^{p'}(\mathcal{U}_\zeta)}$ tends to 0 as α tends to 0, there exists $\delta > 0$ such that:

$$\alpha \le \delta \Rightarrow \int_\Omega |\overline{u}(x,t+h) - \overline{u}(x,t)|\zeta(x)\beta_\alpha dx \le \varepsilon. \tag{18}$$

Suppose now that α has been chosen such that (18) holds. The function $\zeta(1-\beta_\alpha)$ belongs to $\mathcal{D}^+(\Omega)$, and then there exists η such that :$\forall t \in [0, T-\gamma]$, $\forall h \in [0, \min(\eta, T-\gamma-t)]$,

$$\int_\Omega |\overline{u}(x,t+h) - \overline{u}(x,t)|\zeta(x)(1-\beta_\alpha(x))dx \le \varepsilon. \tag{19}$$

Adding (18) and (19) shows that for all t in $[0, T-\gamma-\eta]$, for all $h \in [0,\eta]$,

$$\int_K |\overline{u}(x,t+h) - \overline{u}(x,t)|dx \le 2\varepsilon. \tag{20}$$

So $\overline{u}$ is uniformly continuous from $[0, T-\gamma]$ to $L^1(K)$, and then

$$\overline{u} \in C([0,T); L^1_{\text{loc}}(\overline{\Omega})).$$

■

To conclude this paper, let us give a counter-example to the time continuity in the case where the entropy criterion is not fulfilled for t=0. Consider the inviscid Burgers equation, in the one dimensional case, leading to the following initial value problem.

$$\left\{ \begin{array}{l} \partial_t u - \partial_x \left(u^2\right) = 0, \quad (x,t) \in (\mathbb{R} \times \mathbb{R}_+), \\ u(\cdot, 0) = u_0 = 0. \end{array} \right. \tag{21}$$

Problem (21) admits $u = 0$ as unique entropy solution in the sense of Definition 1.

We define

$$\tilde{u}(x,t) = \left\{ \begin{array}{ll} 0 & \text{if} \quad t = 0, \\ 0 & \text{if} \quad |x| > \sqrt{t}, \\ \dfrac{x}{2t} & \text{if} \quad |x| < \sqrt{t}. \end{array} \right.$$

Then it is easy to check that:

- $\tilde{u} \in L^1_{\text{loc}}(\mathbb{R} \times \mathbb{R}_+)$,
- $\tilde{u}^2 \in L^1_{\text{loc}}(\mathbb{R} \times \mathbb{R}_+)$,
- $\forall \psi \in \mathcal{D}(\Omega \times \mathbb{R}_+)$,

$$\begin{aligned} &\int_0^{+\infty} \int_{\mathbb{R}} \tilde{u}(x,t) \partial_t \psi(x,t) dx dt \\ &+ \int_0^{+\infty} \int_{\mathbb{R}} \tilde{u}^2(x,t) \partial_x \psi(x,t) dx dt = 0, \end{aligned} \tag{22}$$

- $\forall \psi \in \mathcal{D}^+(\Omega \times \mathbb{R}_+^\star)$, $\forall \kappa \in \mathbb{R}$,

$$\begin{aligned} &\int_0^{+\infty} \int_{\mathbb{R}} |\tilde{u} - \kappa|(x,t) \partial_t \psi(x,t) dx dt \\ &+ \int_0^{+\infty} \int_{\mathbb{R}} \text{sign}(\tilde{u} - \kappa) \left(\tilde{u}^2(x,t) - \kappa^2\right) \partial_x \psi(x,t) dx dt = 0. \end{aligned} \tag{23}$$

Thanks to (22), $\tilde{u}$ is a weak solution of (21), and an entropy criterion (23) is fulfilled only for $t > 0$. The fact that the entropy criterion fails for $t = 0$, and that the solution $\tilde{u}$ and the flux $\tilde{u}^2$ are not bounded (see [6, 12]) allows the function $\tilde{u}$ to be discontinuous at $t = 0$. Indeed, for all $t > 0$,

$$\|\tilde{u}(\cdot, t)\|_{L^1(\mathbb{R})} = \frac{1}{2} \neq \|u_0\|_{L^1(\mathbb{R})} = 0.$$

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