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THE EQUATION $-\Delta u + |u|^{\alpha-1}u = f$, FOR $0 \leq \alpha \leq 1$

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1. INTRODUCTION

LET $\alpha > 0$ AND $f \in L^1_{loc}(\mathbb{R}^n)$, $n \geq 1$. We consider the following problem:

$$\left.\begin{aligned} -\Delta u + |u|^{\alpha-1}\, u = f \quad \text{in } \mathcal{D}'(\mathbb{R}^n),\ u \in L^1_{loc}(\mathbb{R}^n), \\ |u|^{\alpha-1}\, u \in L^1_{loc}(\mathbb{R}^n). \end{aligned}\right\} \tag{1}$$

In [1], Brezis showed the following result.

THEOREM [1]. Let $1 < \alpha < \infty$. For every $f \in L^1_{loc}(\mathbb{R}^n)$, there exists a *unique* $u \in L^\alpha_{loc}(\mathbb{R}^n)$ satisfying (1). Moreover, if $f \geq 0$ a.e., then $u \geq 0$ a.e.

In this theorem, no limitation on the growth at infinity of the data f is required for the existence of a solution u, and u is unique without prescribing its behaviour at infinity. The proof uses a device introduced by Baras and Pierre [2] which provides a local estimate of the solutions of (1).

The preceding result fails if $0 < \alpha \leq 1$. Indeed, it is enough to consider, for $n = 1$, the Cauchy problem:

$$\left.\begin{aligned} &u'' = u^\alpha \quad \text{in } \mathbb{R} \\ &u'(0) = 0, \quad u(0) = u_0. \end{aligned}\right\} \tag{2}$$

One sees that for any $u_0 > 0$ (2) has a global positive solution in $\mathbb{R}$, so that no uniqueness is expected even for positive solutions of (1).

Let $0 < \alpha < 1$. We now briefly explain our plan and results: In Section 2 (theorem 1), we prove that if $f \geq 0$ and (1) has a positive solution, then (1) has a "minimal" positive solution u (i.e. $0 \leq u \leq v$ for any other positive solution v).

In Section 3, we focus on the problem of existence for $0 < \alpha < 1$ of positive solutions of (1) for $f \in L^1_{loc}(\mathbb{R}^n)$, $f \geq 0$. This problem is closely related to the following (porous medium equation):

$$\text{for } m > 1, \text{ find } u(x, t) \geq 0 \text{ such that } (\partial u/\partial t) = \Delta(u^m),\ u(x, 0) = u_0(x). \tag{3}$$

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Indeed, the "implicit Euler scheme" associated with (3) is:

$$u(t+h) - u(t) - h\Delta u^m(t+h) = 0, \qquad u(t+h) \geq 0, u(t) \geq 0, h > 0. \tag{4}$$

Setting $u(t+h) = h^{\alpha/\alpha-1}\tilde{u}^\alpha$, where $\alpha = m^{-1}$, we get:

$$-\Delta\tilde{u} + \tilde{u}^\alpha = h^{\alpha/1-\alpha}u(t), \qquad u(t) \geq 0, \tilde{u} \geq 0. \tag{5}$$

We obtain precisely equation (1) with $f = h^{\alpha/1-\alpha}u(t)$.

Our main result is the following (theorem 2, Section 3): We set $B_R = \{x \in \mathbb{R}^n, |x| < R\}$. For any $0 < \alpha < 1$, there exists $c_1 > 0$ and $c_2 > 0$ depending only on n and α such that for $f \in L^1_{\text{loc}}(\mathbb{R}^n)$, $f \geq 0$, one has:

$$\left(\limsup_{R\to\infty} R^{-n-(2\alpha/1-\alpha)} \int_{B_R} f\,dx < c_1\right) \Rightarrow ((1) \text{ has a positive solution}); \tag{6}$$

$$((1) \text{ has a positive solution}) \Rightarrow \left(\limsup_{R\to+\infty} R^{-n-(2\alpha/1-\alpha)} \int_{B_R} f\,dx < c_2\right). \tag{7}$$

Let us compare this result with the results obtained for the porous medium equation. It is known (cf. Bénilan, Crandall and Pierre [3]) that if $u_0 \in L^1_{\text{loc}}(\mathbb{R}^n)$, $u_0 \geq 0$, one has

$$\left(\limsup_{R\to+\infty} R^{-(2/m-1)-n} \int_{B_R} u_0\,dx < \infty\right) \Rightarrow ((3) \text{ has a solution on some interval } [0, T]). \tag{8}$$

The converse of (8) is proved in [4], (cf. also Dahlberg and Kenig [5]).

It is straightforward to see that our result proves that for $u_0 \in L^1_{\text{loc}}(\mathbb{R}^n)$, $u_0 \geq 0$, the porous medium equation (3) has a local solution *if and only if* the "implicit Euler scheme" (5) (with $t = 0$, $\alpha = m^{-1}$) has a positive solution $u(h)$ for $h > 0$ small enough.

It may be asked if it is possible to have $c_1 = c_2$ in (6) and (7). In fact, we can prove, for $n = 1$, that no inequality of the kind $\limsup_{R\to\infty} g(R)^{-1}\int_{B_R} h(x)\,u(x) \leq C$ may be expected to be a necessary and sufficient condition in order to have positive solutions of (1) for $f \in L^1_{\text{loc}}$, $f \geq 0$.

Many of the results stated above can be generalized: we give in Section 4 a proof that Brezis' result ([1]) holds with any odd function $g(u)$, instead of $|u|^{\alpha-1}u$, which verifies: g is convex on $\mathbb{R}^+$, increasing, and

$$\int^\infty \left(\int_0^x g(t)\,dt\right)^{-1/2} dx < +\infty.$$

(This kind of condition already arises in Osserman [6], who studies the inequation $\Delta u \geq f(u)$). For the case $\alpha < 1$, our result can probably be extended to functions $\varphi(u)$, instead of $|u|^{\alpha-1}u$, with more general conditions.

It may be asked whether equation (1) with $0 < \alpha < 1$ has a solution for *any* $f \in L^1_{\text{loc}}(\mathbb{R}^n)$. In fact we can prove that it has always infinitely many solutions (of course without condition of positivity). The proof involves different tools and will be done elsewhere [13].

In Section 5 we return to the case $f \geq 0$, $u \geq 0$, and deal with a characterization in some weighted Banach space of the minimal positive solution of (1). We obtain that if this solution verifies $u(x)\,|x|^{-2/1-\alpha} \to 0$ as $|x| \to +\infty$ (that is the case if $f(x)\,|x|^{-2\alpha/1-\alpha} \to 0$), then every other positive solution of (1) verifies

$$\limsup_{r\to+\infty} r^{-n-(2\alpha/1-\alpha)} \int_{B_r} u > 0.$$

We prove also by an example that this characterization is not available if $f(x)\,|x|^{-2\alpha/1-\alpha} \not\to 0$.

Notation. In what follows, α denotes always a real number such that $0 < \alpha < 1$. If Ω is an open subset of $\mathbb{R}^n$, we often write: $\int_\Omega u\,dx$ for $\int_\Omega u(x)\,dx$, where dx is the Lebesgue measure in $\mathbb{R}^n$. By $\{|x| \leq R\}$ we mean: $\{x \in \mathbb{R}^n, |x| \leq R\}$. We usually set $B_R = \{|x| < R\}$, $S_t = \{|x| = t\}$, and $\int_{S_t} u\,d\sigma = \int_{S_t} u(\sigma)\,d\sigma$, where $d\sigma$ is the usual superficial measure on S_t. If Ω is an open set, $W^{k,p}(\Omega)$ ($k \in \mathbb{N}, p \geq 1$) is the Sobolev space of all measurable functions in Ω whose derivatives in the distribution sense $D^l u$ verify $D^l u \in L^p(\Omega)$ for $|l| \leq k$. $\mathcal{D}(\Omega)$ denotes the set of all C^∞-functions with compact support in Ω. $W_0^{k,p}(\Omega)$ is the closure of $\mathcal{D}(\Omega)$ in $W^{k,p}(\Omega)$, and $W^{k,p}_{\text{loc}}(\Omega)$ is the set of all functions u such that $\varphi u \in W^{k,p}(\Omega)$ for any φ in $\mathcal{D}(\Omega)$. For the Sobolev inclusions, we refer to Adams [8], and for the regularity properties of the Laplacian, to Agmon, Douglis and Nirenberg [9]. Finally for f and $u \in L^1_{\text{loc}}$, "$-\Delta u + u^\alpha = f$ in $\mathcal{D}'(\Omega)$" means that for any $\varphi \in \mathcal{D}(\Omega)$ one has: $-\int_\Omega u\Delta\varphi\,dx + \int_\Omega u^\alpha\varphi\,dx = \int_\Omega f\varphi\,dx$. This makes sense, since $u \in L^1_{\text{loc}} \Rightarrow u^\alpha \in L^1_{\text{loc}}$ for $\alpha < 1$. The support of a function f will be denoted by $\operatorname{supp} f$.

2. PRELIMINARY RESULTS

Let $0 < \alpha < 1$, $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ with $f \geq 0$. Consider the problem:

(P) Find $u \in L^1_{\text{loc}}(\mathbb{R}^n)$, $u \geq 0$, such that $-\Delta u + u^\alpha = f$ in $\mathcal{D}'(\mathbb{R}^n)$.

We shall say that v is a supersolution of (P) if $v \in L^1_{\text{loc}}(\mathbb{R}^n)$, $v \geq 0$, and for some $g \in L^1_{\text{loc}}(\mathbb{R}^n)$

$$-\Delta v + v^\alpha = g \geq f \quad \text{in} \quad \mathcal{D}'(\mathbb{R}^n).$$

We now prove the following result:

THEOREM 1. Let $0 < \alpha < 1$, $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ with $f \geq 0$. If (P) has a supersolution v, it has also a "minimal" solution $u \leq v$. (That is, a solution u such that $0 \leq u \leq w$ for any other solution w of (P).)

In fact, theorem 1 is a consequence of the maximum principle. But the lack of regularity generates some difficulties, which we solve by using the following lemma (proved in the Appendix):

LEMMA 1. Let $u \in W^{1,1}(B_R)$, $-\Delta u \in L^1(B_R)$, and $u \leq 0$ on ∂B_R. If $P \in C^1 \cap L^\infty(\mathbb{R})$, with $P = 0$ on $\mathbb{R}^-$ and $P' \geq 0$, then

$$-\int \Delta u P(u)\,dx \geq 0.$$

Proof of theorem 1. Let $v \in L^1_{\text{loc}}(\mathbb{R}^n)$ be such that $v \geq 0$ and $-\Delta v + v^\alpha = g \geq f$, with $g \in L^1_{\text{loc}}(\mathbb{R}^n)$. We shall prove the existence of a solution u of (P) with $u \leq v$. For this purpose, define the sequence $(u_k)_k$ of solutions of:

$$(\mathrm{P}_k) \qquad \begin{cases} -\Delta u_k + |u_k|^{\alpha-1} u_k = f & \text{in } \mathscr{D}'(B_k), \\ u_k \in W_0^{1,1}(B_k). \end{cases}$$

The existence, uniqueness of u_k and the fact that $u_k \geq 0$ (since $f \geq 0$) are proved in Brezis and Strauss [10]. (Another proof can be found in [11]). Note that $u_k - v$ verifies the assumptions of lemma 1 (with $B = B_R = B_k$). Let P be as in lemma 1. Multiplying by $P(u_k - v)$ the inequality

$$-\Delta(u_k - v) + |u_k|^{\alpha-1} u_k - |v|^{\alpha-1} v \leq 0$$

and then integrating yields:

$$\int_B -\Delta(u_k - v)\, P(u_k - v)\, \mathrm{d}x + \int_B (|u_k|^\alpha - |v|^\alpha)\, P(u_k - v)\, \mathrm{d}x \leq 0.$$

Thus by lemma 1:

$$\int_B (|u_k|^\alpha - |v|^\alpha)\, P(u_k - v)\, \mathrm{d}x \leq 0,$$

and, choosing $P > 0$ on $\mathbb{R}_+^*$, this gives $P(u_k - v) \leq 0$ a.e., and then $u_k \leq v$ a.e. on B. By the same argument, one obtains $u_k \leq u_{k+1}$ on B_k.

We conclude that $(u_k)_k$ (and, therefore, $(u_k^\alpha)_k$) converges in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$. Passing to the limit in the distribution sense in (P_k) yields that u is solution of (P). Moreover $0 \leq u \leq v$, and we obtain $u \leq w$ for any solution w of (P), since w is also a supersolution.

3. THE MAIN THEOREM

THEOREM 2. Let $0 < \alpha < 1$. There exists $c_1 > 0$ and $c_2 > 0$ (depending only on n and α) such that for $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, $f \geq 0$, one has:

(1) if $\limsup_{R\to\infty} R^{-n-(2\alpha/1-\alpha)} \int_{B_R} f\, \mathrm{d}x < c_1$, then (P) has a solution (and therefore a minimal solution in the sense of theorem 1);

(2) if (P) has a solution then $\limsup_{R\to+\infty} R^{-n-(2\alpha/1-\alpha} \int_{B_R} f\, \mathrm{d}x < c_2$.

3.1. *Proof of part* (1) *of theorem* 2

We first state the following lemma ("compact support lemma") which is an essential tool in our proof. Roughly speaking this lemma is contained in [12]. For a sake of completeness we give a proof of this lemma in the appendix (cf. lemma A.3).

We set for $a, b \in \mathbb{R}$, $\Omega(a, b) = \{a \leq |x| < b\}$.

COMPACT SUPPORT LEMMA (Appendix—lemma A.3). Let $0 < \alpha < 1$. For all $0 \leq a < b$ and for all $\varepsilon > 0$ there exists $c(\varepsilon, a, b) > 0$ such that if

$$f \in L^1_{\mathrm{loc}}(\mathbb{R}^n), f \geq 0, \operatorname{Supp} f \subset \Omega(a, b), \int f\, \mathrm{d}x \leq c(\varepsilon, a, b), \tag{9}$$

then the problem

$$(\mathrm{P}') \qquad \begin{cases} -\Delta u + u^\alpha = f & \text{in } \mathscr{D}'(\mathbb{R}^n) \\ u \in L^1(\mathbb{R}^n), u \geq 0, \operatorname{Supp} u \subset \Omega(a - \varepsilon, b + \varepsilon) \end{cases}$$

has a (unique) solution.

Remark 1. Let $R > 0$. The function u is a solution of (P′) if and only if $\tilde{u}(\cdot) = R^{2/1-\alpha} u(\cdot/R)$ is a solution of (P′) with $\tilde{f}(\cdot) = R^{2\alpha/1-\alpha} f(\cdot/R)$, $\tilde{a} = aR$, $\tilde{b} = bR$, $\tilde{\varepsilon} = \varepsilon R$, instead of f, a, b, ε.

(The proof of this remark is straightforward.)

Remark 2. From the compact support lemma and remark 1, one can easily deduce that for any $f \in L^1(\mathbb{R}^n)$ with compact support, there exists a (unique) $u \in L^1(\mathbb{R}^n)$ with compact support and such that $-\Delta u + u^\alpha = f$ in $\mathscr{D}'(\mathbb{R}^n)$ (cf. theorem 6.1 of [12]).

Using this compact support lemma we are going to prove the existence of a supersolution for (P), which, by theorem 1, is enough to ensure the existence of a solution for (P).

To this end, we set, for $k \in \mathbb{N}$,

$$\Omega_k = \Omega(2^k, 2^{k+1}), \quad \Omega_k' = \Omega(0, 2^k).$$

Let $g \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, $g \geq 0$, and such that for some $K > 0$, $\limsup_{R\to+\infty} R^{-n-(2\alpha/1-\alpha)} \int_{B_R} g\, \mathrm{d}x < K$. Then one has for some $K' \geq 0$,

$$\int_{B_R} g\, \mathrm{d}x \leq K R^{n+(2\alpha/1-\alpha)} + K'. \tag{10}$$

Thus, for $k_0 > 1$ large enough, we get

$$\int_{\Omega_{k_0}'} g\, \mathrm{d}x \leq 2K\, (2^{k_0})^{n+(2\alpha/1-\alpha)}, \tag{11}$$

$$\int_{\Omega_k} g\, \mathrm{d}x \leq 2^{n+(2\alpha/1-\alpha)} K \cdot (2^k)^{n+(2\alpha/1-\alpha)}, \quad \text{for} \quad k \geq k_0. \tag{12}$$

If we assume that $K \leq c_1' = 2^{-n-(2\alpha/1-\alpha)} \operatorname{Inf}(c(1/2, 1, 2), c(1/2, 0, 1))$, we deduce from (11) and (12) (by using the compact support lemma and remark 1) the existence of $\{v_k, k \geq k_0 - 1\}$ such that

$$\left.\begin{array}{l} v_k \geq 0, \quad v_k \in L^1(\mathbb{R}^n), \\ -\Delta v_k + v_k^\alpha = \begin{cases} g 1_{\Omega_k} & \text{if} \quad k \geq k_0 \\ g 1_{\Omega_{k_0}'} & \text{if} \quad k = k_0 - 1 \end{cases} \\ \operatorname{Supp} v_k \subset \Omega_{k-1} \cup \Omega_k \cup \Omega_{k+1}, \quad k \geq k_0 \\ \operatorname{Supp} v_{k_0-1} \subset \Omega_{k_0}' \cup \Omega_{k_0}. \end{array}\right\} \tag{13}$$

Thus one sees that the series

$$\sum_{i=k_0-1}^{\infty} v_i \quad \text{and} \quad \sum_{i=k_0-1}^{\infty} v_i^\alpha$$

converge a.e. and in $L^1_{\text{loc}}(\mathbb{R}^n)$. Furthermore, as $(a^\alpha + b^\alpha + c^\alpha) \leq 3(a+b+c)^\alpha$ for all $a, b, c \geq 0$, one has

$$\left(\sum_{i=k_0-1}^{\infty} v_i^\alpha\right) \leq 3\left(\sum_{i=k_0-1}^{\infty} v_i\right)^\alpha. \tag{14}$$

Setting

$$w = \sum_{i=k_0-1}^{\infty} v_i,$$

we then deduce (with (13) and (14)):

$$\left.\begin{array}{l} -\Delta w + 3w^\alpha \geq g \quad \text{in} \quad \mathcal{D}'(\mathbb{R}^n) \\ w \geq 0 \quad w \in L^1_{\text{loc}}(\mathbb{R}^n). \end{array}\right\} \tag{15}$$

With $z = 3^{1/\alpha-1} w$, this gives

$$\left.\begin{array}{l} -\Delta z + z^\alpha \geq 3^{1/\alpha-1}, g \\ z \in L^1_{\text{loc}}(\mathbb{R}^n), \quad z \geq 0. \end{array}\right\} \tag{16}$$

Setting now $g = 3^{1/1-\alpha} f$, we deduce from (16) that if

$$\limsup_{R\to+\infty} R^{-n-(2\alpha/1-\alpha)} \int_{B_R} f\,dx < c_1 = 3^{1/\alpha-1} c_1',$$

then z is a supersolution of (P). This proves part (1) of theorem 2.

3.2. *Proof of part* (2) *of theorem* 2

Let u be a solution of (P). We first give a formal proof, that is assuming that u is regular enough. We then shall justify the calculations.

Integrating on $B_t = B(0, t)$ the relation $-\Delta u + u^\alpha = f$, and then integrating by parts, we get:

$$-\int_{S_t} \frac{du}{dn} + \int_{B_t} u^\alpha = \int_{B_t} f. \tag{17}$$

We set

$$w(t) = \frac{1}{t^{n-1}} \int_{S_t} u\,d\sigma,$$

and obtain:

$$-t^{n-1} w'(t) + \int_{B_t} u^\alpha = \int_{B_t} f \geq 0. \tag{18}$$

Integrating between 1 and $r \geq 1$ and then integrating the first term by parts yields

$$\int_1^r dt \int_{B_t} f \leq \int_1^r dt \int_{B_t} u^\alpha + (n-1)\int_1^r t^{n-2}\, w(t) + w(1) - r^{n-1}\, w(r). \tag{19}$$

Thus, we have to estimate $w(r)$. Using again (18) and integrating between 1 and $r \geq 1$ yields:

$$w(r) \leq \int_1^r \frac{dt}{t^{n-1}} \int_{B_t} u^\alpha + w(1). \tag{20}$$

Thus, by the Hölder inequality:

$$w(r) \leq |S_1|^{1-\alpha} \int_1^r \frac{dt}{t^{n-1}} \int_0^t \tau^{n-1}\, w(\tau)^\alpha\, d\tau + w(1),$$

where $|S_1| = \int_{S_1} d\sigma$. Then, for $r \geq 1$:

$$w(r) \leq |S_1|^{1-\alpha}\left[r\int_1^r w(\tau)^\alpha\, d\tau + r\int_0^1 \tau^{n-1}\, w^\alpha(\tau)\, d\tau\right] + w(1),$$

that is, again by the Hölder inequality:

$$w(r) \leq \tilde{C}_1 r^{2-\alpha}\left(\int_1^r w(\tau)\, d\tau\right)^\alpha + \tilde{C}_2 r, \tag{21}$$

where $\tilde{C}_1$ only depends on n and α, and $\tilde{C}_2$ is independent of r.

Set $z(r) = a(b+r)^{(2/1-\alpha)+1}$, where

$$a\left(1 + \frac{2}{1-\alpha}\right) = \tilde{C}_1 a^\alpha \left(2 + \frac{2}{1-\alpha}\right)^{-\alpha}.$$

One sees easily that for b large enough one has:

$$\left.\begin{array}{ll} z'(r) \geq \tilde{C}_1 r^{2-\alpha}(z(r))^\alpha + \tilde{C}_2 r, & r \geq 1 \\ z'(1) \geq w(1). & \end{array}\right\}$$

We conclude that $z(r) \geq w(r)$, $\forall r \geq 1$. Returning to (19) we make the same calculations as for obtaining (21) and get:

$$\int_1^r dt \int_{B_t} f \leq \tilde{C}_1 r^{2-\alpha}\left(\int_1^r w(\tau)\, d\tau\right)^\alpha + (n-1)\int_1^r t^{n-2} w(t) + \tilde{C}_3,$$

and using $w(r) \leq z(r)$ we deduce that

$$\int_1^r dt \int_{B_t} f \leq \tilde{C}_4 (b+r)^{n+1+(2\alpha/1-\alpha)} + \tilde{C}_5, \tag{22}$$

where $\tilde{C}_4$ only depends on α and n and $\tilde{C}_5$ is independent of r. But the function $t \to \int_{B_t} f$ being nondecreasing, one has:

$$\int_1^r dt \int_{B_t} f \geq \frac{r}{2} \int_{B_{r/2}} f, \qquad \text{for } r \geq 2. \tag{23}$$

We conclude from (22) and (23) that:

$$\limsup_{r\to\infty} r^{-n-(2\alpha/1-\alpha)} \int_{B_R} f \leqslant c_2,$$

with c_2 depending only on α and n.

To justify the calculations above, it is enough to have $u \in C^1 \cap W^{2,2}(\mathbb{R}^n)$. This is not the case, but let φ_k be a sequence of mollifiers. One has

$$-\Delta(u \star \varphi_k) + u^\alpha \star \varphi_k = f \star \varphi_k,$$

and $u \star \varphi_k \in C^\infty(\mathbb{R}^n)$,

$$\left.\begin{aligned} &u^\alpha \star \varphi_k \to u^\alpha && \text{in } L^1_{\mathrm{loc}} \\ &f \star \varphi_k \to f && \text{in } L^1_{\mathrm{loc}} \\ &u \star \varphi_k \to u && \text{in } W^{1,1}_{\mathrm{loc}}, \text{ and} \\ \text{then: } &u \star \varphi_k \to u && \text{in } L^1(S_t) \text{ for any sphere } S_t. \end{aligned}\right\} \tag{24}$$

The relations (17), (18) have now a sense with $u \star \varphi$ instead of u; $u^\alpha \star \varphi$ instead of u^α, $f \star \varphi$ instead of f. Then (19) and (20) are justified by letting $k \to +\infty$ and using (24).

4. A GENERALIZATION OF BREZIS' RESULT [1]

THEOREM 3. Let $g: \mathbb{R} \to \mathbb{R}$ be continuous odd function, convex on $\mathbb{R}^+$ and verifying

$$g(s)s \geqslant 0, \qquad \forall s \in \mathbb{R}. \tag{25}$$

$$\int^\infty \frac{\mathrm{d}t}{G(t)^{1/2}} < \infty, \quad \text{where } G(s) = \int_0^s g(t)\,\mathrm{d}t. \tag{26}$$

Then for any $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ $(n \geqslant 1)$, there exists a unique function u solution of

$$\left.\begin{aligned} &-\Delta u + g(u) = f \quad \text{in } \mathcal{D}'(\mathbb{R}^n) \\ &u \in L^1_{\mathrm{loc}}(\mathbb{R}^n), \quad g(u) \in L^1_{\mathrm{loc}}(\mathbb{R}^n). \end{aligned}\right\} \tag{27}$$

Moreover, if $f \geqslant 0$ a.e., then $u \geqslant 0$ a.e.

As in Brezis [1], the preceding result can be generalized in the following way.

THEOREM 4. Let $h: \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function such that

$$h(x, s)\,\mathrm{sgn}\,s \geqslant |g(s)|, \quad \forall s \in \mathbb{R}, \quad \text{a.e. in } x \in \mathbb{R}^n, \tag{28}$$

where g verifies the assumptions of theorem 3, and:

$$\sup_{|s| \leqslant t} |h(x, s)| \in L^1_{\mathrm{loc}}(\mathbb{R}^n) \quad \text{for any } t \text{ in } \mathbb{R}. \tag{29}$$

Then for every $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ there exists at least a solution of

$$\left.\begin{aligned} &-\Delta u + h(\cdot, u) = f \quad \text{in } \mathcal{D}'(\mathbb{R}^n), \\ &u \in L^1_{\mathrm{loc}}(\mathbb{R}^n), h(\cdot, u) \in L^1_{\mathrm{loc}}(\mathbb{R}^n). \end{aligned}\right\} \tag{30}$$

(We omit here the proof of theorem 4, which essentially is the same as in [1].)

Remark 3. The results of theorems 3 and 4 are optimal in the following sense: Let $g: \mathbb{R}^+ \to \mathbb{R}^+$ be increasing, with $g(0) = 0$, and assume that g does not verify (26), that is:

$$\int^\infty \left(\int_0^s g(t)\,\mathrm{d}t\right)^{-1/2} \mathrm{d}s = \infty. \tag{31}$$

Then one sees easily that for any $u_0 > 0$, the O.D.E.

$$\left.\begin{aligned} &-u'' + g(u) = 0 \\ &u(0) = u_0, u'(0) = 0 \end{aligned}\right\}$$

has a positive solution in $C^2(\mathbb{R}, \mathbb{R})$. Thus we have infinitely many solutions, and *no local estimate* on them.

4.1. *A local estimate for proving theorem* 3

LEMMA 2. Let g verify the assumptions of theorem 3 and $0 < R < R'$. Let $f \in L^1(B_{R'})$ and $u \in L^1_{\mathrm{loc}}(B_{R'})$ such that

$$\left.\begin{aligned} &-\Delta u + g(u) = f \quad \text{in } \mathcal{D}'(B_{R'}) \\ &g(u) \in L^1_{\mathrm{loc}}(B_{R'}). \end{aligned}\right\} \tag{32}$$

Then:

$$\int_{B_R} |g(u)| \leqslant C\left(1 + \int_{B_{R'}} |f|\right) \tag{33}$$

where C only depends on g, R and R'.

In the case $g(u) = |u|^{\alpha-1}u$, $\alpha > 1$, the proof in [1] is based on a tool of Baras and Pierre [2] using Hölder's inequality. The following lemma will provide a generalization of this tool to a class of convex functions.

LEMMA 3. Let $g: \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous odd function, convex on $\mathbb{R}^+$ and verifying $g(0) = 0$ and $\int^\infty (\int_0^t g(s)\,\mathrm{d}s)^{-1/2}\,\mathrm{d}t < \infty$. Then for any $C > 0$ there exists a convex function $\varphi \in C^2([0, 1], \mathbb{R}^+)$ verifying $\varphi(0) = \varphi'(0) = 0$ and

$$t(\varphi''(x) + \varphi'(x)) \leqslant \frac{1}{2C} g(t)\,\varphi(x) + 2 \tag{34}$$

for any $x \in [0, 1]$ and $t \in \mathbb{R}^+$. (φ depends only on g and C.)

Proof of lemma 3. Set:

$$k(y) = \left[g^{-1}\left(\frac{4C}{y}\right)\right]^{-1} \quad \text{for } y > 0 \quad \text{and } k(0) = 0. \tag{35}$$

We seek for a solution of the Cauchy problem:

$$\left.\begin{aligned} &\varphi''(x) = k(\varphi(x)), \qquad \varphi(0) = \varphi'(0) = 0,\\ &\varphi(x) > 0 \quad \text{for } x \in]0, \alpha[, \quad \alpha > 0. \end{aligned}\right\} \tag{36}$$

Such a nontrivial solution exists since (26) implies

$$\int_0 \left(\int_0^t k(s)\,ds\right)^{-1/2} dt < \infty. \tag{37}$$

Indeed, (37) allows to define a solution (φ, α) of (36) by:

$$\left.\begin{aligned} &\int_0^{\varphi(x)} \left(\int_0^t k(s)\,ds\right)^{-1/2} dt = \sqrt{2}x \quad \text{for } x \in [0, \alpha], \quad \text{with}\\ &\alpha = \min\left(\frac{1}{\sqrt{2}}\int_0^1 \left(\int_0^t k(s)\,ds\right)^{-1/2} dt, \qquad 1\right). \end{aligned}\right\}$$

(For a proof of (26) ⇒ (37), see lemma 4 below.)

By (35) one has

$$\frac{k(y)}{y} = \frac{1}{4C}\frac{g(1/k(y))}{1/k(y)},$$

and, since $t \to g(t)/t$ is nondecreasing we get:

$$\frac{k(y)}{y} \leqslant \frac{1}{4C}\frac{g(t)}{t} \quad \text{for any } t, y \in \mathbb{R}^+ \quad \text{such that } tk(y) \geqslant 1.$$

Thus,

$$tk(y) \leqslant \frac{1}{4C}g(t)y + 1 \quad \text{for any } t, y \geqslant 0. \tag{38}$$

Setting $y = \varphi(x)$ and using (36) yields

$$t\varphi''(x) \leqslant \frac{1}{4C}g(t)\varphi(x) + 1 \quad \text{for } x \in [0, \alpha].$$

Extending φ to [0, 1] by:

$$\varphi(x) = \varphi(\alpha) + (x - \alpha)\varphi'(\alpha) + \frac{(x-\alpha)^2}{2}\varphi''(\alpha),$$

we get:

$$t\varphi''(x) \leqslant \frac{1}{4C}g(t)\varphi(x) + 1 \quad \text{for } x \in [0, 1]. \tag{39}$$

Note that φ'' is nondecreasing on [0, 1]. Thus

$$\varphi'(x) = \int_0^x \varphi''(t)\,dt \leqslant \varphi''(x) \quad \text{for } x \in [0, 1],$$

and, therefore, (34) is a consequence of (39).

Proof of lemma 2. Multiplying (32) by sgn u and using Kato's inequality we get:

$$-\Delta|u| + |g(u)| \leqslant |f| \quad \text{in } \mathcal{D}'(B_{R'}).$$

For any $\psi \in \mathcal{D}(B_{R'})$ with $\psi \geqslant 0$, we then have:

$$-\int_{B_{R'}} |u|\Delta\psi + \int_{B_{R'}} |g(u)|\psi \leqslant \int_{B_{R'}} |f|\psi. \tag{40}$$

By a density argument, (40) remains true as $\psi \in C_c^2(B_{R'})$ and $\psi \geqslant 0$.

Applying now (40) with $\varphi \circ \psi$ instead of ψ, where φ is given by lemma 2 and $\psi \equiv 1$ on B_R, $0 \leqslant \psi \leqslant 1$ in $B_{R'}$. We obtain:

$$-\int_{B_{R'}} |u|(\varphi''(\psi)|\nabla\psi|^2 + \varphi'(\psi)\Delta\psi) + \int_{B_{R'}} |g(u)|\varphi(\psi) \leqslant \int |f|\varphi(\psi).$$

Thus,

$$\int_{B_{R'}} |g(u)|\varphi(\psi) \leqslant C\int_{B_{R'}} |u|(\varphi''(\psi) + \varphi'(\psi)) + \int_{B_{R'}} |f|\varphi(\psi),$$

where C only depends on ψ. Using (34):

$$\int_{B_{R'}} |g(u)|\varphi(\psi) \leqslant \tfrac{1}{2}\int_{B_{R'}} |g(u)|\varphi(\psi) + C_1 + C_2\int_{B_{R'}} |f|,$$

where C_1 and C_2 only depend on ψ, R, R', g.

The use of (28) gives finally (33). (Note that ψ may be chosen in order to depend only on R and R'.)

It remains to prove the relation (26) ⇒ (37):

LEMMA 4. Let $g : \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous convex function and $k : \mathbb{R}^+ \to \mathbb{R}^+$ defined by $k(s) = [g^{-1}(c/s)]^{-1}$, where c is some positive constant. Then

$$\int^\infty \left(\int_0^t g(s)\,ds\right)^{-1/2} dt < \infty \quad \text{implies} \int_0 \left(\int_0^t k(s)\,ds\right)^{-1/2} < \infty.$$

Proof of lemma 4. g and k being nondecreasing, one has

$$\frac{t}{2}g\left(\frac{t}{2}\right) \leqslant \int_0^t g(s)\,ds \leqslant tg(t)$$

and the same inequalities for k. Thus

$$\int^{\infty}\left(\int_0^t g(s)\,ds\right)^{-1/2} dt < \infty \Leftrightarrow \int^{\infty}\frac{dt}{t^{1/2}g(t)^{1/2}} < \infty, \tag{41}$$

and

$$\int_0\left(\int_0^t k(s)\,ds\right)^{-1/2} dt < \infty \Leftrightarrow \int_0 \frac{dt}{t^{1/2}k(t)^{1/2}} < \infty \Leftrightarrow \int_0 \frac{dt}{t^{1/2}k(ct)^{1/2}} < \infty$$

$$\Leftrightarrow \int_0 \frac{[g^{-1}(1/t)]^{1/2}}{t^{1/2}} < \infty \Leftrightarrow \int^{\infty}\frac{u^{1/2}g'(u)}{g(u)^{3/2}} < \infty \quad \left(\text{where } t = \frac{1}{g(u)}\right)$$

$$\Leftrightarrow [-2u^{1/2}g(u)^{-1/2}]^{\infty} + \int^{\infty}\frac{du}{u^{1/2}g(u)^{1/2}} < \infty$$

(integrating by parts).

Since g is convex, $(g(t))/t$ is increasing, and then $\int^{\infty} t^{-1/2}g(t)^{-1/2} < \infty$ implies $(g(t))/t \to +\infty$ as $t \to +\infty$.

Thus,

$$[-2u^{1/2}g(u)^{-1/2}]^{\infty} + \int^{\infty}\frac{du}{u^{1/2}g(u)^{1/2}} < \infty \Leftrightarrow \int^{\infty}\frac{du}{u^{1/2}g(u)^{1/2}} < \infty.$$

This gives lemma 4.

4.2. *Existence of solutions for theorem* 3

The proof is now the same as in Brezis [1]. Instead of the Baras–Pierre estimate, we use lemma 1. So we just recall the main steps of the proof.

(1) For any $n \in \mathbb{N}^*$ there exists $u_n \in W_0^{1,1}(B_n)$ solution of

$$\left.\begin{aligned} -\Delta u_n + g(u_n) &= f \quad \text{in } \mathcal{D}'(B_n) \\ g(u_n) &\in L^1(B_n). \end{aligned}\right\}$$

(This is proved in [11].)

(2) By lemma 2, $(g(u_n))_n$ is bounded in $L^1_{\text{loc}}(\mathbb{R}^N)$ and then Δu_n and u_n are bounded in $L^1_{\text{loc}}(\mathbb{R}^N)$ (indeed, $|g(t)| \geq \varepsilon|t|$ for some $\varepsilon > 0$ and t large enough). Using the same method as in [1], we conclude that $u_n \to u$ and $g(u_n) \to g(u)$ for some u in $L^1_{\text{loc}}(\mathbb{R}^n)$. Thus, u is solution of (30). This gives the part "existence" of theorem 3.

4.3. *End of the proof of theorem* 3

(1) LEMMA 5. Let g be as in theorem 3. There exists a family of functions $(u_R)_{R \geq R_0}$ such that:

(i) $u_R \in C^2(B_R)$, $u_R \geq 0$, $u_R(x) \to +\infty$ as $|x| \to R$;
(ii) $u_R \to 0$ in $L^\infty_{\text{loc}}(\mathbb{R}^n)$ as $R \to +\infty$;
(iii) $-\Delta u_R + g(u_R) \geq 0$.

Proof. Set $H_{u_0}(u) = \int_{u_0}^{u} (G(t) - G(u_0))^{-1/2}\,dt$, where $G(t) = \int_0^t g(s)\,ds$, and

$$u_{R(u_0)}(x) = H_{u_0}^{-1}(\sqrt{2}|x|),$$

with

$$R(u_0) = \int_{u_0}^{\infty} (G(t) - G(u_0))^{-1/2} \quad \text{and } u_0 > 0. \tag{42}$$

One sees easily that $-\Delta u_{R(u_0)} + g(u_{R(u_0)}) \geq 0$. Since g is convex, $\int_0^\infty G(t)^{-1/2}\,dt = +\infty$, and therefore $R(u_0) \to \infty$ as $u_0 \to 0$. Thus, we may define u_R for any $R \geq R_0$ with R_0 large enough. The property (i) is clear, and to prove (ii), note that

$$\sqrt{2}|x| = \int_{u_0}^{u_R(x)} (G(t) - G(u_0))^{-1/2}\,dt \quad \text{for } R = R(u_0). \tag{43}$$

As $R \to +\infty$, one has by (42) $u_0 \to 0$. As $u_0 \to 0$, (43) shows that $u_R(x) \to 0$ uniformly in x as x remains in a bounded set.

(2) *If u is a solution of* (27), *then $f \leq 0$ implies $u \leq 0$.*

Proof. Here again we follow the method of [1], using the comparison functions constructed in lemma 5.

One has by (iii):

$$-\Delta(u - u_R) + g(u) - g(u_R) \leq 0 \quad \text{in } B_R.$$

Thus by Kato's inequality

$$-\Delta(u - u_R)^+ \leq 0. \tag{44}$$

Again by Kato's inequality, we get $-\Delta u^+ \leq 0$ from $-\Delta u^+ + g(u)^+ \leq 0$. That implies $u^+ \in L^\infty_{\text{loc}}(\mathbb{R}^n)$. Then (i) yields $(u - u_R)^+ = 0$ near ∂B_R, and this jointed to (44) gives $u \leq u_R$ a.e. By (ii) we obtain $u \leq 0$ a.e.

(3) *The solution u of* (27) *is unique.* Let u and v be two solutions of (27). Then $-\Delta(u - v) + g(u) - g(v) = 0$. Thus by Kato's inequality:

$$-\Delta|u - v| + |g(u) - g(v)| \leq 0.$$

Note that if u and v have the same sign,

$$|g(u) - g(v)| = \left|\int_u^v |g'(t)|\,dt\right| \geq \int_0^{|v-u|} |g'(t)|\,dt = g(|v - u|),$$

and if $uv \leq 0$ and, for instance, $|v| \geq |u|$:

$$|g(u) - g(v)| = |g(u)| + |g(v)| \geq |g(v)| \geq \left|g\left(\frac{|v| + |u|}{2}\right)\right| = \left|g\left(\frac{v - u}{2}\right)\right|.$$

We then get: $|g(u) - g(v)| \geq |g((u - v)/2)|$, $\forall u, v \in \mathbb{R}$. Thus,

$$-\Delta|u - v| + g\left(\left|\frac{u - v}{2}\right|\right) \leq 0.$$

By the preceding result (2) applied with the function $t \to g(t/2)$, we get $|u - v| \leq 0$, that is $u = v$ a.e.

5. REMARKS ON THE BEHAVIOUR OF POSITIVE SOLUTIONS OF (1) AT INFINITY

We saw in theorem 1 that if (1) has a positive solution for $f \geq 0$, it has also a minimal positive solution. An interesting question is as to whether this minimal solution can be distinguished from the others by its behaviour at infinity. (That is the case if $\alpha = 1$.) We first give a positive answer to this question as the behaviour of f at infinity is subcritical (i.e. $f(x)|x|^{-2\alpha/(1-\alpha)} \to 0$).

THEOREM 5. Let u be the minimal positive solution of (1) for some $f \geq 0$, and assume that

$$|x|^{-2/(1-\alpha)}|u(x)| \to 0 \quad \text{as} \quad |x| \to +\infty.$$

Then for every other positive solution v of (1) one has:

$$\limsup_{r\to+\infty} r^{-n-(2/1-\alpha))} \int_{B_r} v > 0.$$

Remarks. (1) If $|x|^{-(2\alpha/(1-\alpha))}f(x) \to 0$ at infinity, one sees easily that the supersolution constructed in the existence proof of theorem 2, and, therefore, the minimal solution u, verify $|x|^{-2/(1-\alpha)}u(x) \to 0$ as $|x| \to +\infty$. Thus, as $|x|^{-(2\alpha/(1-\alpha))}f(x) \to 0$, we get a simple characterization of the minimal positive solution of (1).

(2) If instead of $|x|^{-(2/(1-\alpha))}u(x) \to 0$ one only has $r^{-n-(2/(1-\alpha))} \int_{B_r} u \to 0$, the result of theorem 4 seems to fail.

LEMMA 6. Let $w \in L^1_{\text{loc}}(\mathbb{R}^n)$ verifying $|\Delta w| \leq w^\alpha$. Then there exists a constant C depending only on α such that:

$$\forall a \geq 0, \left(\limsup_{r\to+\infty} r^{-(2/(1-\alpha))-n} \int_{B_r} w \leq a \Rightarrow \limsup_{|x|\to+\infty} |x|^{-(2/(1-\alpha))} w(x) \leq Ca\right).$$

Proof. Assume that $\limsup_{r\to\infty} r^{-(2/(1-\alpha))-n} \int_{B_r} u < a$. Then for $r \geq r_0$ large enough one has

$$r^{-(2/(1-\alpha))-n} \int_{B_r} u \leq a. \tag{45}$$

Set now $u_r(x) = r^{-(2/(1-\alpha))} u(rx)$. Then one sees immediately that u_r verifies:

$$|\Delta u_r| \leq u_r^\alpha \quad \text{and} \quad \int_{B_1} u_r \leq a. \tag{46}$$

Using classical interior estimates and a straight bootstrap argument, the relations (46) imply that:

$$|u_r|_{L^\infty(B_{1/2})} \leq Ca, \quad \text{with } C \text{ depending only on } \alpha.$$

Thus, for $|x| \leq r/2$, $u(x) = r^{2/(1-\alpha)} u_r(x/r) \leq Ca\, r^{2/(1-\alpha)}$.

We conclude that $u(x) \leq C2^{2/(1-\alpha)}a|x|^{2/(1-\alpha)}$.

Proof of theorem 5. Let u be a minimal positive solution of (1) such that $u(x)|x|^{-(2/(1-\alpha))} \to 0$ as $|x| \to +\infty$, and $u + w$ another positive solution. Thus,

$$\Delta w = (u+w)^\alpha - u^\alpha \geq 0. \tag{47}$$

Assume by contradiction that $r^{-(2/(1-\alpha))-n} \int_{B_r} w \to 0$ as $r \to +\infty$.

Since (47) implies $|\Delta w| \leq w^\alpha$, we obtain by lemma 7 that $|x|^{-(2/(1-\alpha))}w(x) \to 0$ as $|x| \to +\infty$. If $a, b \geq 0$, one verifies easily that there exists a constant C depending only on α such that:

$$(a+b)^\alpha \geq C \inf(a^{\alpha-1}b, b^\alpha).$$

Applying this relation to u and w and taking into account that $u(x)|x|^{-(2/(1-\alpha))}$ and $w(x)|x|^{-(2/(1-\alpha))} \to 0$ as $|x| \to +\infty$, we obtain:

$$\forall \varepsilon > 0, r_0, \qquad |x| \geq r_0 \Rightarrow (u+w)^\alpha - u^\alpha \geq C\varepsilon^{-1}|x|^{-2}w. \tag{48}$$

In what follows, we make formal calculations, assuming w as smooth as necessary. We shall justify them later. Integrating (47) on B_r and using (48) yields:

$$\int_{\partial B_r} \frac{dw}{dn} \geq \int_{B_{r_0}} (u+w)^\alpha - u^\alpha + C\varepsilon^{-1} \int_{B_r \setminus B_{r_0}} |x|^{-2}\, w. \tag{49}$$

Setting $C_0 = \int_{B_{r_0}} (u+w)^\alpha - u^\alpha > 0$ if r_0 is large enough, we get

$$r^{n-1} \frac{d}{dr}\left(\frac{1}{r^{n-1}} \int_{S_r} w\right) \geq C_0 + C\varepsilon^{-1} \int_{B_r \setminus B_{r_0}} |x|^{-2} w. \tag{50}$$

Set $\bar{w}(r) = \frac{1}{r^{n-1}} \int_{S_r} w$. We obtain:

$$r^{n-1}\bar{w}'(r) \geq C_0(\varepsilon) + C\varepsilon^{-1} \int_{r_0}^r t^{n-3}\bar{w}(t)\, dt \quad \text{for } r \geq r_0(\varepsilon).$$

By Gronwall's lemma, this implies that for any $k > 0$, $r^{-k}\bar{w}(r) \to +\infty$ as $r \to +\infty$, which contradicts the assumption $|x|^{-(2/(1-\alpha))}w(x) \to 0$ as $|x| \to +\infty$.

We now justify the above formal calculations: as in 3.2, we can use a sequence of mollifiers φ_k: we can write relations (47)–(48) with $w \star \varphi_k$ instead of w and $((u+w)^\alpha - u^\alpha) \star \varphi_k$ instead of $(u+w)^\alpha - u^\alpha$. With these modifications, (49) and (50) remain true. Call $(50)_k$ the modified relation. Let now $k \to +\infty$. Since $w \in W^{1,1}_{\text{loc}}(\mathbb{R}^n)$, $\bar{w} \in A.C.\ (\mathbb{R}^+, \mathbb{R})$, and passing to the limit in $(50)_k$ yields (50) in the classical sense.

The following proposition proves that if f has the limit behaviour at infinity which ensures existence of positive solutions, the result of theorem 4 is not available.

PROPOSITION 1. There exists two positive constants a and b such that every positive solution of

$$-u'' + u^\alpha = cr^{2\alpha(1-\alpha)} \tag{51}$$

verifies

$$a < \limsup_{r\to\infty} |r|^{-(2/(1-\alpha))} u(r) < b. \tag{52}$$

Proof. (1) By theorem 2, (51) admits positive solutions for c small enough. Thus the preceding proposition is nonempty. Assume now by contradiction that the left inequality in

(52) is false. We get

$$\limsup_{r\to\infty} |r|^{-(2/(1-\alpha))} u(r) = 0,$$

and by (1) : $u''(r) \simeq -cr^{2\alpha/(1-\alpha)}$ as $r \to \infty$. This implies $u < 0$ for r large enough and yields a contradiction.

(2) From (51) we get $u'' \leq u^\alpha$, and this implies easily the second inequality of (52).

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REFERENCES

1. BREZIS H., Semilinear equations in $\mathbb{R}^n$ without condition at infinity *Appl. Math. Optim.* (to appear).
2. BARAS P. & PIERRE M., Singularités éliminables d'équations elliptiques semi-linéaires, *Ann. Inst. Fourier* (to appear). (Cf. also *C.r. hebd. sèanc. Acad. Sci. Paris* **295**, 519–522 (1982).)
3. BENILAN PH., CRANDALL M. G. & PIERRE M., Solutions of the porous medium equation under optimal conditions on initial values, *Indiana Univ. Math. J.* (to appear).
4. ARONSON D. G. & CAFFARELLI L. A., The initial trace of a solution of the porous medium equation, *Trans. Am. math. Soc.* (July 1983).
5. DAHLBERG B. E. T. & KENIG C. E., Nonnegative solutions of the porous medium equation, preprint.
6. OSSERMAN R., On the inequality $\Delta u \geq f(u)$, *Pacif. J. Math.* **7**, 1641–1647 (1957).
7. DIAZ-DIAZ J. I., Solutions with compact support for some degenerate parabolic problems, *Nonlinear Analysis* **3**, 831–847 (1979).
8. ADAMS R. A., *Sobolev Spaces*, Academic Press, New York (1975).
9. AGMON S., DOUGLIS A. & NIRENBERG L., Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions, *Communs pure appl. Math.* **12**, 623–727 (1959).
10. BREZIS H. & STRAUSS W. A., Semilinear second-order elliptic equations in L^1, *J. Math. Soc. Japan* **25**, (1973).
11. GALLOUËT T. & MOREL J. M., Resolution of a semilinear equation in L^1. *Proc. R. Soc. Edinb.* (to appear). (Cf. also *C.r. hebd. séanc. Acad., Sci. Paris* Série I, **296**, 493–496 (1983).)
12. BENILAN PH., BREZIS H. & CRANDALL M. G., A semilinear equation in L^1, *Annuli Scu. norm. sup. Pisa* **2**, 523–555 (1975).
13. MOREL J. M., Runge property and multiplicity of solutions for elliptic equations in $\mathbb{R}^N$ with a monotone nonlinearity *Manuscripta math.*, **54**, 165–185 (1985).

APPENDIX

In this appendix we prove several lemmas used in the preceding sections. Let $n \geq 1$.

LEMMA A.1. Let Ω be a bounded regular open set of $\mathbb{R}^n$ and let $u \in W^{1,1}(\Omega)$ be such that $u \leq 0$ a.e. on $\partial\Omega$ (in the sense of traces) and $\Delta u \in L^1(\Omega)$. Then there exists $u_k \in \mathcal{D}(\bar{\Omega})$ such that $u_k \leq 0$ on $\partial\Omega$, $u_k \to u$ in $L^1(\Omega)$ and $\Delta u_k \to \Delta u$ in $L^1(\Omega)$.

Proof of lemma A.1. Let $f_k \in C^\infty(\bar{\Omega})$ and $g_k \in C^\infty(\partial\Omega)$ such that

$$\left.\begin{aligned} f_k &\to f = -\Delta u \quad \text{in } L^1(\Omega), \quad \text{as } k \to +\infty,\\ g_k &\to g = u_{|\partial\Omega} \quad \text{in } L^1(\partial\Omega), \quad \text{as } k \to +\infty. \end{aligned}\right\}$$

There exists $u_k \in C^\infty(\bar{\Omega})$ such that

$$\left.\begin{aligned} -\Delta u_k &= f_k \quad \text{on } \bar{\Omega},\\ u_k &= g_k \quad \text{on } \Omega. \end{aligned}\right\}$$

Step 1. *Estimates on* u_k.
Let $\varphi \in W^{2,q}(\Omega)$, $\varphi = 0$ on $\partial\Omega$ $(1 < q \leq +\infty)$. One has

$$\int_\Omega f_k\varphi \,dx = -\int_\Omega \Delta u_k \varphi \,dx = -\int_\Omega u_k \Delta\varphi \,dx + \int_{\partial\Omega} g_k \frac{\partial\varphi}{\partial n} \,d\sigma.$$

Then

$$\int_\Omega u_k \Delta\varphi \,dx = -\int_\Omega f_k\varphi \,dx + \int_{\partial\Omega} g_k \frac{\partial\varphi}{\partial n} \,d\sigma. \tag{A1}$$

Let $\psi \in L^\infty(\Omega)$, there exists $\varphi \in W^{2,q}(\Omega)$ $(1 \leq q < +\infty)$ such that

$$\left.\begin{aligned} -\Delta\varphi &= \psi \quad \text{on } \Omega\\ \varphi &= 0 \quad \text{on } \partial\Omega, \end{aligned}\right\} \tag{A2}$$

and furthermore one has (cf. [9]):

$$\|\varphi\|_{W^{2,q}} \leq C\|\psi\|_q \leq C\|\psi\|_\infty$$

(where C denote some "constants" only depending on Ω and q).
Taking some $q > n$ we conclude (using the Sobolev injection theorem)

$$\left.\begin{aligned} \|\varphi\|_\infty &\leq C\|\psi\|_\infty\\ \left\|\frac{\partial\varphi}{\partial n}\right\|_\infty &\leq C\|\psi\|_\infty. \end{aligned}\right\}$$

Then (A1) gives, for all $\psi \in L^\infty(\Omega)$

$$\left|\int_\Omega u_k\psi \,dx\right| \leq C(\|f_k\|_1 + \|g_k\|_1)\|\psi\|_\infty.$$

Thus,

$$\|u_k\|_1 \leq C(\|f_k\|_1 + \|g_k\|_1).$$

From this inequality (with $u_k - u_q$, instead of u_k) we conclude that $\{u_k, k \in \mathbb{N}\}$ is a Cauchy sequence in $L^1(\Omega)$. Then there exists v such that $u_k \to v$ in $L^1(\Omega)$.

Step 2. *We show that* $v = u$ *a.e.*
Let $v_k \in \mathcal{D}(\Omega)$ such that $v_k \to v$ in $W^{1,1}(\Omega)$ and $\Delta v_k \to \Delta v$ in $L^1(\Omega)$. For $\varphi \in C^2(\bar{\Omega})$ such that $\varphi = 0$ on $\partial\Omega$, one has

$$-\int_\Omega \Delta\varphi v_k \,dx = -\int_{\partial\Omega} \frac{\partial\varphi}{\partial n} v_k \,dx - \int_\Omega \Delta v_k \varphi \,dx.$$

Since $-\Delta v_k \to f$ in $L^1(\Omega)$ and $v_k \to g$ in $L^1(\partial\Omega)$ (since $v_k \to u$ in $W^{1,1}(\Omega)$) we conclude by letting $k \to +\infty$ that

$$-\int_\Omega \Delta\varphi\, u \,dx = \int_\Omega f\varphi \,dx - \int_{\partial\Omega} \frac{\partial\varphi}{\partial n} g \,dx. \tag{A3}$$

In (A1), let $k \to +\infty$. Since $u_k \to v$ in $L^1(\Omega)$ we also have

$$\int_\Omega v\, \Delta\varphi \,dx = -\int_\Omega f\varphi \,dx + \int_{\partial\Omega} \frac{\partial\varphi}{\partial n} g \,dx. \tag{A4}$$

(A3) and (A4) give:

$$\int_\Omega (u - v)\,\Delta\varphi \,dx = 0 \tag{A5}$$

for all $\varphi \in C^2(\bar{\Omega})$ with $\varphi = 0$ on $\partial\Omega$.
Let $\psi \in \mathcal{D}(\Omega)$. There exists $\varphi \in C(\bar{\Omega})$ such that

$$\begin{aligned} \Delta\varphi &= \psi \quad \text{on} \quad \Omega\\ \varphi &= 0 \quad \text{on} \quad \partial\Omega. \end{aligned}$$

Thus (A5) gives $\int_\Omega (u - v)\,\psi \,dx = 0$ for all $\psi \in \mathcal{D}(\Omega)$, and, therefore, $u = v$ a.e.

LEMMA A.2. Let Ω be a bounded regular open set of $\mathbb{R}^n$. Let $u \in W^{1,1}(\Omega)$ be such that $u \le 0$ a.e. on $\partial\Omega$ and $-\Delta u \in L^1(\Omega)$. Let $P \in C^1 \cap L^\infty(\mathbb{R})$, $P = 0$ on $\mathbb{R}^-$, $P' \ge 0$. Then one has

$$\int_\Omega -\Delta u\, P(u)\, dx \ge 0.$$

Proof of lemma A.2. By lemma A.1 there exists $\{u_k, k \in \mathbb{N}\} \subset \mathcal{D}(\bar{\Omega})$ such that

$$\begin{aligned} u_k &\to u && \text{in } L^1(\Omega) && \text{as } k \to +\infty, \\ -\Delta u_k &\to -\Delta u && \text{in } L^1(\Omega) && \text{as } k \to +\infty, \\ u_k &\le 0 && \text{on } \partial\Omega. \end{aligned}$$

We clearly have

$$\int_\Omega -\Delta u_k\, P(u_k) = \int_\Omega |\nabla u_k|^2\, P'(u_k)\, dx \ge 0.$$

We can assume (after extracting a subsequence) that

$$\begin{aligned} &u_k \to u \quad \text{a.e.} \\ &-\Delta u_k \to -\Delta u \quad \text{a.e.} \\ &|\Delta u_k| \le g \quad \text{a.e., with some } g \in L^1(\Omega). \end{aligned}$$

By the dominated convergence theorem we conclude that

$$\int_\Omega -\Delta u\, P(u)\, dx \ge 0.$$

For $a, b \in \mathbb{R}$, we set $\Omega(a, b) = \{a \le |x| < b\}$.

LEMMA A.3. (Compact support lemma—cf. [12, theorem 6.1] and [7].) Let $0 < \alpha < 1$. For all $0 \le a < b$ and for all $\varepsilon > 0$, there exists $C(\varepsilon, a, b) > 0$ such that if

$$f \in L^1_{\text{loc}}(\mathbb{R}^n), f \ge 0, \operatorname{Supp} f \subset \Omega(a, b), \int f\, dx \le C(\varepsilon, a, b), \tag{9}$$

then the problem

$$\left.\begin{aligned} &-\Delta u + u^\alpha = f \quad \text{in} \quad \mathcal{D}';\mathbb{R}^n) \\ &u \in L^1(\mathbb{R}^n), u \ge 0, \operatorname{Supp} u \subset \Omega(a - \varepsilon, b + \varepsilon) \end{aligned}\right\} \tag{P'}$$

has a (unique) solution.

Proof of lemma A.3. Let $0 < \alpha < 1$. We prove lemma A.3 with, for instance, $a = 2$, $b = 3$, and some $0 < \varepsilon < 1$. (This case contains all the difficulties.)

We set $\Omega = \Omega(2, 3)$, $B = \{1 < |x| < 4\}$ and $\Omega_\varepsilon = \Omega(2 - \varepsilon, 3 + \varepsilon)$.

Let $f \in L^1(\mathbb{R}^n)$, $f \ge 0$, $\operatorname{Supp} f \subset \Omega$ and u be the unique solution (cf. [10]) of

$$\left.\begin{aligned} &-\Delta u + u^\alpha = f \quad \mathcal{D}'(B) \\ &u \in W_0^{1,1}(B), \quad u \ge 0. \end{aligned}\right\} \tag{P''}$$

Our aim is to prove that for $\int_B f\, dx$ small enough one has $\operatorname{Supp} u \subset \Omega_\varepsilon$. We then obtain of course a (unique) solution for (P').

Step 1. *Estimates on u.* One has (cf. [10]) $\int_B u^\alpha\, dx \le \int_B f\, dx$ and then $\int_B |\Delta u|\, dx \le 2 \int_B f\, dx$, this gives for some C_1 (only depending on n)

$$\int_B u\, dx \le c_1 \int_B f\, dx. \tag{A6}$$

Since $\Delta u = u^\alpha$ on $B\backslash\bar{\Omega}$, a classical "bootstrap" argument gives $u \in W^{2,p}_{\text{loc}}(B\backslash\bar{\Omega})$ (and then, taking $p > n$, $u \in C^1(B\backslash\bar{\Omega})$).

Let $1 < R_1 < R_2 < 2 < 3 < R_3 < R_4 < 4$. Then there exists C_2 (only depending on n, R_1, R_2, R_3, R_4) such that

$$\operatorname{Sup}\{u(x), R_1 \le |x| \le R_2 \quad \text{or} \quad R_3 \le |x| \le R_4\} \le C_2 \int_B f\, dx.$$

In what follows we take $R_1 = 1 + \varepsilon/4$, $R_2 = 2 - \varepsilon/4$, $R_3 = 3 + \varepsilon/4$, $R_4 = 4 - \varepsilon/4$.

Step 2. *Estimates on Supp u.* In this step we construct on $\{1 < |x| \le R_2\}$ and $\{R_3 \le |x| < 4\}$ some radial comparison functions.

Let $M_1 = \operatorname{Max}\{u(x), |x| = R_2\}$.

One has, by step 1,

$$M_1 \le C_2 \int_B f\, dx.$$

We set, for some $A > 0$ and $R_5 = R_2 - \varepsilon/4 = 2 - \varepsilon/2$,

$$\left.\begin{aligned} v(x) &= A(|x| - R_5)^{2/(1-\alpha)} && \text{if } R_5 \le |x| \le R_2 \\ v(x) &= 0 && \text{if } 1 \le |x| \le R_5, \end{aligned}\right\}$$

an easy calculation yields

$$\begin{aligned} -\Delta v(x) + v^\alpha(x) &= \left(A^\alpha - A\frac{2(1+\alpha)}{(1-\alpha)^2}\right)(|x| - R_5)^{2\alpha/(1-\alpha)} \\ &\quad - \frac{A}{|x|}(n-1)\frac{2}{1-\alpha}(|x| - R_5)^{(1+\alpha)/(1-\alpha)} \quad \text{if } R_5 \le |x| \le R_2 \\ &= 0 \quad \text{if } 1 < |x| \le R_5. \end{aligned}$$

Since $R_2 - R_5 < 1$ and $0 < \alpha < 1$ one has

$$\frac{1}{|x|}(|x| - R_5)^{(1+\alpha)/(1-\alpha)} \le \frac{1}{R_5}(|x| - R_5)^{2\alpha/(1-\alpha)} \quad \text{if } R_5 \le |x| \le R_2,$$

that is

$$-\Delta v(x) + v^\alpha(x) \ge \left(A^\alpha - A\left(\frac{2(1+\alpha)}{(1-\alpha)^2} - \frac{2A(n-1)}{(1-\alpha)R_5}\right)(|x| - R_5)^{2\alpha/(1-\alpha)}\right), \quad \text{if } R_5 \le |x| \le R_2.$$

Taking $A = A_\varepsilon > 0$ small enough (A_ε only depends on n, α and ε) one has

$$-\Delta v + v^\alpha \ge 0 \quad \text{on} \quad 1 \le |x| \le R_2. \tag{A7}$$

Taking also

$$\int_B f\, dx \le \frac{1}{C_2} A_\varepsilon (R_2 - R_5)^{2/(1-\alpha)} = C_\varepsilon^\star > 0, \tag{A8}$$

one has

$$u(x) \le M_1 \le C_2 \int_B f\, dx \le v(x) \quad \text{if} \quad |x| = R_2. \tag{A9}$$

(A7) and (A9) then give:

$$\left.\begin{aligned} &-\Delta(u - v) + u^\alpha - v^\alpha \le 0 \quad \text{on} \quad 1 < |x| \le R_2 \\ &u - v \in W^{1,1}(1 < |x| < R_2) \\ &u - v \le 0 \quad \text{on} \quad \{|x| = 1\} \cup \{|x| = R_2\} \end{aligned}\right\}$$

Thus, by lemma A.2, one has for $P \in C^1 \cap L^\infty(\mathbb{R})$, $P = 0$ on $\mathbb{R}^-$, $P > 0$ on $\mathbb{R}_+^\star$, $P' \ge 0$,

$$\int_{1<|x|<R_2} (u^\alpha - v^\alpha)\, P(u - v) \le 0.$$

This proves $u(x) \le v(x)$ if $1 < |x| < R_2$.

Then we conclude $u(x) = 0$ if $1 < |x| \leq R_5 = 2 - \varepsilon/2$.

By a similar method (a little bit simpler) we prove that there exists $C_\varepsilon^{\star\star} > 0$ (only depending on n, α and ε) such that if

$$\int_B f\,dx \leq C_\varepsilon^{\star\star} \tag{A10}$$

then

$$u(x) = 0 \quad \text{if} \quad 3 + \varepsilon/2 = R_6 \leq |x| < 4.$$

Hence, if f satisfies (A8) and (A10) (that is if $\int_B f\,dx \leq C_\varepsilon = \text{Inf}(C_\varepsilon^\star, C_\varepsilon^{\star\star})$) one has

$$\text{Supp}\, u \subset \Omega_\varepsilon = \Omega(2 - \varepsilon, 3 + \varepsilon).$$

This proves lemma A.3 for $a = 2$ and $b = 3$.

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ANALYSIS OF NONLINEAR INTEGRO-DIFFERENTIAL EQUATIONS ARISING IN AGE-DEPENDENT EPIDEMIC MODELS

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1. INTRODUCTION

BASICALLY, there are two modes for directly transmitting an infectious disease within a single population: vertical transmission and horizontal transmission. Vertical transmission is defined as the direct transfer of infection from a parent organism to its offsprings. Horizontal transmission is any transfer of infection except that which is vertically transmitted. For example AIDS is both vertically and horizontally transmitted while malaria is horizontally transmitted.

Vertically transmitted diseases have seldom been considered in mathematical models of epidemics. Examples of previous such models are found in Anderson and May [1], Cooke and Busenberg [7], Busenberg and Cooke [3], Busenberg, Cooke and Pozio [4], Fine [10] and Re'gniere [17].

Likewise age-dependent diseases has been presented by Cooke and Busenberg [7] and Dietz [8]. Age-dependence introduces a coupling of age-structure and vertical transmission which can produce novel dynamic behavior.

In this paper, a system of nonlinear integro-differential equations which model an age-dependent epidemic of a disease with vertical transmission is investigated. This model treats the simple $S \to I$ type of epidemic in this new setting. Existence and uniqueness are proved under suitable hypotheses and the asymptotic behavior of the system is determined. A renewal theorem is used to study the behavior of the model equations in various pertinent parameter ranges. A numerical method for integrating this system of equations is developed and is used to obtain approximations of its solutions for some special cases which illustrate the results obtained via analytical means. Moreover, numerical integrations of the equations are used to study some phenomena that were not treated analytically.

2. A MODEL OF A VERTICALLY TRANSMITTED DISEASE

Consider an age-structured population of variable size exposed to a disease which is both horizontally and vertically transmitted with the following assumptions on the model.

(a) Let $s(a, t)$ and $i(a, t)$, respectively, denote the age-density for susceptibles and infectives of age a at time t. Then

$$\int_{a_1}^{a_2} s(a, t)\,da = \text{total number of susceptibles at time } t \text{ of ages between } a_1 \text{ and } a_2.$$

$$\int_{a_1}^{a_2} i(a, t)\,da = \text{total number of infectives at time } t \text{ of ages between } a_1 \text{ and } a_2.$$