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RESONANCE FOR JUMPING NON-LINEARITIES.

by

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0. Let $\Omega \subset \mathbb{R}^N$ be an open set. We look for conditions on $h \in L^2(\Omega)$ in order to ensure existence of solution to the problem :

(1) $\quad u \in D(A), \quad Au = \alpha u^+ - \beta u^- + \gamma(.,u) + h$

where $u^+ = \max(u,0)$, $u^- = \max(-u,0)$, A and $\gamma(.,.)$ satisfying :

(2) $\begin{cases} \text{A is a linear self-adjoint operator with compact resolvent,} \\ \text{the domain of A is } D(A) \subset L^2(\Omega)\text{, and A maps } D(A) \text{ into } L^2(\Omega). \end{cases}$

(3) $\begin{cases} \gamma : \Omega \times \mathbb{R} \to \mathbb{R} \text{ is a measurable function with respect to} \\ x \in \Omega\text{, continuous with respect to } s \in \mathbb{R} \text{ and } \lim_{|s|\to\infty} \frac{\gamma(.,s)}{s} = 0, \\ \sup_{s \in \mathbb{R}} \left| \frac{\gamma(.,s)}{s} \right| \in L^\infty(\Omega). \end{cases}$

(When A does not satisfy (2) we assume that :

$$(2+)\begin{cases} A \text{ is a linear self-adjoint operator}, R(A) \text{ is closed in } L^2(\Omega), \\ D(A) \subset L^2(\Omega), \text{ and the inclusion of } D(A) \cap R(A) \text{ (equipped} \\ \text{with the graph norm) in } L^2(\Omega) \text{ is compact.} \end{cases}$$

$$(3+)\begin{cases} \gamma \text{ satisfies (3), and } s \to \alpha s^+ - \beta s^- + \gamma(.,s) = f(.,s) \\ \text{is monotone.)} \end{cases}$$

Finally α and β belong to R and we denote by Sp(A) the spectrum of A : $Sp(A) = \{\lambda_n ; n \in I\}$ (notice that card (I)$\leq$ card N). The following is well-known :

i) if $\alpha = \beta \notin Sp(A)$, for every $h \in L^2(\Omega)$, (1) has at least one solution.

ii) if $\alpha = \beta \in Sp(A)$, (1) is said to be at resonance. Let $\gamma_+(.) = \lim_{s \to +\infty} \gamma(.,s)$, $\gamma_-(.) = \lim_{s \to -\infty} \gamma(.,s)$ a.e. in Ω and in $L^2(\Omega)$ exists and, for instance, $\gamma_+ > \gamma_-$ a.e. in Ω; then (1) has at least one solution for every $h \in L^2(\Omega)$ satisfying the following Landesman-Lazer condition :

$$\forall \varphi \in N_\alpha, \varphi \neq 0, \int h \varphi \, dx < \int \gamma_+ \varphi^- \, dx - \int \gamma_- \varphi^+ \, dx$$

(cf. Landesman-Lazer [1], Brézis-Nirenberg [2]; for $a \in Sp(A)$, N_a is the corresponding eigenspace). The above condition is "almost necessary" in the sense that if $\forall s \in R, \gamma_-(x) \leq \gamma(x,s) \leq \gamma_+(x)$, a.e in Ω, and if (1) has one solution, then $h \in L^2(\Omega)$ satisfies:

$$\forall \varphi \in N_\alpha, \int h \varphi \, dx \leq \int \gamma_+ \varphi^- \, dx - \int \gamma_- \varphi^+ \, dx.$$

iii) if $\alpha \neq \beta$, say $\alpha < \beta$, and $[\alpha,\beta] \cap Sp(A) = \emptyset$, (1) has at least one solution for every $h \in L^2(\Omega)$ (cf. Kazdan-Warner [3], Gallouët-Kavian [4]).

From now on we suppose that α and β satisfy :

(4) $[\alpha,\beta] \cap Sp(A) = \{\lambda\}$, where λ is a simple eigenvalue of A.

Let $\underline{\lambda}$ and $\overline{\lambda}$ be defined as follows :

$$\underline{\lambda} = \operatorname{Sup}\{\lambda_n ; \ \lambda_n < \lambda, \ n \in I\}, \quad \overline{\lambda} = \inf\{\lambda_n ; \lambda_n > \lambda, n \in I\}.$$

We denote by φ a normalised eigenfunction for λ and I_λ the open interval $]\underline{\lambda},\overline{\lambda}[$. In [4] we have proved the following :

__LEMMA I.__ *For every* $(a,b) \in I_\lambda \times I_\lambda$, *there exists a unique* (u,C) *belonging to* $D(A) \times R$ *such that :*

(5) $Au = au^+ - bu^- + C\varphi$ *and* $\int u \varphi \, dx = 1.$

The function $C(.,.) : I_\lambda \times I_\lambda \to R$ *has the following properties :*

a) for every $a \in I_\lambda$, $C(a,a) = \lambda - a.$

b) if, for instance, $\varphi^- = 0$, i.e. $\varphi \geq 0$ *on* Ω, *then* $C(a,b) = \lambda - a.$

c) if $\varphi^- \neq 0$ *and* $\varphi^+ \neq 0$, $C(.,.)$ *is decreasing in each variable.*

d) $\Gamma = \{(a,b) \in I_\lambda \times I_\lambda ; C(a,b) = 0\}$ *is a continuous curve passing through the point* (λ,λ) *of* $I_\lambda \times I_\lambda$.

Let N(a,b) be the set $\{u \in D(A) ; Au = au^+ - bu^-\}$; then

$$N(a,b) = \{0\} \text{ if and only if, } C(a,b).C(b,a) \neq 0$$

(note that $N(\lambda,\lambda) = N_\lambda$). The question of existence or non-existence of solution for (1) when $N(\alpha,\beta) = \{0\}$ has been studied in [4]. In the present paper we study the case where $N(\alpha,\beta) \neq \{0\}$:

1) if $C(\alpha,\beta) = C(\beta,\alpha) = 0$ we have "resonance" ;

2) if $C(\alpha,\beta) = 0 \neq C(\beta,\alpha)$ (or $C(\alpha,\beta) \neq 0 = C(\beta,\alpha)$ we have "semi-resonance".

1. The main result.

In what follows we assume that $\gamma : \Omega \times R \to R$ satisfies :

$$(6)\quad \begin{cases} \sup_{s \in R} |\gamma(.,s)| \in L^2(\Omega) \\ \lim_{s \to +\infty} \gamma(.,s) = \gamma_+(.) \ , \quad \lim_{s \to +\infty} \gamma(.,s) = \gamma_-(.) \quad \text{a.e. in } \Omega \end{cases}$$

Let w_1 and w_2 be defined as follows :

$$(7)\quad Aw_1 = \alpha w_1^+ - \beta w_1^- \quad , \quad \int w_1 \varphi \, dx = 1 \quad (\text{when } C(\alpha,\beta) = 0)$$

$$(8)\quad Aw_2 = \alpha w_2^+ - \beta w_2^- \quad , \quad \int w_2 \varphi \, dx = -1 \quad (\text{when } C(\beta,\alpha) = 0)$$

(notice that $u = -w_2$ satisfies $Au = \beta u^+ - \alpha u^-$, $\int u \varphi dx = 1$)

The main theorem is :

<u>*THEOREM 1*</u>. *Assume that (2), (3), (or (2+) (3+)), and (4), (6) are fulfilled. For* $h \in L^2(\Omega)$ *we define* $A_i(h)$

$$A_i(h) = \int h\, w_i \, dx + \int \gamma_+ w_i^+ \, dx - \int \gamma_- w_i^- \, dx \quad , \ i = 1,2.$$

i) *If* $C(\alpha,\beta) = C(\beta,\alpha) = 0$, *(1) has at least one solution for every* $h \in L^2(\Omega)$ *such that* $A_1(h).A_2(h) > 0$.

ii) *If* $C(\alpha,\beta) = 0 \neq C(\beta,\alpha)$ *(resp.* $C(\alpha,\beta) \neq 0 = C(\beta,\alpha)$, *(1) has at least one solution for every* $h \in L^2(\Omega)$ *such that* $C(\beta,\alpha).A_1(h) < 0$ *(resp.* $C(\alpha,\beta).A_2(h) < 0$*).*

<u>Remarks</u>.1) When $\alpha = \beta$, $C(\alpha,\beta) = 0$ means $\alpha \in Sp(A)$, and the condition $A_1(h).A_2(h) > 0$ is precisely the Landesman-Lazer sufficient condition (notice that in this case $w_1 = -w_2 = \varphi$).

2) When $\alpha \neq \beta$ and $C(\alpha,\beta) = C(\beta,\alpha) = 0$, the sufficient condition $A_1(h).A_2(h) > 0$ is not "almost necessary".

3) For $h = h_0 + t\varphi$ (where $t \in R$ and $h_0 \in \varphi^\perp$), in the case i) of the theorem, (1) has at least one solution when t belongs to an open bounded interval depending on h_0.

4) When $C(\alpha,\beta) = C(\beta,\alpha) = 0$, we have $N(\alpha,\beta) = R_+w_1 \cup R_+w_2$; when $C(\alpha,\beta) = 0 \neq C(\beta,\alpha)$ (resp. $C(\alpha,\beta) \neq 0 = C(\beta,\alpha)$ we have $N(\alpha,\beta) = R_+ w_1$ (resp. $R_+ w_2$).

2. Proof of the theorem when the resolvent of A is compact.

The proof relies on the invariance, by compact homotopy, of the topological degree of Leray-Schauder [5].

When $C(\alpha,\beta) = C(\beta,\alpha) = 0$, assume, for instance, $\alpha < \beta$ and $A_i(h) < 0$ for $i = 1,2$. Let $z \in R\text{-}Sp(A)$ and $B = (A - zI)^{-1}$. For $t \in [0,1]$ and $u \in L^2(\Omega)$ we define :

$$T(t,u) = B(\alpha u^+ - \beta u^- + t\gamma(.,u) + (\beta-\alpha)(1-t)u^-) - zBu.$$

Clearly the following problems are equivalent :

$$(9)\quad u \in D(A), \quad Au = \alpha u^+ - \beta u^- + t\gamma(.,u) + (\beta-\alpha)(1-t)u^- + h$$

$$(10)\quad u \in L^2(\Omega), \quad u = T(t,u) + Bh.$$

We shall use the following lemma, which is proved in §3 :

<u>*LEMMA 2*</u>. *Under the assumptions of theorem 1 i) and assuming that* $A_i(h) < 0$ *(i = 1,2), there exists* $R > 0$ *such that for* $t \in [0,1]$ *and* $u \in L^2(\Omega)$ *satisfying (10) we have* $\|u\|_{L^2} < R$.

We denote by $B(0,R)$ the ball of radius R in $L^2(\Omega)$ and by $d(.;.;.)$ the topological degree of Leray-Schauder. The resolvent B being compact, $T(.,.) : [0,1] \times L^2(\Omega) \to L^2(\Omega)$ is compact (here we assume that A satisfies (2)) : by invariance of the topological degree we have :

$$\forall t \in [0,1],\ d(Id - T(t,.)\ ;\ B(0,R)\ ;\ Bh) = \text{constant}.$$

In particular :

$$d(Id - T(1,.)\ ;\ B(0,R)\ ;\ Bh) = d(Id-T(0,.);B(0,R);Bh) = \pm 1$$

(when $t = 0$, (9) becomes $Au = \alpha u + h$ and $\alpha \notin Sp(A)$: so the last degree is necessarily ± 1). In this way one sees that (10) has a solution for $t = 1$, i.e. (1) has at least one solution. It remains to establish lemma 2.

3. Proof of lemma 2.

Assuming that A satisfies (2), we make a proof by contradiction. Suppose there exists a sequence $(t_n,u_n)_n$ such that : $t_n \in [0,1]$, $u_n \in D(A)$, $b_n = \|u_n\|_{D(A)} \to \infty$ as n goes to infinity, and

$$(11)\quad Au_n = \alpha u_n^+ - \beta u_n^- + t_n\gamma(.,u_n) + (1-t_n)(\beta-\alpha)u_n^- + h$$

Let $v_n = \dfrac{u_n}{b_n}$; the inclusion $D(A) \subset L^2(\Omega)$ being compact we can extract a subsequence (still denoted by v_n, t_n) such that $v_n \to v$ in $L^2(\Omega)$ and a.e., $t_n \to t$. Moreover

$$(12)\quad Av_n = \alpha v_n^+ - \beta v_n^- + t_n\frac{\gamma(.,b_nv_n)}{b_n} + (1-t_n)(\beta-\alpha)v_n^- + \frac{h}{b_n}$$

and the right-hand side of (12) tends to f in $L^2(\Omega)$, with f

$$f = \alpha v^+ - \beta v^- + (1-t)(\beta-\alpha)v^-$$

Hence $Av_n \to f$ in L^2, $v_n \to v$ in L^2 : A being a closed operator, we conclude that $v \in D(A)$ and

$$(13)\quad Av = \alpha v^+ - \beta v^- + (1-t)(\beta-\alpha)v^- \ ,\quad \|v\|_{D(A)} = 1.$$

If $\int v\,\varphi\,dx = 0$, then v and Av belong to $\varphi^\perp$ and using proposition 2.1 of [4], we see that $v = 0$: this is in contradiction with $\|v\|_{D(A)} = 1$. Hence $\int v\,\varphi\,dx \neq 0$.

We claim that $t = 1$. If $t < 1$, $C(.,.)$ being decreasing (because we assume that $\alpha < \beta$ and $C(\alpha,\beta) = C(\beta,\alpha) = 0$: this means that $\varphi^+ \neq 0$ and $\varphi^- \neq 0$; lemma 1c) implies that $C(.,.)$ is decreasing) we have the following :

$$(14)\begin{cases} 0 = C(\alpha,\beta) < C(\alpha,\ \beta+(1-t)(\alpha-\beta) \\ 0 = C(\beta,\alpha) < C(\beta+(1-t)(\alpha-\beta)\ ,\alpha) \end{cases}$$

But (13) means that :

. if $\int v\,\varphi\,dx > 0$, $\quad C(\alpha,\beta+(1-t)(\alpha-\beta)) = 0$;

. if $\int v\,\varphi\,dx < 0$, $\quad C(\beta+(1-t)(\alpha-\beta),\alpha) = 0$;

this is in contradiction with (14). Hence we have $t = 1$ and

$$(15)\quad Av = \alpha v^+ - \beta v^- \ ,\quad \|v\|_{D(A)} = 1\ ,\quad v \in N(\alpha,\beta).$$

Then we can write :

$$v = aw_1 \quad \text{if} \quad a = \int v\,\varphi\,dx > 0$$

$$v = aw_2 \quad \text{if} \quad -a = \int v\,\varphi\,dx < 0.$$

We assume that, for instance, $\int v\,\varphi\,dx < 0$, and we define $a_n \in R$, $Z_n \in D(A)$ as follows :

$$a_n = -\int v_n\,\varphi\,dx\ ,\quad Z_n = v_n - a_nw_2$$

in such a way that :

(16) $v_n = a_n w_2 + Z_n$, $a_n \to a > 0$, $\|Z_n\|_{D(A)} \to 0$, $Z_n \in \varphi^\perp$

We claim that :

(17) there exists $M > 0$ such that $\forall\, n \geq 1$, $b_n \|Z_n\|_{D(A)} \leq M$.

Suppose that (17) is established and multiply (11) on both sides by w_2 (remember that $u_n = b_n v_n$) :

$$b_n \int Av_n . w_2 dx = b_n \int (\alpha v_n^+ - \beta v_n^-) w_2 dx + (1-t_n)(\beta-\alpha) b_n \int v_n^- w_2 dx$$

$$+ t_n \int \gamma(x, b_n v_n) . w_2 dx + \int h w_2 dx$$

For n large enough, $\int v_n^- w_2 dx \leq 0$, because $v_n^- \to a w_2^-$ in L^2 and $a > 0$; hence

(18) $\int h w_2 dx + t_n \int \gamma(x, b_n v_n) . w_2 dx \geq b_n \int (Av_n . w_2 - (\alpha v_n^+ - \beta v_n^-) w_2 dx$

First of all notice that the limit of the left-hand side of (18) is equal to $A_2(h)$. Next, notice that :

$$E_n = \int [\, Av_n . w_2 - (\alpha v_n^+ - \beta v_n^-) w_2]\, dx$$

(19) $E_n = \int Av_n . w_2 dx - \int (\alpha v_n^+ - \beta v_n^-) w_2 dx$

$$E_n = \int v_n . A w_2 dx - \int (\alpha v_n^+ - \beta v_n^-) w_2 dx$$

because $A = A^*$; now remember that $Aw_2 = \alpha w_2^+ - \beta w_2^-$

$$E_n = \alpha \int (v_n w_2^+ - v_n^+ w_2) dx - \beta \int (v_n w_2^- - v_n^- w_2) dx$$

$$E_n = \alpha \int (v_n^+ w_2^- - v_n^- w_2^+) dx - \beta \int (v_n^+ w_2^- - v_n^- w_2^+) dx$$

that is $E_n = (\alpha-\beta) \int (v_n^+ w_2^- - v_n^- w_2^+) dx$, and we have :

(20) $|E_n| \leq |\beta-\alpha| \left(\int v_n^+ w_2^- dx + \int v_n^- w_2^+ dx \right)$

As $a_n \to a$ and $a > 0$, for n large enough we have $a_n > 0$.

If $x \in \Omega$ is such that

$$w_2(x) \geq 0 \quad \text{and} \quad v_n(x) = a_n w_2(x) + Z_n(x) \leq 0$$

then :

$$Z_n(x) \leq v_n(x) \leq 0 \quad \text{and} \quad 0 \leq w_2(x) = \frac{v_n(x) - Z_n(x)}{a_n} \leq \frac{|Z_n(x)|}{a_n}$$

and hence $v_n^-(x) w_2^+(x) \leq \frac{1}{a_n} |Z_n(x)|^2$, a.e. in Ω.

Using the same arguments, one can see that :

$$v_n^+(x) . w_2^-(x) \leq \frac{1}{a_n} |Z_n(x)|^2 \quad \text{a.e. in } \Omega.$$

From these inequalities and (20) we deduce

(21) $|E_n| \leq |\beta-\alpha| \frac{2}{a_n} \|Z_n\|^2_{L^2}$

hence, using (17) :

$$b_n |E_n| \leq |\beta-\alpha| \frac{2}{a_n} M . \|Z_n\|_{D(A)}$$

so that $\lim_{n\to\infty} b_n |E_n| = 0$. Now coming back to (18) and passing to the limit we find :

$$A_2(h) = \int h w_2 dx + \int \gamma_+ w_2^+ dx - \int \gamma_- w_2^- dx \geq \lim_{n\to\infty} b_n E_n = 0,$$

which is in contradiction with $A_i(h) < 0$ $(i = 1,2)$: then the proof of lemma 2 will be complete once we have established (17). If (17) does not hold, there is a subsequence (still denoted by $b_n \|Z_n\|_{D(A)}$) such that $\lim_{n\to\infty} b_n \|Z_n\|_{D(A)} = +\infty$. Let $C_n = \|Z_n\|_{D(A)}$, $y_n = \frac{Z_n}{C_n}$. The inclusion $D(A) \subset L^2(\Omega)$ being compact there is a subsequence (still denoted by) $(y_n)_n$ such that :

$$(22)\quad \begin{cases} y_n \overset{L^2}{\to} y \quad , \quad Ay_n \rightharpoonup Ay \quad \text{in } L^2 \text{ weak}, \quad y \in \varphi^1 \\ y_n(x) \to y(x) \quad \text{a.e. in } \Omega, \text{ and there exists } g \in L^2(\Omega) \text{ such} \\ \text{that} \quad |y_n(x)| \leq g(x) \quad \text{a.e.} \end{cases}$$

On the other hand, $v_n = a_n w_2 + Z_n$ satisfies (12) ; multiplying (12) by $\frac{w_2}{C_n}$ and using the fact that $A = A^*$, we get :

$$(23)\quad \frac{1}{C_n} \int v_n A w_2 dx = \frac{1}{C_n} \int (\alpha v_n^+ - \beta v_n^-) w_2 dx + t_n \int \frac{\gamma(\cdot, b_n v_n)}{b_n C_n} w_2 dx + \int \frac{h}{b_n C_n} w_2 dx + \frac{1-t_n}{C_n}(\beta-\alpha) \int v_n^- w_2 dx$$

E_n being defined in (19), it follows from (23) that :

$$(\beta-\alpha) \frac{1-t_n}{C_n} \int v_n^- w_2 dx = \frac{1}{C_n} E_n - t_n \int \frac{\gamma(x, b_n v_n)}{b_n C_n} w_2 dx - \int \frac{h}{b_n C_n} w_2 dx$$

Applying (21) we get $\lim_{n\to\infty} \frac{1}{C_n} E_n = 0$. As $\sup_{s\in\mathbb{R}} |\gamma(\cdot,s)| \in L^2(\Omega)$, and $b_n C_n$ going to infinity we conclude that :

$$(\beta-\alpha) \lim_{n\to\infty} \frac{1-t_n}{C_n} \int v_n^- w_2 dx = 0.$$

On the other hand $\lim_{n\to\infty} \int v_n^- w_2 dx = -a \int |w_2^-|^2 dx \neq 0$, since w_2 satisfies (8) and $\alpha \notin Sp(A)$, hence $w_2^- \neq 0$. As $\beta-\alpha \neq 0$, we find that

$$(24)\quad \lim_{n\to\infty} \frac{1-t_n}{C_n} = 0.$$

From (8), (12) and $y_n = \frac{Z_n}{C_n}$, $v_n = a_n w_2 + Z_n$ we get :

$$(25)\quad Ay_n = \alpha \left[\left(\frac{a_n}{C_n} w_2 + y_n\right)^+ - \frac{a_n}{C_n} w_2^+ \right] - \beta \left[\left(\frac{a_n}{C_n} w_2 + y_n\right)^- - \frac{a_n}{C_n} w_2^- \right] + t_n \frac{\gamma(\cdot, b_n v_n)}{b_n C_n} + \frac{h}{b_n C_n} + (\beta-\alpha) \frac{1-t_n}{C_n} v_n^-$$

when n goes to infinity, $b_n C_n \to \infty$, and the last three terms of (25) go to zero in $L^2(\Omega)$ (notice that for the last term we use (24)).

On the other hand one can easily check that a.e. in Ω, the following inequalities hold (remember that by (22) we have $|y_n| \leq g$ a.e. in Ω) :

$$(26)\quad \begin{aligned} &\left| \left(\frac{a_n}{C_n} w_2 + y_n\right)^+ - \frac{a_n}{C_n} w_2^+ \right| \leq |y_n| \leq g \\ &\left| \left(\frac{a_n}{C_n} w_2 + y_n\right)^- - \frac{a_n}{C_n} w_2^- \right| \leq |y_n| \leq g \end{aligned}$$

Moreover, extracting a subsequence, we may assume that the last three terms of (25) go to zero a.e. in Ω, and there exists $g_1 \in L^2(\Omega)$ such that

$$\left| t_n \frac{\gamma(x, b_n v_n(x))}{b_n C_n} + \frac{h(x)}{b_n C_n} + (\beta-\alpha) \frac{1-t_n}{C_n} v_n^-(x) \right| \leq g_1(x) \text{ a.e.}$$

in Ω .

Hence applying (25), (26) and the above inequality , we have

$$(27)\quad |Ay_n(x)| \leq 2\,\mathrm{Max}(|\alpha|,|\beta|).g(x) + g_1(x)$$

Let $\rho(x)$ be defined a.e. in Ω as follows :

$\rho(x) = \alpha$ if $w_2(x) > 0$, or if $w_2(x) = 0$ and $y(x) \geq 0$

$\rho(x) = \beta$ if $w_2(x) < 0$, or if $w_2(x) = 0$ and $y(x) < 0$, then, from (25) and the fact that $C_n \to 0$ one can see that

$Ay_n(x) \to \rho(x)y(x)$ a.e. in Ω, and using (27) and Lebesgue convergence theorem we conclude that :

$$Ay_n \overset{L^2}{\to} \rho y \quad , \quad y_n \overset{L^2}{\to} y.$$

The operator A being closed we have : $y_n \overset{D(A)}{\to} y \quad y \in D(A)$,

$Ay = \rho y, \quad y \in \varphi^{\perp}$, $\| y \|_{D(A)} = 1$.

Since ρ satisfies : $\underline{\lambda} < \alpha \leqslant \rho \leqslant \beta < \overline{\lambda}$, applying proposition (2.1) of [4], we conclude that $y = 0$: this is in contradiction with $\| y \|_{D(A)} = 1$, and hence, (17) is established.

<u>Remarks</u> :

1) For $\alpha = \beta = \lambda \in Sp(A)$, that is for the classical resonant problem, the same method can be applied, and actually, the proof of Landesman-Lazer theorem is quite simple. In fact, in this case, λ has not to be a simple eigenvalue of A ,and we do not need to assume that $\gamma_+ \leqslant \gamma_-$ nor $\gamma_- \leqslant \gamma_+$; the sufficient condition for existence of solution to (1) reads:

$\forall\phi \in N_\lambda$, $\phi \neq 0$, $A_\phi(h) = \int h\phi dx + \int \gamma_+ \phi^+ dx - \int \gamma_- \phi^- dx \neq 0$, and $A_\phi(h)$ has the same sign for all $\phi \in N_\lambda$, $\phi \neq 0$.

Suppose that, for instance, $\forall\phi \in N_\lambda, \phi \neq 0$, $A_\phi(h) < 0$. We choose $\varepsilon > 0$ such that $[\lambda - \varepsilon, \lambda[\cap Sp(A) = \emptyset$; we have to prove that there exists $R > 0$ such that for (t,u) satisfying:

(9)' $\quad t \in [0,1]$, $u \in D(A)$, $\quad Au = \lambda u + t\gamma(.,u) - (1-t)\varepsilon u + h$

we have $\| u \|_{D(A)} < R$. If not, there is a sequence $(t_n,u_n)_n$ satisfying (9)' such that $t_n \in [0,1]$, $b_n = \| u_n \|_{D(A)} \to +\infty$ as $n \to \infty$; Extracting a subsequence we may suppose $\frac{u_n}{b_n} = v_n \to v$ in L^2, $t_n \to t$, (t,v) satisfying (cf § 3)

$$Av = (\lambda - (1-t)\varepsilon)v \quad , \quad \| v \|_{D(A)} = 1.$$

This implies $t = 1$, $\quad v = \varphi \in N_\lambda$. But

$$\int Au_n . \varphi dx = \lambda \int u_n \varphi dx + t_n \int \gamma(.,b_n v_n)\varphi dx + \int h\varphi dx - $$
$$- (1-t_n)\varepsilon b_n \int v_n \varphi dx$$

that is

$$\int h\varphi dx + t_n \int \gamma(.,b_n v_n)\varphi dx = (1-t_n)\varepsilon b_n \int v_n \varphi dx$$

The right-hand side is non-negative for n large enough (because $\int v_n \varphi dx \to 1$) ; the left-hand side tends to :

$$\int h\varphi dx + \int \gamma_+ . \varphi^+ dx - \int \gamma_- \varphi^- dx \geqslant 0$$

which is in contradiction with

$$\forall \varphi \in N_\lambda \ , \ \varphi \neq 0 \ , \quad \int h\varphi dx + \int \gamma_+ \varphi^+ dx - \int \gamma_- \varphi^- dx < 0.$$

2) The proof of the theorem in other cases is quite the same : instead of $T(t,u)$ defined in § 2, we should define :

$$T_o(t,u) = B(\alpha u^+ - \beta u^- + t\gamma(.,u) + \tilde{f}(t,u)) - zBu$$

where $\tilde{f}(t,u) = \pm (\beta-\alpha)(1-t)u^{\pm}$, and the sign + or - is to be chosen according to the sign of $(\beta-\alpha)$, $A_i(h)$, and $C(\alpha,\beta)$ (or $C(\beta,\alpha)$).

3) When $\Omega \subset R$ is an open interval and $Au = -u''$ with Dirichlet boundary condition, E.N. Dancer [6] finds similar sufficient conditions for existence of solution to the resonant semi-linear problem, and he points out that these sufficient conditions are not necessary in general.

4. Proof of the theorem in the non-compact case.

We shall give the proof when $0 \leqslant \underline{\lambda} < \alpha < \lambda < \beta < \overline{\lambda}$ and $C(\alpha,\beta) = 0 = C(\beta,\alpha)$ and $A_i(h) < 0$ $(i = 1,2)$, the assumptions (2+), (3+), (4), (6) being satisfied. Moreover, for convenience, we shall assume the following :

(28) $\delta > 0$, $\forall\, u, v \in R$, $(f(.,v) - f(.,u))(v-u) \geqslant \delta|u-v|^2$

Let $g_t(u) = \alpha u^+ - \beta u^- + t\gamma(.,u) + (1-t)(\beta-\alpha)u^-$, for $u \in L^2$ and $t \in [0,1]$; let Q be the orthogonal projection of L^2 on $N(A)$ $(\neq\{0\})$ and $B = (A-zI)^{-1}$ for $z \in R\text{-}Sp(A)$.

Then we have the following lemma which is analoguous to lemma 5.2 of [4] and can be proved in the same way (remember that

$$\forall\, u, v \in L^2, \forall\, t \in [0,1], \ (g_t(u)-g_t(v))(u-v) \geqslant \delta|u-v|^2,$$

a.e. in Ω) :

<u>LEMMA 4</u>. *Let (2+), (3+), (28) be satisfied. For* $t \in [0,1]$ *and* $u \in L^2$, *define* $D_t u = (I-Q)u + \frac{1}{z}Qg_t(u)$.

Then $D_t : L^2 \to L^2$ *is invertible,* $D_t^{-1} : L^2 \to L^2$ *is continuous, and bounded on bounded sets.*

Then, for $(t,u) \in [0,1] \times L^2$ we define

$$S(t,u) = B(I-Q)g_t(u) - zB(I-Q)u$$

$$T(t,u) = D_t^{-1}S(t,u) + D_t^{-1}Bh.$$

By (2+), $B(I-Q)$ is compact and so $T: [0,1] \times L^2 \to L^2$ is compact.

Clearly the following problems are equivalent :

(29) $u \in D(A)$, $Au = \alpha u^+ - \beta u^- + t\gamma(.,u) + (1-t)(\beta-\alpha)u^- + h$

(30) $u \in L^2(\Omega)$, $u = T(t,u)$,

and to use the invariance of the topological degree, it suffices to prove

<u>LEMMA 5</u>. *There exists* $R > 0$ *such that* $\forall\, (t,u) \in [0,1] \times L^2$ *satisfying (30), we have* $\|u\|_{L^2} < R$.

The proof is made by contradiction : assume there is a sequence $(t_n,u_n)_n \in [0,1] \times D(A)$, such that $b_n = \|u_n\|_{D(A)} \to +\infty$.

Extracting a subsequence we may assume :

$$t_n \to t \in [0,1] \ ; \ v_n = \frac{u_n}{b_n} \rightharpoonup v \quad \text{in } D(A) \text{ weak :}$$

$$v_{1n} = (I-Q)\frac{u_n}{b_n} \in D(A) \cap R(A), \quad v_{1n} \to v_1 \quad \text{in } L^2$$ (the inclusion $D(A) \cap R(A) \subset L^2$ is compact)

$$v_{2n} = Q\frac{u_n}{b_n} \in N(A), \quad v_{2n} \rightharpoonup v_2 \quad \text{in } L^2 \text{ weak.}$$

It is easy to see that

(31) $\forall u,v \in R$, $\forall\, t \in [0,1]$, $(g_t(u)-g_t(v))(u-v) \geqslant \delta|u-v|^2$ and v_n satisfies :

$$Av_n = \frac{1}{b_n}g_{t_n}(b_nv_n) + \frac{h}{b_n} = Av_{1n} \quad (v_n = v_{1n} + v_{2n},$$

$v_{2n} \in N(A)$)

Using (31) we have, for $w \in L^2(\Omega)$:

(32) $\frac{1}{b_n}\int(g_{t_n}(b_nv_n) - g_{t_n}(b_nw))(v_n-w)dx \geqslant \delta\|v_n - w\|^2_{L^2}$

But $\frac{1}{b_n}g_{t_n}(b_nv_n) = Av_{1n} - \frac{h}{b_n}$; hence, denoting by $(.|.)$ the scalar product in $L^2(\Omega)$, we have

(33) $(Av_{1n}|v_n-w) - (\frac{h}{b_n} + \frac{1}{b_n}g_{t_n}(b_nw)|v_n-w) \geqslant 0$

Righting $w = w_1 + w_2$ with $w_1 \in R(A)$, $w_2 \in N(A)$, we have

. $(Av_{1n}|v_n-w) = (Av_{1n}|v_{1n}-w_1) \to (Av_1|v_1-w_1)$

because $Av_{1n} \rightharpoonup Av_1$ in L^2 weak

$$v_{1n}-w_1 \to v_1-w_1 \quad \text{in } L^2.$$

$$\frac{h}{b_n} + \frac{1}{b_n} g_{t_n}(b_n w) \to \alpha w^+ - \beta w^- + (1-t)(\beta-\alpha)w^-, \text{ in } L^2 \text{ and hence}$$

$$\left(\frac{h}{b_n} + \frac{1}{b_n} g_{t_n}(b_n w)\middle| v_n - w\right) \to (\alpha w^+ - \beta w^- + (1-t)(\beta-\alpha)w^- | v-w).$$

So, passing to the limit in (33) we have

$$(Av_1 - (\alpha w^+ - \beta w^- + (1-t)(\beta-\alpha)w^-)|v-w) \geqslant 0$$

and using Minty's trick (replace w by $v + \varepsilon w$ and make $\varepsilon \to 0$) :

(34) $Av = \alpha v^+ - \beta v^- + (1-t)(\beta-\alpha)v^-$

Now, coming back to (32), we put $w = v$ and passing to the limit we find

$$\overline{\lim_{n\to\infty}}\ \delta \|v_n - v\|^2 \leqslant 0$$

i.e. $v_n \to v$ in L^2

$$Av_n = \frac{1}{b_n} g_{t_n}(b_n v_n) + \frac{h}{b_n} \to \alpha v^+ - \beta v^- + (1-t)(\beta-\alpha)v^-, \text{ in } L^2,$$

so that $v_n \to v$ in $D(A)$ and $\|v\|_{D(A)} = 1$.

Using the fact that $C(.,.)$ is decreasing, one can prove in the same way as in §3 , that $(v|\varphi) \neq 0$ and $t = 1$. Hence

$$v = aw_1 \quad \text{if} \quad a = (v|\varphi) > 0$$

$$v = aw_2 \quad \text{if} \quad -a = (v|\varphi) < 0.$$

Assume that, for instance, $-a = (v|\varphi) < 0$ and $v = aw_2$. Define

$a_n = -(v_n|\varphi)$, $Z_n = v_n - a_n w_2$

(35) $v_n = a_n w_2 + Z_n$, $a_n \to a > 0$, $\|Z_n\|_{D(A)} \to 0$

$Z_n \in \varphi^\perp$.

We can prove that the analoguous of (17) holds and hence the proof of the theorem, in the non-compact case, can be carried out in the same way as in §3.

Remark. Here we have supposed that $f(.,.)$ satisfies (28) ;in fact if $f(.,.)$ satisfies (3+),(4),(6) but not (28) ,one can find,for $n \in N$ large enough , $u_n \in D(A)$ such that

(36) $\quad Au_n = f(.,u_n) \pm u_n/n + h$

where the sign + or − is to be chosen according to the sign of $C(\alpha,\beta)$, $C(\beta,\alpha)$,and $A_1(h)$.(With this choice we have $C(\alpha \pm 1/n, \beta \pm 1/n).C(\beta \pm 1/n, \alpha \pm 1/n) > 0$ and the existence of one solution to (36) is given by [4]).Then one can find a priori estimate for $(u_n)_n$ in $D(A)$,and making $n \to$ we find a solution to $Au = f(.,u) + h$. Actually the existence of a priori estimate in this case relies on the fact that f is strongly monotone near infinity : for instance,if $0<\alpha\leqslant\beta$ one has : $f(.,s).s \geqslant \alpha s^2 - s \sup_{s\in R}|\gamma(.,s)|$

$$f(.,s).s \geqslant \frac{\alpha}{2} s^2 - k(.)$$

where $k(.) = \frac{1}{2\alpha} \sup_{s\in R} |\gamma(.,s)| \in L^1(\Omega)$ (cf. §4 of [4])

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GLOBALLY HYPOELLIPTIC VECTOR FIELDS ON COMPACT SURFACES

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1. Introduction

The main purpose of this paper is to classify complex, globally hypoelliptic, non-singular vector fields on compact, connected, orientable, two-dimensional smooth manifolds M^2. Consider a complex vector field L on M^2.

Def. 1.1: L is said to be globally hypoelliptic if for every distribution $u \in \mathcal{D}'(M^2)$ such that $Lu \in C^\infty(M^2)$, it follows that $u \in C^\infty(M^2)$.

The principal symbol ℓ of L is defined on the cotangent bundle $T^*(M^2)$ by the identity

$$\ell(d\phi) = L(\phi) \ , \ \phi \in C^\infty(M^2;R)$$

Def. 1.2: L is said to satisfy condition (P) in M^2 if