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Resolution of a semilinear equation in L^1

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Synopsis

Let $\Omega=\mathbb{R}^N$ or Ω be a bounded regular open set of $\mathbb{R}^N$ and let $\gamma(x,s)\colon\Omega\times\mathbb{R}\to\mathbb{R}$ be a continuous nondecreasing function in s, measurable in x, such that $\gamma(x,0)=0$ almost everywhere. We solve, for $f\in L^1(\Omega)$, the problem (P): $-\Delta u+\gamma(.,u)=f$ in Ω, $u=0$ on $\partial\Omega$. (In fact, for this result, instead of assuming that γ is nondecreasing in s we need only that $\gamma(x,s)s\geqq 0$.) We deduce an "almost" necessary and sufficient condition on $f\in\mathscr{D}'(\Omega)$, in order that (P) has a solution. Roughly speaking, this condition is $f=-\Delta V+g$, with $g\in L^1(\Omega)$ and $\gamma(.,V)\in L^1(\Omega)$.

1. Introduction

1.1. Problem and results

Let $\Omega\subset\mathbb{R}^N$ be an open set and $\gamma(x,s)\colon\Omega\times\mathbb{R}\to\mathbb{R}$ a function measurable in x, and continuous nondecreasing in s, with $\gamma(x,0)=0$ almost everywhere. This paper treats first the problem

$$(\mathrm{P})\begin{cases}-\Delta u+\gamma(.,u)=f \text{ in } \mathscr{D}'(\Omega),\\ u=0 \text{ on } \partial\Omega,\end{cases}$$

where $f\in L^1(\Omega)$. We shall consider two cases: when $\Omega=\mathbb{R}^N$ and when Ω is a smooth bounded set of $\mathbb{R}^N$. When $\gamma(x,s)=\gamma(s)$ does not depend on x, Problem (P) has been completely solved in [**1**] for the case where $\Omega=\mathbb{R}^N$, and in [**2**], for the case where Ω is a bounded open subset of $\mathbb{R}^N$ with smooth boundary. In the second case, the proofs involve properties of the maximal monotone graph of γ in $\mathbb{R}\times\mathbb{R}$. They cannot be extended to include the dependence of γ on x, and, moreover, apply to $\Omega=\mathbb{R}^N$ only under a coerciveness condition on γ of the type $\gamma(u).u\geqq\varepsilon u^2$.

Let us now explain the difference between our technique and that employed in [**1**]. It is easy to find, by a variational argument, a solution for a regularized version of (P) given by

$$(\mathrm{P}_n) - \Delta u_n + \frac{1}{n} u_n + \gamma_n(.,u_n) = f_n,$$

where γ_n is a truncation of γ and $f_n \in L^2(\Omega)$. The difficulty in passing to the limit as $(1/n) \to 0$, is to prove weak precompactness in $L^1_{\mathrm{loc}}(\Omega)$ of the sequence $(\gamma_n(.,u_n))_n$.

In [**1**], the authors make use of a classical compactness theorem of Kolmogorov. Moreover, they describe the functional framework for solving problems which, like (P), involve the Laplacian with range in $L^1(\mathbb{R}^N)$. We shall use their results extensively, and our proofs often follow theirs step by step. But, instead of Kolmogorov's theorem, we base our proof on a compactness argument due to Vitali. The main idea is to obtain equiintegrability of the sequence $\gamma_n(.,u_n)$ by proving the relation

$$\int_{[|u_n|\geqq t]} |\gamma_n(.,u_n)|\,dx \leqq \int_{[|u_n|\geqq t]} |f_n|\,dx.$$

Since, for the case $\Omega = \mathbb{R}^N$, we will need to use, as in [**1**] properties of Δ^{-1} on $L^1(\mathbb{R}^N)$, it is not surprising that the fundamental solution of the Laplacian plays an important role. In partricular, it will be necessary to handle separately the cases $N \geqq 3$, $N=2$, $N=1$.

We now give the plan of this paper, and summarize the main results.

In section 2, we first solve (P) in $\mathbb{R}^N$ for $N \geqq 3$. We assume that the function $x \to \gamma(x,t)$ is in $L^1_{\mathrm{loc}}(\mathbb{R}^N)$, for all $t \in \mathbb{R}$, and then show that (P) admits a unique solution u in $M^{N/(N-2)}(\mathbb{R}^N)$, with $\gamma(.,u) \in L^1(\mathbb{R}^N)$, where $M^p(\mathbb{R}^N)$ denotes the Marcinkiewicz (or weak-L^p) space (see below). The existence part of this result does not require γ to be nondecreasing in s, but only that $\gamma(x,s)s \geqq 0$ (assuming moreover that $\sup_{|s|\leqq t} |\gamma(x,s)| \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$ for all $t \in \mathbb{R}^+$).

Later in section 2, we deduce an "almost" necessary and sufficient condition on $f \in \mathscr{D}'(\mathbb{R}^N)$ for problem (P) to have a solution; roughly speaking, this condition is $f = -\Delta V + g$ with $g \in L^1(\mathbb{R}^N)$ and $\gamma(.,V) \in L^1(\mathbb{R}^N)$.

In section 3, we solve (P) in $\mathbb{R}^N$ for $N=2$ and $N=1$, $f \in L^1$. In addition to the assumption of section 2, we require as in [**1**] some coerciveness on the nonlinear term, namely:

$$\exists C_1, C_2 > 0, \text{meas}\,\{x, \gamma(x,C_1) \leqq C_2 \text{ or } \gamma(x,-C_1) \geqq -C_2\} < \infty.$$

For $N=2$, we obtain a solution, unique up to a constant, to the problem

$$\begin{cases} -\Delta u + \gamma(.,u) = f, \\ \gamma(.,u) \in L^1(\mathbb{R}^2), \quad |\nabla u| \in M^2(\mathbb{R}^2), \quad u \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^2). \end{cases} \tag{1}$$

If $N=1$, we prove the same result for the problem

$$\begin{cases} -u'' + \gamma(.,u) = f, \\ \gamma(.,u) \in L^1(\mathbb{R}), \quad u \in L^1_{\mathrm{loc}}(\mathbb{R}). \end{cases} \tag{2}$$

As in section 2, the assumption that γ is nondecreasing can be weakened to the assumption that $\gamma(x,s)s \geqq 0$. We can also treat the case $f \in \mathscr{D}'(\mathbb{R}^N)$.

In section 4, we solve (P) on a bounded regular open set of $\mathbb{R}^N$ ($N \geqq 1$). Under the assumption $\gamma(., t) \in L^1_{\text{loc}}(\Omega)$ for all $t \in \mathbb{R}$, we obtain, for all $f \in L^1(\Omega)$, a unique solution to the problem

$$\begin{cases} -\Delta u + \gamma(., u) = f \text{ in } \mathscr{D}'(\Omega), \\ u \in W_0^{1,1}(\Omega), \quad \gamma(., u) \in L^1(\Omega). \end{cases} \tag{3}$$

As in the preceding sections, for the existence part of this result, the assumption that γ is nondecreasing in s can again be weakened to $\gamma(x, s)s \geqq 0$. Then we deduce an "almost" necessary and sufficient condition on $f \in \mathscr{D}'(\Omega)$ for (3) to have a (unique) solution. Roughly speaking, this condition is $f = -\Delta V + g$ with $g \in L^1(\Omega)$ and $\gamma(., V) \in L^1(\Omega)$.

As a corollary, we show that the problem

$$\begin{cases} -\Delta u + |u|^{p-1}u = \mu, \\ u \in L^p(\Omega), \quad u \in W_0^{1,1}(\Omega), \end{cases}$$

has a solution if and only if the distribution μ is in $L^1(\Omega) + W^{-2,p}(\Omega)$ We examine the relation of this proposition to a theorem of [**4**].

1.2. Some notation and definitions

When $\Omega \subset \mathbb{R}^N$ is Lebesgue measurable, we denote its measure by meas Ω. For $f \in L^1(\Omega)$, $\int_\Omega f(x)\, dx$ denotes the integral of f over Ω with respect to Lebesgue measure but this is shortened to $\int_\Omega f dx$, or $\int f$ when $\Omega = \mathbb{R}^N$. The norm in $L^p(\Omega)$ is denoted by $\|.\|_{L^p}$ or $\|.\|_p$, $1 \leqq p \leqq \infty$. Let u be a measurable function on $\mathbb{R}^N$, $1 < p < \infty$ and $(1/p') + (1/p) = 1$. Set

$$\|u\|_{M^p} = \min\left\{C \in [0, \infty], \quad \int_K |u(x)|\, dx \leqq C(\text{meas } K)^{1/p'}, \quad K \text{ measurable} \subset \mathbb{R}^N\right\}.$$

M^p ($\mathbb{R}^N$) is the set of measurable functions u on $\mathbb{R}^N$ satisfying $\|u\|_{M^p} < \infty$. It is easy to verify that M^p ($\mathbb{R}^N$) is a Banach space under the norm $\|.\|_{M^p}$; it is called the Marcinkiewicz space. If u is a function on $\mathbb{R}^N$, $[|u| > \lambda]$ denotes $\{x \in \mathbb{R}^N, |u(x)| > \lambda\}$. If $k \geqq 0$ is an integer and $1 \leqq p \leqq \infty$, $W^{k,p}(\Omega)$ is the Sobolev space of functions u on the open set $\Omega \subseteq \mathbb{R}^N$ for which $D^l u \in L^p(\Omega)$ when $|l| \leqq k$, with its usual norm. $W_0^{k,p}(\Omega)$ is the closure of $\mathscr{D}(\Omega) = C_0^\infty(\Omega)$ in $W^{k,p}(\Omega)$.

If $p = 2$, we write H^k for $W^{k,2}$. A function u lies in $W^{k,p}_{\text{loc}}(\Omega)$ if $\zeta u \in W^{k,p}(\Omega)$ for all $\zeta \in \mathscr{D}(\Omega)$. We denote by $\mathcal{M}(\Omega)$ the set of bounded Radon measures on Ω, with the norm: $\|\mu\|_{\mathcal{M}} = |\mu|\, (\Omega)$.

2. The equation $-\Delta u + \gamma(., u) = f$ in $L^1(\mathbb{R}^N)$ with $N \geqq 3$

THEOREM 1. *Let $\gamma(x, s)\colon \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ be a function measurable in x which is continuous and nondecreasing in s. Assume that $\gamma(x, 0) = 0$ almost everywhere and that the function $x \to \gamma(x, t)$ is in $L^1_{\text{loc}}(\mathbb{R}^N)$ for all t in $\mathbb{R}$. Then, for every f in $L^1(\mathbb{R}^N)$, the following problem has a unique solution:*

$$\begin{cases} -\Delta u + \gamma(., u) = f, \text{ in } \mathscr{D}'(\mathbb{R}^N), \\ \gamma(., u) \in L^1(\mathbb{R}^N), \quad u \in M^{N/(N-2)}(\mathbb{R}^N). \end{cases} \tag{4}$$

Remark 1. In Theorem 1, instead of $-\Delta$ we can have a more general uniformly elliptic second order operator.

In the proof of existence in Theorem 1, we do not need the assumption that $\gamma(x, s)$ is nondecreasing in s. It is enough to assume that $\gamma(x, s)s \geqq 0$ almost everywhere in $x \in \mathbb{R}^N$, for all $s \in \mathbb{R}$; and $\operatorname{Sup}_{|s| \leqq t} |\gamma(x, s)| \in L^1_{\text{loc}}(\mathbb{R}^N)$ for all $t \in \mathbb{R}$.

Proof of Theorem 1. (a) *Uniqueness of the solution.* Let u and v be two solutions of (4). Then $(u-v) \in M^{N/(N-2)}$ and $-\Delta(u-v) \in L^1$ Thus it is "easy" to show (see [**1**, Lemma A.10]) that, for every function p in $\mathscr{C} = \{p \in C^1 \cap L^\infty(\mathbb{R}),\ p' \geqq 0, p(0) = 0\}$, one has

$$\int_{\mathbb{R}^N} -\Delta(u-v)p(u-v)\, dx \geqq 0.$$

Thus, $\int (\gamma(., u) - \gamma(., v))p(u-v) \leqq 0$.

Since $\gamma(x, s)$ is nondecreasing in s and we can choose p increasing, we deduce that $\gamma(., u) = \gamma(., v)$ almost everywhere, and therefore $u = v$ almost everywhere. Indeed, from $\gamma(., u) = \gamma(., v)$ almost everywhere, we deduce $-\Delta(u-v) = 0$ almost everywhere, and since $u - v \in M^{N/(N-2)}$, we can conclude that $u = v$ almost everywhere (see [**1**, Lemma A.5]).

(b) *Existence of a solution for Problem* (4). We first solve a penalized version of Problem (P).

LEMMA 1. *Assume only* $N \geqq 1$. *Let* γ *be as in Theorem* 1, *but assume moreover that* $\sup_{s \in \mathbb{R}} |\gamma(., s)| \in L^2(\mathbb{R}^N)$. *Let* f *be in* $L^2(\mathbb{R}^N)$. *Then, for all* $\varepsilon > 0$, *the following problem has a unique solution*

$$\begin{cases} -\Delta u + \varepsilon u + \gamma(., u) = f, \\ u \in H^1(\mathbb{R}^N). \end{cases} \tag{5}$$

Proof. Set $j(x, s) = \int_0^s \gamma(x, t)\, dt$ and

$$E(u) = \frac{1}{2} \int |\nabla u|^2\, dx + \frac{\varepsilon}{2} \int |u|^2\, dx + \int j(., u)\, dx - \int fu\, dx.$$

This functional is well defined for u in $H^1(\mathbb{R}^N)$. Indeed, one has $0 \leqq j(., u) \leqq u\gamma(., u) \in L^1(\mathbb{R}^N)$ (since $u \in L^2(\mathbb{R}^N)$). Note that E is strictly convex and that $E(u) \to +\infty$ as $\|u\|_{H^1} \to +\infty$. Thus, by minimizing E on $H^1(\mathbb{R}^N)$, we obtain a unique solution u for the associated Euler equation (5). Moreover, from $u \in L^2(\mathbb{R}^N)$, $\Delta u \in L^2(\mathbb{R}^N)$, we deduce $u \in H^2(\mathbb{R}^N)$.

Next, we provide some estimates on the solution of (5) for $N \geqq 3$.

LEMMA 2. *Let* $N \geqq 3$. *Let* γ *and* f *be as in Lemma* 1, *and assume moreover that* $f \in L^1(\mathbb{R}^N)$. *Let* $\varepsilon > 0$. *Then, the solution* u *of* (5) *satisfies the following inequalities (with* $\gamma_\varepsilon(., u) = \varepsilon u + \gamma(., u)$*):*

$$\int_{\{|u| \geqq t\}} |\gamma_\varepsilon(., u)|\, dx \leqq \int_{\{|u| \geqq t\}} |f|\, dx \text{ for all } t \text{ in } \mathbb{R}, \tag{6}$$

$$\|u\|_{M^{N/(N-2)}} \leqq c_N \|f\|_{L^1} \tag{7}$$

and

$$\|\nabla u\|_{M^{N/(N-1)}} \leqq d_N \|f\|_{L^1},$$

where c_N *and* d_N *depend only on the dimension* $N \geqq 3$.

Proof. Let $p \in \tilde{\mathscr{C}} = \{p \in C^1(\mathbb{R}), p(0) = 0, \ p' \geqq 0, p' \in L^\infty\}$. Since $u \in H^1$, we have $p(u) \in H^1(\mathbb{R}^N)$, and since $u \in H^2$, on applying Green's formula we obtain

$$\int (-\Delta u) p(u) \, dx \geqq 0.$$

On multiplying (5) by $p(u)$ and integrating, we obtain

$$\int \gamma_\varepsilon(., u) p(u) \, dx \leqq \int f p(u) \, dx. \tag{8}$$

We can easily find a sequence $(p_n)_n$ *in* $\tilde{\mathscr{C}}$, such that

$$\begin{cases} p_n(x) \uparrow 1, & \text{for} \quad x > t, \\ p_n(x) = 0, & \text{for} \quad -t \leqq x \leqq t, \\ p_n(x) \downarrow -1, & \text{for} \quad x < -t. \end{cases}$$

By using the monotone convergence theorem and the dominated convergence theorem, we can pass to the limit in (8) with p_n instead of p. Hence, we obtain (6) Then (7) is a direct application of the following lemma, which has already been used in the first part of the proof of Theorem 1.

LEMMA A. ([**1**, Lemma A.5]). *Let* $N \geqq 3$, $u \in L^1_{\text{loc}}(\mathbb{R}^N)$, $\Delta u \in L^1(\mathbb{R}^N)$, *and let* u *satisfy*

$$\lim_{n \to \infty} n^{-N} \int_{n \leqq |y| \leqq 2n} |u(y)| \, dy = 0.$$

Then, $u \in M^{N/(N-2)}$, $|\nabla u| \in M^{N/(N-1)}(\mathbb{R}^N)$, *and* $\|u\|_{M^{N/(N-2)}} \leqq c'_N \|\Delta u\|_{L^1}$, $\|\nabla u\|_{M^{N/(N-1)}} \leqq d'_N \|\Delta u\|_{L^1}$, for some constants c'_N and d'_N independent of u.

Indeed, on applying (6) with $t = 0$ and using (5), we obtain $\|\Delta u\|_{L^1} \leqq 2 \|f\|_{L^1}$, and by Lemma A we obtain (7) with $c_N = 2c'_N$ and $d_N = 2d'_N$.

As a final step, we have to pass to the limit in (5) to obtain a solution to (4).

Let $f_n \in \mathscr{D}(\mathbb{R}^N)$ be a sequence such that $f_n \to f$ in $L^1(\mathbb{R}^N)$ as $n \to +\infty$. Choose $\varphi \in \mathscr{D}(\mathbb{R}^N)$, with $0 \leqq \varphi \leqq 1$, $\varphi \equiv 1$ on $B_1 = \{x \in \mathbb{R}^N, |x| < 1\}$, $\varphi \equiv 0$ on $B_2^c = \{x \in \mathbb{R}^N, |x| \geqq 2\}$, and set

$$\varphi_n(x) = \varphi\left(\frac{x}{n}\right), \quad n \in \mathbb{N}^*, \quad x \in \mathbb{R}^N,$$

$$\begin{cases} \gamma_n(x, s) = \gamma(x, s)\varphi_n(x), & \text{if} \quad |\gamma(x, s)| \leqq n, \\ \gamma_n(x, s) = n \operatorname{sgn}(\gamma(x, s))\varphi_n(x), & \text{if} \quad |\gamma(x, s)| > n. \end{cases}$$

According to Lemma 1, there exists $u_n \in H^2$ such that $-\Delta u_n + (1/n)u_n + \gamma_n(., u_n) = f_n$. By applying Lemma 2, with u_n and $(1/n)u_n + \gamma_n(., u_n)$ instead of u and γ_ε, one sees that the sequence $(u_n)_n$ is bounded in $M^{N/(N-2)}$, and $(\nabla u_n)_n$ in $M^{N/(N-1)}$. Thus $(u_n)_n$ and $(\nabla u_n)_n$ are bounded in $L^1_{\text{loc}}(\mathbb{R}^N)$, and therefore $(u_n)_n$ is bounded in $W^{1,1}_{\text{loc}}(\mathbb{R}^N)$. So $(u_n)_n$ is a precompact sequence in $L^1_{\text{loc}}(\mathbb{R}^N)$, and we may assume that

$$u_n \to u \text{ in } L^1_{\text{loc}} \text{ and almost everywhere.}$$

Moreover, applying Fatou's lemma to u_n yields $u \in M^{N/(N-2)}$. But we have $\gamma_n(.,u_n) \to \gamma(.,u)$ almost everywhere, and then using (6) with $t=0$, we see that $(\gamma_n(.,u_n))_n$ is a bounded sequence in $L^1(\mathbb{R}^N)$. Then, by Fatou's lemma, we also have $\gamma(.,u) \in L^1(\mathbb{R}^N)$. In order to prove the convergence of $(\gamma_n(.,u_n))_n$ to $\gamma(.,u)$ in $L^1_{\text{loc}}(\mathbb{R}^N)$, it remains (by a classical theorem of Vitali) to show that $(\gamma_n(.,u_n))_n$ is equiintegrable on every bounded measurable set B of $\mathbb{R}^N$.

Let B be a bounded measurable set of $\mathbb{R}^N$ and $\varepsilon > 0$. We want to prove that there exists $\delta > 0$ such that for every measurable set K of B one has

$$\text{meas } K \leqq \delta \Rightarrow \int_K |\gamma_n(.,u_n)|\, dx \leqq \varepsilon, \quad \forall n \in \mathbb{N}^*.$$

We first remark that, since $(u_n)_n$ is bounded in $M^{N/(N-2)}$, we have meas $(\{|u_n| \geqq t\}) \to 0$ as $t \to +\infty$, *uniformly in* $n \in \mathbb{N}^*$. Then, since $(f_n)_n$ is equiintegrable on $\mathbb{R}^N$, we have, for some $t_0 > 0$,

$$t \geqq t_0 \Rightarrow \int_{\{|u_n| \geqq t\}} |f_n|\, dx \leqq \varepsilon, \quad \forall n \in \mathbb{N}^*.$$

By using (6), we obtain

$$t \geqq t_0 \Rightarrow \int_{\{|u_n| \geqq t\}} |\gamma_n(.,u_n)|\, dx \leqq \varepsilon, \quad \forall n \in \mathbb{N}^*. \tag{9}$$

Now, for $t = t_0$ and for every measurable set $K \subset B$, we have

$$\int_{K \cap \{|u_n| < t_0\}} |\gamma_n(.,u_n)|\, dx \leqq \int_K (|\gamma(.,t_0)| + |\gamma(.,-t_0)|)\, dx + \frac{t_0}{n} \text{meas } K.$$

Since $\gamma(.,t) \in L^1_{\text{loc}}(\mathbb{R}^N)$ for all $t \in \mathbb{R}$, there exists $\delta > 0$ such that

$$\int_K (|\gamma(.,-t_0)| + |\gamma(.,t_0)|)\, dx + t_0 \text{ meas } (K) < \varepsilon \text{ for meas } (K) \leqq \delta. \tag{10}$$

From relations (9) and (10), we conclude that

$$\text{meas } K \leqq \delta \Rightarrow \int_K |\gamma_n(.,u_n)|\, dx \leqq 2\varepsilon \text{ for all } n \text{ in } \mathbb{N}^*.$$

Thus, $(\gamma_n(.,u_n))_n$ is equiintegrable on B, and, therefore, $\gamma_n(.,u_n) \to \gamma(.,u_n)$ in $L^1_{\text{loc}}(\mathbb{R}^N)$. Thus, all the terms of equation (5) tend to the terms of equation (4), and therefore (4) is verified by u in the sense of distributions.

We now give some consequences of Theorem 1.

Corollary 1. Let $\gamma(x,s): \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ be measurable in x, continuous and nondecreasing in s. Let $V \in L^1_{\text{loc}}(\mathbb{R}^N)$ be such that $\gamma(x, V(x)) \in L^1(\mathbb{R}^N)$, and for every t in $\mathbb{R}$, $\gamma(x, t+V(x)) \in L^1_{\text{loc}}(\mathbb{R}^N)$. Then, for every f in $L^1(\mathbb{R}^N)$, there exists a unique function u such that

$$\begin{cases} -\Delta u + \gamma(., u) = -\Delta V + f \text{ (in } \mathscr{D}'(\mathbb{R}^N)), \\ u - V \in M^{N/(N-2)}(\mathbb{R}^N),\ \gamma(., u) \in L^1(\mathbb{R}^N). \end{cases} \tag{11}$$

Proof. On setting $w = u - V$, we see that (11) is equivalent to

$$\begin{cases} -\Delta w + \gamma(., w + V) - \gamma(., V) = f - \gamma(., V), \\ w \in M^{N/(N-2)}, \quad \gamma(., w + V) - \gamma(., V) \in L^1(\mathbb{R}^N). \end{cases}$$

This is equation (4), with $\tilde{f}(x) = f(x) - \gamma(x, V(x)) \in L^1$ instead of $f(x)$, and $\tilde{\gamma}(x, s) = \gamma(x, s + V(x)) - \gamma(x, V(x))$ instead of γ. We still have $\tilde{\gamma}(x, 0) = 0$ almost everywhere, and $\tilde{\gamma}(x, t) \in L^1_{\text{loc}}(\mathbb{E}^N)$ for all t in $\mathbb{R}$.

Note that the result of Corollary 1 has an "almost full" generality. Indeed, let $g \in \mathscr{D}'(\mathbb{R}^N)$, so that if there exists a solution u to the problem

$$\begin{cases} -\Delta u + \gamma(., u) = g, \\ \gamma(., u) \in L^1, \end{cases}$$

then g has the form indicated in Corollary 1, namely,

$$g = -\Delta V + f, \quad \text{with} \quad f \in L^1(\mathbb{R}^N) \quad \text{and} \quad \gamma(., V) \in L^1(\mathbb{R}^N).$$

(We may take, for instance, $V = u$ and $f = \gamma(., u)$.)

This corollary can be used to prove the following result of Bénilan and Brezis:

THEOREM A (Brezis [**3**]). *Assume γ is nondecresing and continuous, $\gamma(0) = 0$, and $\gamma(\pm 1/|x|) \in L^1$ near $x = 0$. Then, for every bounded measure $\mu \in \mathcal{M}(\mathbb{R}^3)$, there exists a unique u solution to*

$$\begin{cases} -\Delta u + \gamma(u) = \mu, \\ u \in M^3(\mathbb{R}^3), \quad \gamma(u) \in L^1(\mathbb{R}^3). \end{cases}$$

Remark. The proof of Theorem A as a consequence of Corollary 1 is rather technical. Let us give this proof in the easy case where μ has compact suport. We choose a function $\varphi \in \mathscr{D}(\mathbb{R}^N)$ with $\varphi \equiv 1$ near 0 and write $\mu = f - \Delta V$ with $V = \mu * \varphi/|\cdot|$ and $f = -\Delta(\mu * 1 - \varphi/|\cdot|)$. The fact that $f \in L^1(\mathbb{R}^N)$ is an easy consequence of $\varphi \equiv 1$ near 0. Since $\varphi/|\cdot| \in M^3(\mathbb{R}^N)$, we have $V \in M^3(\mathbb{R}^N)$. (Recall that $M^3 * \mathcal{M} \subset M^3$.) Then, the assumption on γ gives $\gamma(V) \in L^1(\mathbb{R}^N)$. (In fact, the assumption $\gamma(\pm 1/|x|) \in L^1$ near $x = 0$ gives $\gamma(W) \in L^1_{\text{loc}}(\mathbb{R}^N)$ for all $W \in M^3(\mathbb{R}^N)$. This is proved in [**3**, pp. 58–59]).

3. The cases $N = 2$ and $N = 1$

THEOREM 2. *Let $\gamma(x, s): \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ be a function measurable in x, and continuous and nondecreasing in s. Assume that $\gamma(x, 0) = 0$ almost everywhere and that the function $x \to \gamma(x, t)$ is in $L^1_{\text{loc}}(\mathbb{R}^N)$ for all t in $\mathbb{R}$.*

(1) *If $N = 2$, assume moreover that for some C_1 and C_2 in $\mathbb{R}^{+*}$ one has*

$$\text{meas}\,\{x, \gamma(x, C_1) \leqq C_2 \quad or \quad \gamma(x, -C_1) \geqq -C_2\} < +\infty. \tag{12}$$

Then the following problem has a solution, unique up to a constant:

$$\begin{cases}-\Delta u+\gamma(.,u)=f\\ \gamma(.,u)\in L^1(\mathbb{R}^2), \quad |\nabla u|\in M^2(\mathbb{R}^2), \quad u\in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^2).\end{cases} \tag{13}$$

(2) *If* $N=1$, *assume also that for some* C_1 *and* C_2 *in* $\mathbb{R}^{+*}$, *one has* (12). *Then the following problem has a solution, unique up to a constant*:

$$\begin{cases}-u''+\gamma(.,u)=f,\\ \gamma(.,u)\in L^1(\mathbb{R}), \quad u\in L^1_{\mathrm{loc}}(\mathbb{R}).\end{cases} \tag{14}$$

Remark 2. Same as Remark 1 (with an obvious modification of (12) for the second part of Remark 1).

Proof of Theorem 2 *in the case* $N=2$. The proof is similar to that of Theorem 1, so we only indicate the necessary modifications.

(a) *Uniqueness* (*up to a constant*). Instead of [**1**, Lemma A.10], we use the following:

LEMMA B [**1**, Lemma A.13]. *Let* $u\in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^2)$, $|\nabla u|\in M^2(\mathbb{R}^2)$, $\Delta u\in L^1(\mathbb{R}^2)$, *and let* p *be a function in* $\mathscr{C}=\{p\in C^1\cap L^\infty(\mathbb{R}),\ p'\geqq 0\}$. *If there is a* $k>0$ *for which* meas $[|u|>k]<\infty$, *then*

$$\int_{\mathbb{R}^2} p'(u)\,|\mathrm{grad}\,u|^2+\int_{\mathbb{R}^2}\Delta u p(u)\leqq 0.$$

To apply this lemma to $u-v$, where u and v are two solutions to (13), we have only to show that meas $[|u|>k]<\infty$ for some k when u is a solution to (13). But $\gamma(.,u)$ is in $L^1(\mathbb{R}^2)$, and so meas $[|\gamma(x,u)|>C_2]$ is finite. Thus, by (12), meas $[|u|>C_1]$ is also finite. On taking p increasing, we obtain $-\int_{\mathbb{R}^2}\Delta(u-v)p(u-v)\geqq 0$ and then $\int_{\mathbb{R}^2}(\gamma(.,u)-\gamma(.,v))p(u-v)\leqq 0$. Consequently, $\gamma(.,u)=\gamma(.,v)$ almost everywhere and therefore $\Delta(u-v)=0$. We conclude the proof by applying the following lemma (instead of [**1**, Lemma A.5] for $N\geqq 3$).

LEMMA C ([**1**, Lemma A.11]). *Let* $u\in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^2)$, $\Delta u\in L^1(\mathbb{R}^2)$ *and* $\lim_{n\to\infty} n^{-2}\int_{n\leqq|x|\leqq 2n}|\nabla u(x)|\,dx=0$. *Then* $|\mathrm{grad}\,u|\in M^2(\mathbb{R}^2)$ *and* $\|\mathrm{grad}\,u\|_{M^2}\leqq d_2'\,\|\Delta u\|_{L^1}$, *where* d_2' *is independent of* u.

By applying Lemma C, we obtain $\nabla u=\nabla v$ almost everywhere and hence u and v differ by a constant.

(b) *Existence of a solution to Problem* (13). Let us first observe that since $N=2$, Lemma 1 in the discussion of problem (4) continues to hold.

Now set $\gamma_\varepsilon(.,u)=\varepsilon u+\gamma(.,u)$. We have

LEMMA 2 *bis. Let* γ *and* f *be as in Lemma* 1, *and assume moreover that* $f\in L^1(\mathbb{R}^2)$. (*Here* $N=2$.) *Then the solution* u *of* (5) *satisfies the following inequalities:*

$$\int_{\{|u|\geqq t\}}|\gamma_\varepsilon(.,u)|\,dx\leqq\int_{\{|u|\geqq t\}}|f|\,dx \text{ for all } t \text{ in } \mathbb{R}, \tag{6}$$

$$\|\nabla u\|_{M^2(\mathbb{R}^2)}\leqq d_2\,\|f\|_{L^1}, \text{ with } d_2=2d_2' \text{ independent of } u. \tag{15}$$

Proof. For (6), identical to that of (6) in Lemma 2, for (15), by application of Lemma C instead of Lemma A.

Next, we define γ_n and u_n as in the final part of the proof of Theorem 1. The arguments are identical, except for the three following relations:

R1. The sequence $(u_n)_n$ is bounded in $W^{1,1}_{\text{loc}}(\mathbb{R}^2)$.

R2. meas $(B \cap \{|u_n| \geqq t\}) \to 0$ as $t \to +\infty$, uniformly in $n \in \mathbb{N}^*$.

R3. The limit u verifies $|\nabla u| \in M^2(\mathbb{R}^2)$.

Proof of R1. We apply the following lemma.

LEMMA D ([**1**, Lemma A.16]). *Let B be a ball of radius R in $\mathbb{R}^N$ and $u \in W^{1,p}(B)$, with $1<p<N$. Then there is a constant C depending only on p and N such that if $\sigma = \text{meas}\,[|u|<\lambda]>0$, then*

$$\|u\|_{L^{p^*}(B)} \leqq \lambda\, (\text{meas } B)^{1/p^*} + C\left(\left(\frac{\text{meas } B}{\sigma}\right)^{1/p^*} + 1\right) \|\nabla u\|_{L^p(B)}$$

where $1/p^* = (1/p) - (1/N)$.

Recall that u_n is in $H^2(\mathbb{R}^2)$, and therefore in $W^{1,1}_{\text{loc}}(B)$ for $1 \leqq p < +\infty$. By (15), $(\nabla u_n)_n$ is bounded in $M^2(\mathbb{R}^2)$. Thus it is bounded in $L^p(B)$ for all $1 \leqq p < 2$. To prove that $(u_n)_{n \in N}$ is bounded in $L^{p^*}(B)$ (and therefore in $L^1(B)$), we just have to show that for some $\varepsilon > 0$ and some $\lambda > 0$, meas $[|u_n| < \lambda] > \varepsilon$, independently of n. Since by (6) $(\gamma_n(.\,, u_n))_n$ is bounded in L^1, one sees that $\alpha_n =$ meas $(\{x, |\gamma_n(x, u_n)| \geqq C_2\})$ is bounded independently of n. Set $E = \{x, \gamma(x, C_1) \leqq C_2$ or $\gamma(x, -C_1) \geqq -C_2\}$ and recall that

$$\gamma_n(x, u_n(x)) = \gamma(x, u_n(x)) \quad \text{if} \quad |\gamma(x, u_n(x))| \leqq n \quad \text{and} \quad |x| \leqq n,$$
$$\gamma_n(x, u_n(x)) = n \,\text{sgn}\, (\gamma(x, u_n(x))) \quad \text{if} \quad |\gamma(x, u_n(x))| \geqq n \quad \text{and} \quad |x| \leqq n.$$

Thus, on choosing $n_0 > C_2$, and setting $B_n = B(0, n)$, one has $(x \in B_n \backslash E$ and $|u_n(x)| > C_1) \Rightarrow (||\gamma_n(x, u_n(x))| > C_2)$ for $n \geqq n_0$. Thus, by (12), meas $\{x \in B_n, |u_n(x)| > C_1\} \leqq \alpha_n + \text{meas } E \leqq M$ independent of n, and therefore meas $(\{x \in B_n, |u_n(x)| < C_1\}) \to \infty$ as $n \to +\infty$. By applying Lemma D, we conclude that $(u_n)_n$ is bounded in $L^{p^*}_{\text{loc}}$, and therefore in L^1_{loc}.

Since $(\nabla u_n)_n$ is bounded in M^2, it is bounded in L^1_{loc}, and so $(u_n)_n$ is a bounded sequence in $W^{1,1}_{\text{loc}}(\mathbb{R}^2)$.

Proof of R2. The proof is immediate, since we have seen that $(u_n)_n$ is a bounded sequence in $L^1(B)$ for every ball B.

Proof of R3. R3 is a direct consequence of the following.

LEMMA. *Assume $u_n \to u$ in $\mathscr{D}'(\mathbb{R}^2)$ and that the sequence $(\nabla u_n)_n$ is bounded in $M^2(\mathbb{R}^2)$. Then $\nabla u \in M^2(\mathbb{R}^2)$.*

Proof. As $(\nabla u_n)_{n \in \mathbb{N}}$ is bounded in $M^2(\mathbb{R}^2)$, it is also bounded in $L^p_{\text{loc}}(\mathbb{R}^2)$ for all $1 \leqq p < 2$. So we may assume, eventually by extracting a subsequence, that $\nabla u_n \to \nabla u$ in L^p_{loc}-weak, and therefore in L^1_{loc}-weak. Set $\varphi = \text{sgn}\,(\partial u/\partial x_i) 1_K$ for $i = 1, 2$ and K compact in $\mathbb{R}^2$. We have $\int (\partial u_n/\partial x_i) \,.\, \varphi \leqq \|\nabla u_n\|_{M^2}\, (\text{meas } K)^{\frac{1}{2}} \leqq M$

(meas $K)^{\frac{1}{2}}$. On letting $n \to +\infty$, we obtain

$$\int_K \left|\frac{\partial u}{\partial x_i}\right| \leqq M\,(\text{meas } K)^{\frac{1}{2}}.$$

By the monotone convergence theorem, this relation is still true for any $K \subset \mathbb{R}^2$ with finite measure. We hence conclude that $\nabla u \in M^2(\mathbb{R}^2)$.

Proof of Theorem 2 *in the Case* $N=1$. (i) Following a method given in **[1]**, we first obtain some simple estimates on a solution to (14). It follows from $u'' \in L^1(\mathbb{R})$ that $u' \in L^\infty(\mathbb{R})$ and the limits $u'(\pm\infty)$ exist If, e g $u'(+\infty) \neq 0$, then $|u(x)| \to +\infty$ as $x \to +\infty$. By using (12), it is easy to see that this contradicts $\gamma(.,u) \in L^1(\mathbb{R})$. Thus $u'(\pm\infty)=0$, and so

$$\|u'\|_{L^\infty} \leqq \|u''\|_{L^1}. \tag{16}$$

(ii) *Uniqueness up to a constant.* Let $p \in C^1(\mathbb{R},\mathbb{R})$ be a nondecreasing function with p and $p' \in L^\infty(\mathbb{R})$, $p(0)=0$. It is not difficult to show that if $u'' \in L^1(\mathbb{R})$ and $u' \in L^\infty$, then $p'(u)u'^2 \in L^1(\mathbb{R})$ and

$$\int p'(u)u'^2 + \int p(u)u'' \leqq 0. \tag{17}$$

By using (17) in the same way as Lemma B, we deduce that the solution u to (14) is unique up to a constant.

(iii) *Existence of a solution for problem* (14). We proceed as in the case $N=2$, by again first noting that Lemma 1 holds for $N=1$. Secondly, we have

Lemma 2, *ter. Let* $N=1$ *and* f *and* γ *be as in Lemma* 1, *and assume in addition that* $f \in L^1(\mathbb{R})$. *Then the solution* u *of* (5) *satisfies the following inequality:*

$$\int_{|u|\geqq t} |\gamma_\varepsilon(.,u)|\,dx \leqq \int_{|u|\geqq t} |f|\,dx \text{ for all } t \text{ in } \mathbb{R}. \tag{6}$$

Proof. This is identical to that of (6) in Lemma 2.

We conclude in similar fashion to the case $N=2$, but use (16) and (6) to obtain u' bounded in L^p_{loc}. Indeed, from (6) with $t=0$ and (16), we deduce that the solution u_n of (5) verifies $\|u_n'\|_{L^\infty} \leqq 2\,\|f_n\|_{L^1}$.

4. The equation $-\Delta u + \gamma(.,u) = f$ in $L^1(\Omega)$

In this section, Ω is a bounded open set in $\mathbb{R}^N$ ($N \geqq 1$), and we assume that its boundary is smooth.

Our main result is the following.

Theorem 3. *Let* $\gamma(x,t): \Omega\times\mathbb{R} \to \mathbb{R}$ *be a function measureable in* x, *continuous and nondecreasing in* t. *Assume that* $\gamma(x,0)=0$ *almost everywhere in* Ω, *and that* $\gamma(.,t) \in L^1_{\text{loc}}(\Omega)$ *for every* $t \in \mathbb{R}$. *Then, for any* f *in* $L^1(\Omega)$, *the following problem has a unique solution:*

$$\begin{cases} -\Delta u + \gamma(.,u) = f \text{ in } \mathscr{D}'(\Omega) \\ u \in W_0^{1,1}(\Omega), \quad \gamma(.,u) \in L^1(\Omega). \end{cases} \tag{18}$$

Remark 3. This is the same as Remark 1 with Ω instead of $\mathbb{R}^N$.

Proof. Here, once again, we follow step by step the proof of Theorem 1. Let us first give a lemma similar to [**1**, Lemma A.10].

LEMMA 3. *Let $p \in C^1(\mathbb{R}, \mathbb{R})$, with $p \in L^\infty(\mathbb{R})$, $p' \geqq 0$, $p(0) = 0$. Then, for all u in $W_0^{1,1}(\Omega)$ with $-\Delta u \in L^1(\Omega)$, one has*

$$\int_\Omega -\Delta u p(u) \geqq 0.$$

Proof. Let $u_n \in \mathcal{D}(\bar{\Omega})$ be a sequence such that

$$u_n \to u \text{ in } W_0^{1,1}(\Omega),$$
$$-\Delta u_n \to -\Delta u \text{ in } L^1(\Omega).$$

By Green's formula, we have $\int_\Omega -\Delta u_n p(u_n) = \int_\Omega |\nabla u_n|^2 p'(u_n) \geqq 0$. Since $p \in L^\infty$, on passing to the limit, we obtain

$$-\int \Delta u p(u) \geqq 0.$$

Uniqueness of a solution to (18). By applying Lemma 3 to the difference $u - v$ of two solutions of (18), we obtain

$$\int_\Omega (\gamma(.,u) - \gamma(.,v)) p(u - v) \leqq 0.$$

On assuming $p' > 0$, we deduce $\gamma(.,u) = \gamma(.,v)$ almost everywhere, and therefore $-\Delta(u - v) = 0$ almost everywhere. Since $u - v \in W_0^{1,1}(\Omega)$, this implies $u = v$ almost everywhere.

Existence of a solution to (18). Lemma 1 still holds with Ω instead of $\mathbb{R}^N$, $\varepsilon = 0$ and $H_0^1(\Omega)$ instead of $H^1(\mathbb{R}^N)$. The proof is identical. Thus we obtain the following result.

LEMMA 4. *Let γ be as in Theorem* 3, *and assume in addition that $\sup_{s \in \mathbb{R}} |\gamma(.,s)| \in L^2(\Omega)$. Let f be in $L^2(\Omega)$. Then the following problem has a unique solution:*

$$\begin{cases} -\Delta u + \gamma(.,u) = f \\ u \in H_0^1(\Omega). \end{cases} \tag{19}$$

Note that since $-\Delta u \in L^2(\Omega)$, we still have $u \in H^2(\Omega)$.

Instead of the estimates in Lemma 2, we shall need those given in the following.

LEMMA 5. *Let γ and f be as in Lemma* 4, *and assume in addition that $f \in L^1(\Omega)$. Then the solution u to* (19) *satisfies the following inequalities:*

$$\int_{\Omega \cap [|u| \geqq t]} |\gamma(.,u)| \leqq \int_{\Omega \cap [|u| \geqq t]} |f| \, dx \text{ for } t \in \mathbb{R}, \tag{20}$$

$$\|u\|_{W^{1,p}(\Omega)} \leqq C \|f\|_{L^1(\Omega)}, \text{ where } p \in [1, N/N-1[, \text{ and } C \text{ only depends on } p \text{ and } \Omega. \tag{21}$$

Proof. The proof of (20) is identical to that of (6) in Lemma 2 (with Ω instead of $\mathbb{R}^N$, $\varepsilon = 0$, and $H_0^1(\Omega)$ instead of $H^1(\mathbb{R}^N)$). On using (20) with $t = 0$, we obtain $\|-\Delta u\|_{L^1} \leqq 2\|f\|_{L^1}$. Using the continuous embedding $L^1(\Omega) \hookrightarrow W^{-1,p}(\Omega)$ for $1 \leqq p < N/(N-1)$ and a classical result of regularity for the Laplacian, we have

$$\|u\|_{W^{1,p}(\Omega)} \leqq C'\,\|-\Delta u\|_{W^{-1,p}(\Omega)} \leqq C''\,\|-\Delta u\|_{L^1(\Omega)},$$

where C' and C'' only depend on p and Ω.

Let $f_n \in L^2(\Omega)$ with $f_n \to f$ in $L^1(\Omega)$ as $n \to \infty$. Set, for $n \geqq 1$,

$$\begin{cases} \gamma_n(x, s) = \gamma(x, s) \text{ if } |\gamma(x\ s)| \leqq n, \\ \gamma_n(x, s) = n \operatorname{sgn}(\gamma(x, s)) \text{ if } |\gamma(x, s)| > n. \end{cases}$$

According to Lemma 4, the following problem has a solution u_n:

$$\begin{cases} -\Delta u_n + \gamma_n(., u_n) = f_n \text{ in } \mathscr{D}'(\Omega), \\ u_n \in H_0^1(\Omega). \end{cases} \tag{22}$$

By applying relations (20) and (21) of Lemma 5, one sees that $(u_n)_n$ is bounded in $W_0^{1,p}(\Omega)$ for $1 \leqq p < N/(N-1)$. Thus, we may assume $u_n \to u$ in $L^1(\Omega)$, $W_0^{1,p}$-weak, and almost everywhere. The proof that $(\gamma_n(., u_n))_n \to \gamma(., u)$ in $L^1_{\text{loc}}(\Omega)$ is identical to that of the last step in Theorem 1. Then, on passing to the limit in (22) and applying Fatou's lemma, a solution u to (18) is obtained.

We now give some consequences of Theorem 3.

COROLLARY 1. *Let $\gamma(x, s)\colon \Omega \times \mathbb{R} \to \mathbb{R}$ be a function measurable in x, continuous and nondecreasing in s. Let $V \in L^1_{\text{loc}}(\Omega)$ be such that $\gamma(x, V(x)) \in L^1(\Omega)$ and $\gamma(x, y + V(x)) \in L^1_{\text{loc}}(\Omega)$ for all t in $\mathbb{R}$. Let $f \in L^1(\Omega)$. Then there exists a unique solution to*

$$\begin{cases} -\Delta u + \gamma(., u) = f - \Delta V, \\ \gamma(., u) \in L^1(\Omega), \quad (u - V) \in W_0^{1,1}(\Omega). \end{cases} \tag{23}$$

Proof. Apply Theorem 3 with $\tilde{\gamma}(x, s) = \gamma(x, s + V(x)) - \gamma(x, V(x))$ instead of $\gamma(x, s)$ and $\tilde{f}(x) = f(x) - \gamma(x, V(x))$ instead of $f(x)$.

Note that, as for Corollary 1 to Theorem 1, this result has an "almost full" generality. In fact, assume that $g \in \mathscr{D}'(\Omega)$ is such that there exists a solution to

$$\begin{cases} -\Delta u + \gamma(., u) = g, \\ \gamma(., u) \in L^1(\Omega), \quad u \in W_0^{1,1}(\Omega). \end{cases} \tag{23$^+$}$$

Then one has $g = f - \Delta V$ with $f = \gamma(., u) \in L^1(\Omega)$ and $V = u \in W_0^{1,1}(\Omega)$ is such that $\gamma(., u) \in L^1(\Omega)$. Then, using Corollary 1, the necessary and "almost" sufficient condition on $g \in \mathscr{D}'(\Omega)$ for (P) to have a solution is $g = f - \Delta V$ with $f \in L^1(\Omega)$ and $V \in W_0^{1,1}(\Omega)$ such that $\gamma(., V) \in L^1(\Omega)$. We say "almost" sufficient because, in the hypothesis of Corollary 1, we also need $\gamma(., t + V) \in L^1_{\text{loc}}(\Omega)$ for all $t \in \mathbb{R}$. This condition is a consequence of $\gamma(., V) \in L^1(\Omega)$ if $\gamma(x, s)$ is not "too rapidly" increasing in s as $s \to +\infty$.

Let us apply this corollary to the case $\gamma(x, u) = |u|^{p-1}u$ $(p > 1)$.

COROLLARY 2. *Let $p>1$, $\mu \in \mathcal{D}'(\Omega)$. Consider the problem*

$$\begin{cases} -\Delta u + |u|^{p-1}u = \mu, \\ u \in L^p(\Omega), \quad u \in W_0^{1,1}(\Omega). \end{cases} \tag{24}$$

Then

(a) *If $\mu \notin L^1(\Omega) + W^{-2,p}(\Omega)$, (24) has no solution,*
(b) *If $\mu \in L^1(\Omega) + W^{-2,p}(\Omega)$, (24) has a unique solution.*

Proof. (a) If (24) has a solution u, *one has* $u \in L^p(\Omega)$, and then $-\Delta u \in W^{-2,p}(\Omega)$. (b) If $\mu = f + g$ with $f \in L^1$ and $g \in W^{-2,p}$, then there exists $V \in L^p(\Omega) \cap W_0^{1,1}(\Omega)$ such that $-\Delta V = g$. (This is a classical result.) So (24) is equivalent to

$$\begin{cases} -\Delta u + |u|^{p-1}u = f - \Delta V, \\ u \in L^p(\Omega), \quad u - V \in W_0^{1,1}(\Omega). \end{cases} \tag{25}$$

(25) can be solved by direct application of Corollary 1.

Corollary 2 may be considered as a generalization of the following result (see Baras and Pierre [**4**]).

THEOREM B ([**4**]). *Let μ be a bounded measure on Ω. Then the problem*

$$\begin{cases} -\Delta u + |u|^{p-1}u = \mu \text{ in } \mathcal{D}'(\Omega), \quad (p>1), \\ u \in L^p(\Omega) \cap W_0^{1,1}(\Omega), \end{cases} \tag{26}$$

has a solution if and only if μ satisfies the following condition:

$$|\mu|\,(A) = 0 \text{ for every subset of } \Omega \text{ whose } W^{2,p'}\text{-capacity is zero.} \tag{27}$$

Remark. Let $K \subset \mathbb{R}^N$ be a compact set. The $W^{2,p'}$-capacity of K is $C_{2,p'}(K) = \inf\{\|\varphi\|^{p'}_{W^{2,p'}},\ \varphi \in \mathcal{D}(\mathbb{R}^N),\ \varphi = 1 \text{ on a neighbourhood of } K\}$. If ω is open, we set $C_{2,p'}(\omega) = \sup_{K \subset \omega} \{C_{2,p'}(K)\}$, and finally for $E \subset \mathbb{R}^N$, $C_{2,p'}(E) = \inf_{\omega \supset E} \{C_{2,p'}(\omega)\}$.

In fact, Corollary 2 is more general than Theorem B, but Theorem B provides a very useful criterion for a measure μ to have the form $f - \Delta V \in L^1 + W^{-2,p}$.

Let us prove Theorem B from Corollary 2. We just have to prove that the following properties are equivalent for all measures $\mu \in \mathcal{M}(\Omega)$:

$$|\mu|\,(A) = 0 \text{ for every subset } A \text{ of } \Omega \text{ with } C_{2,p'}(A) = 0, \tag{27}$$

$$\mu \in L^1(\Omega) + W^{-2,p}(\Omega). \tag{28}$$

The proof that (28) $\Rightarrow$ (27) is straightforward (see [**4**, Lemma 4.1]). To show the converse we use an argument due to Ancona. Assume $\mu \geqq 0$. (If not, we write $\mu = \mu^+ - \mu^-$.) As a consequence of the Hahn–Banach theorem (see [**4**, Lemma 4.2]), one has $\mu = \sum_{n=0}^{\infty} \mu_n$; where the sum converges in $\mathcal{M}(\Omega)$ and μ_n is a bounded positive measure on Ω, with compact support, and $\mu_n \in W^{-2,p}(\Omega)$. Let $\rho_m \in \mathcal{D}(\mathbb{R}^N)$ be a sequence of mollifiers. One has $\mu_n * \rho_m \to \mu_n$ in $W^{-2,p}$ as $m \to +\infty$, and

$$\|\mu_n * \rho_m\|_{L^1(\mathbb{R}^N)} \leqq \|\mu_n\|_{\mathcal{M}(\mathbb{R}^N)}.$$

Choose a sequence $m_n \to +\infty$ such that the sum $g = \sum_n (\mu_n - \mu_n * \rho_{m_n})$ is absolutely convergent in $W^{-2,p}(\Omega)$. As the sum $f = \sum_n \mu_n * \rho_{m_n}$ is absolutely convergent in $L^1(\Omega)$, we obtain $\mu = f + g$, with $f \in L^1(\Omega)$ and $g \in W^{-2,p}(\Omega)$.

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