On the Regularity of Solutions to Elliptic Equations

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1 Introduction

In this paper, we prove that the $W^{1,p}$-estimate, $p > 2$, of any solution to the Dirichlet problem for a linear elliptic equation with discontinuous coefficients, due to N.G. Meyers [12] can be generalized to other boundary conditions, and for an open set with a Lipschitz continuous boundary. For regular operator in Lipschitz domains, G. Savaré obtain optimal regularity results in [14]. In [9], Konrad Gröger shows the result for a mixed boundary value problem, by using a fixed-point technique (he cannot mimic Meyers' proof for he's interested in monotonous non-linear operators).

Our proof works differently and very simply. The technique is to reduce the problem, by using local coordinates and reflection arguments, to a Dirichlet problem in a ball and to apply known results.

This paper is organized as follows. In section 2, we introduce our notations and recall Meyers' Theorem. Section 3 is devoted to the study of Neumann problem. Section 4 is devoted to other boundary condition, mainly Fourier condition and the mixed boundary value problem. The last section is about an application of our main result, namely, the uniqueness (up to a constant) of the weak solution of Neumann problem for a linear elliptic equation, in a bounded connected open set of $\mathbb{R}^2$, whose right-hand side is a measure.

2 Definitions and Preliminary Results

Let Ω be a bounded connected open set of $\mathbb{R}^N$, $N \geq 2$. $\mathbb{R}^N$ is considered with its euclidean norm, denoted $|.|$ and $.$ denotes the inner product. We consider the following linear elliptic equation

$$-\mathrm{div}(A(x)\nabla u(x)) = f(x),\ x \in \Omega, \tag{1}$$

where A is an element of $(L^\infty(\Omega))^{N\times N}$ which satisfies the following condition (ellipticity and boundedness):

$$\exists\ \alpha, \beta > 0 \ \text{ such that }\ \forall \xi \in \mathbb{R}^N,\ \alpha|\xi|^2 \leq A(x)\xi.\xi \ \text{ for a.e. } x \in \Omega, \ \text{ and } ||A||_\infty \leq \beta, \tag{2}$$

and f is a given function. We start from the following result, due to N.G. Meyers [12], for Dirichlet problem, if the boundary of Ω is smooth enough.

Theorem 1 (Meyers) *Let Ω be a bounded connected open set of C^2-class and A in $(L^\infty(\Omega))^{N\times N}$ satisfy (2). There is a real number p_0, $p_0 > 2$, such that if u is the weak solution of (1), i.e.*

$$\begin{cases} u \in H_0^1(\Omega), \\ \displaystyle\int_\Omega A(x)\nabla u(x).\nabla\varphi(x)\, dx = \langle f, \varphi\rangle_{H^{-1},H_0^1}, \forall \varphi \in H_0^1(\Omega), \end{cases}$$

and f belongs to $W^{-1,p}(\Omega)$, $p \in [2, p_0[$, then $u \in W_0^{1,p}(\Omega)$ and there is a $C(p)$ such that

$$||u||_{W_0^{1,p}} \leq C(p)||f||_{W^{-1,p}}.$$

Moreover, p_0 only depends on A and Ω and $C(p)$ on A, Ω and p, not on f.

The proof of this theorem uses a regularity theorem of Agmon-Douglis-Nirenberg (see [2] and [3]), and in particular the open set needs to be regular enough (of C^2-class is sufficient). Here, a simplified version of this theorem is only needed thereafter, the open set considered being a ball centered on zero with radius $R > 0$. Indeed, for an open set with Lipschitz continuous boundary, we use local maps and reflection to get back to a Dirichlet problem on the unit ball. Furthermore, our method works also for a large choice of boundary conditions. So let us recall a definition of an open set with Lipschitz continuous boundary.

Definition 1 *Let $\Omega \subset \mathbb{R}^N$ be a bounded open set. Its boundary $\partial\Omega$ is Lipschitz continuous if, for all $a \in \partial\Omega$, there exists an orthonormal coordinates system $\mathcal{R}_a$, a neighbourhood of a, $V = \prod_{i=1}^{N}]\alpha_i, \beta_i[= V' \times]\alpha, \beta[$ in these coordinates, and a Lipschitz continuous function $\eta : V' \to]\alpha, \beta[$ such that*

$$\begin{aligned} V \cap \Omega &= \{(y', y_N) \in V \mid y_N > \eta(y')\}, \\ V \cap \partial\Omega &= \{(y', \eta(y'))\,,\ y' \in V'\}. \end{aligned}$$

With this definition, we can prove the following proposition, where $B = \{x \in \mathbb{R}^N,\ |x| < 1\}$.

Proposition 1 *Let $\Omega \subset \mathbb{R}^N$ be a bounded open set with a Lipschitz continuous boundary. Then there exists a family $(U_0,\ U_1,\ \ldots,\ U_k)$ of open sets of $\mathbb{R}^N$, satisfying*

$$\overline{\Omega} \subset \bigcup_{i=0}^{k} U_i,\ \overline{U_0} \subset \Omega, \tag{3}$$

and $(J_1,\ \ldots,\ J_k)$ functions such that for $i = 1,\ \ldots,\ k$, $J_i : U_i \to B$ is an homeomorphism, J_i and J_i^{-1} are Lipschitz continuous and

$$\begin{aligned} J_i(U_i \cap \Omega) &= B \cap \{(x', x_N) \mid x' \in \mathbb{R}^{N-1},\ x_N > 0\} = B_+, \\ J_i(U_i \cap \partial\Omega) &= B \cap \{(x', 0) \mid x' \in \mathbb{R}^{N-1}\} = B^{N-1}. \end{aligned} \tag{4}$$

Remarks:

1. This definition of Lipschitz continuous boundary allows us to define properly the outward normal of Ω and to integrate on the boundary. That is actually necessary in section 4 for the Fourier condition.

2. Because the Rademacher Theorem, it is possible to make a change of variable with Lipschitz continuous functions. Indeed, if J is a Lipschitz continuous homeomorphism, mapping an open set U onto an open set V, the Jacobian matrix of J, denoted $\mathrm{D}J$, is defined almost everywhere and we have the classical formulae of change of variable (see [4] or [7]).

   Moreover, if J^{-1} is Lipschitz continuous too, the operator $T_J : W^{1,p}(V) \to W^{1,p}(U)$, defined by $T_J(u) = u \circ J$, is linear continuous. The norm of T_J only depends on the "Lipschitz contents" of J and J^{-1} and N.

Let us finally set $p^* = \dfrac{Np}{N-p}$ if $N > p$ and $p^* = \infty$ if $N \leq p$.

3 Meyers' Theorem for Neumann Problem

Let Ω be bounded connected open set of $\mathbb{R}^N$, with a Lipschitz continuous boundary. Let us consider the Neumann problem for the equation (1) where A satisfies conditions (2).
Define

$$H^1_*(\Omega) = \{u \in H^1(\Omega); \int_\Omega u(x)dx = 0\}.$$

A weak formulation of this problem is expressed by

$$\left\{ \begin{array}{l} u \in H^1_*(\Omega), \\ \displaystyle\int_\Omega A(x)\nabla u(x).\nabla\varphi(x)\ dx = \langle f, \varphi\rangle_{(H^1)', H^1}, \forall \varphi \in H^1(\Omega), \end{array} \right. \tag{5}$$

where f is in $(H^1(\Omega))'$ with $\langle f, 1\rangle_{(H^1)',H^1} = 0$ (note that this condition is necessary to obtain a solution of (5)). By the Lax-Milgram theorem and the Poincaré inequality with a null mean, there exists a unique solution u in $H^1_*(\Omega)$ to (5).
If f belongs to $(W^{1,q}(\Omega))'$, with $p > 2$ and $q = p/(p-1)$, there is u in $H^1_*(\Omega)$ solution to (5) (indeed $(W^{1,q}(\Omega))' \subset (H^1(\Omega))'$). The following theorem improves the regularity of u.

Theorem 2 (Meyers Neumann) *Let Ω be a bounded connected open set of $\mathbb{R}^N$, with a Lipschitz continuous boundary. Let A in $(L^\infty(\Omega))^{N\times N}$ satisfy (2). For $p \geq 2$ and $q = p/p - 1$, let T_p be the operator defined by $T_p(f) = u$, for all $f \in (W^{1,q}(\Omega))'$, with $\langle f, 1\rangle_{(H^1)',H^1} = 0$, where u is the unique solution to (5). Then, there is a real number p_M, $2^* > p_M > 2$, such that, for all p, $2 < p < p_M$, the operator T_p is linear continuous from $(W^{1,q}(\Omega))'$ to $W^{1,p}(\Omega)$. Moreover, the norm of T_p only depends on p, α, β and Ω and p_M on α, β and Ω, not on f.*

Proof. Let p be fixed as greater than or equal to 2 and less than 2^*. Let $f \in (W^{1,q}(\Omega))'$. As previously seen, we can consider $u = T_p(f)$. So, u belongs to $H^1_*(\Omega)$.

Step 1 (Localization) Let us now consider a set of local maps, given by the proposition 1. We associate a partition of unity $(\theta_i)_i$ to the open sets $(U_i)_{i=0,\ ...,\ k}$; that is, functions $\theta_0,\ \theta_1,\ ...,\ \theta_k$ of $C^\infty(\mathbb{R}^N)$ such that

$$0 \leq \theta_i \leq 1,\ \forall i = 0,\ 1,\ ...,\ k \text{ and } \sum_{i=0}^k \theta_i = 1 \text{ on } \overline{\Omega},$$

and

$$\text{supp}\theta_i \text{ is compact and included in } U_i,\ \forall i = 0,\ ...,\ k.$$

Let then

$$u_i = \theta_i u.$$

For all $\varphi \in H^1(\Omega)$, u_i satisfies

$$\begin{aligned} \int_\Omega A\nabla u_i.\nabla\varphi\ dx &= \int_\Omega \theta_i A\nabla u.\nabla\varphi\ dx + \int_\Omega uA\nabla\theta_i.\nabla\varphi\ dx \\ &= \int_\Omega A\nabla u.(\nabla(\theta_i\varphi) - \varphi\nabla\theta_i)dx + \int_\Omega uA\nabla\theta_i.\nabla\varphi\ dx \\ &= \langle \theta_i f - \text{div}(uA\nabla\theta_i) - A\nabla u.\nabla\theta_i, \varphi\rangle_{(H^1(\Omega))',H^1(\Omega)} \end{aligned}$$

where $\langle\theta_i f, \varphi\rangle = \langle f, \theta_i\varphi\rangle$ and, if a function F belongs to $(L^2)^N$, let

$$\langle -\text{div}(F), \varphi\rangle_{(H^1(\Omega))',H^1(\Omega)} = \int_\Omega F(x).\nabla\varphi(x)dx.$$

To define properly a linear form f_i on $H^1(U_i \cap \Omega)$, let us consider a function γ of $C^\infty(\mathbb{R}^N)$ such that $\text{supp}\theta_i \subset \text{supp}\gamma \subset U_i$ and $\gamma(x) = 1$ on $\text{supp}\theta_i$. For all function φ in $H^1(U_i \cap \Omega)$, we define f_i

$$\langle f_i, \varphi\rangle_{(H^1(U_i\cap\Omega))',H^1(U_i\cap\Omega)} = \langle\theta_i f - \text{div}(uA\nabla\theta_i) - A\nabla u.\nabla\theta_i, \gamma\varphi\rangle_{(H^1(\Omega))',H^1(\Omega)},$$

where $\gamma\varphi$ is the function extended by zero on Ω. So, we have got for all $\varphi \in H^1(U_i \cap \Omega)$

$$\int_{U_i\cap\Omega} A\nabla u_i.\nabla\varphi\ dx = \langle f_i, \varphi\rangle_{(H^1(U_i\cap\Omega))',H^1(U_i\cap\Omega)}.$$

Of course $\theta_i f$ is in $(W^{1,q}(U_i \cap \Omega))'$ and we have $||\theta_i f||_{(W^{1,q})'} \leq C_{\theta_i}||f||_{(W^{1,q})'}$.
If u belongs to $H^1(\Omega)$, then $A\nabla u.\nabla\theta_i$ belongs to L^2. According to Sobolev's injection theorem, a function L^2 is also in $(W^{1,q})'$ if $q^* > 2$, *i.e.* $p < 2^*$. By the continuity of the Sobolev imbedding,

$$||A\nabla u.\nabla\theta_i||_{(W^{1,q})'} \leq C_0||u||_{H^1} \leq C_1||f||_{(H^1)} \leq C_2||f||_{(W^{1,q})'}.$$

In the same way, $\text{div}(uA\nabla\theta_i)$ belongs to $(W^{1,q}(U_i \cap \Omega))'$ if $uA\nabla\theta_i$ is in $(L^p)^N$ (*i.e.* $p < 2^*$), and

$$||\text{div}(uA\nabla\theta_i)|| \leq C_0'||u||_{(L^p)^N} \leq C_1'||u||_{H^1} \leq C_2'||f||_{(W^{1,q})'}.$$

Finally f_i is in $(W^{1,q}(U_i \cap \Omega))'$ and there exists a real M_i positive such that

$$||f_i||_{(W^{1,q})'} \leq M_i||f||_{(W^{1,q})'}.$$

Interior estimates. Consider first u_0.
Let B_R a ball with radius R large enough to allow $U_0 \subset B_R$ (Ω is bounded...). With the function γ used before we can also extend f_0 on $H_0^1(B_R)$. We extend u_0 by zero outside U_0. Then, u_0 is a solution of the following problem

$$\begin{cases} \displaystyle\int_{B_R} A\nabla u_0.\nabla\varphi \, dx = \langle f_0, \varphi\rangle_{H^{-1}(B_R), H_0^1(B_R)}, \forall\varphi \in H_0^1(B_R), \\ u_0 = 0, \text{ on } \partial B_R. \end{cases}$$

Note that f_0 is in $W^{-1,p}(B_R)$. Hence, according to Meyers' Theorem, there is a $2^* > p_0 > 2$, such that, if $p \in [2, p_0[$, then $u_0 \in W_0^{1,p}(B_R)$, and even in $W_0^{1,p}(U_0)$, by definition of θ_0. Moreover, there is a real positive $C_0(p)$ such that

$$||u_0||_{W_0^{1,p}} \leq C_0(p)||f_0||_{(W^{1,q})'} \leq M_0 C_0(p)||f||_{(W^{1,q})'},$$

$C_0(p)$ and p_0 only depends on α, β and Ω, not on f.

Estimates near the boundary. Let us now consider $v = u_i$ and $g = f_i$, for a fixed i. We will avoid recalling indices i throughout this proof. As seen previously, v satisfies

$$\int_{U\cap\Omega} A\nabla v.\nabla\varphi \, dx = \langle g, \varphi\rangle_{(H^1(U\cap\Omega))', H^1(U\cap\Omega)}, \forall\varphi \in H^1(U \cap \Omega),$$

where the mapping g is an element of $(W^{1,q}(U \cap \Omega))'$ (where $q = p/p - 1$), as soon as $p < 2^*$.

Step 2 (Transport) Now, we make the change of variable $y = J(x)$, where J is the Lipschitz continuous function given by Proposition 1. Let $H = J^{-1}$. $\text{D}J$ (resp. $\text{D}H$) denotes the Jacobian matrix of J (resp. H), *i.e.* the matrix with general term $\partial J_i/\partial x_j$. ${}^{\text{t}}M$ denotes the transpose matrix of the matrix M. Let $w(y) = v \circ H(y)$, for all $y \in B_+ = \{x \in \mathbb{R}^N, \ |x| < 1, \ x_N > 0\}$. Let $\psi \in H^1(B_+)$, and $\varphi = \psi \circ J$. Then,

$$\nabla v(x) = {}^{\text{t}}\text{D}J(x)\nabla w(J(x)), \text{ and } \nabla\varphi(x) = {}^{\text{t}}\text{D}J(x)\nabla\psi(J(x)).$$

Hence,

$$\begin{aligned} A(x)\nabla v(x).\nabla\varphi(x) &= A(x){}^{\text{t}}\text{D}J(x)\nabla w(J(x)).{}^{\text{t}}\text{D}J(x)\nabla\psi(J(x)) \\ &= \text{D}J(x)A(x){}^{\text{t}}\text{D}J(x)\nabla w(J(x)).\nabla\psi(J(x)). \end{aligned}$$

Let

$$\Lambda(y) = |\det \text{D}H(y)| \ \text{D}J(H(y))A(H(y)){}^{\text{t}}\text{D}J(H(y)). \tag{6}$$

According to the formulae of change of variable, we have

$$\int_{U\cap\Omega} A(x)\nabla v(x).\nabla\varphi(x)\ dx\ = \int_{B+} \Lambda(y)\nabla w(y).\nabla\psi(y)\ dy$$

The mappings J and H both are Lipschitz continuous, hence the Jacobian matrices and $|\det \mathrm{D}H|$ are bounded with respect to the supremum norm. Hence, the matrix Λ is in $(L^\infty(B_+))^{N\times N}$.

Λ also satisfies the uniform ellipticity condition. Indeed, there exist reals m, M such that

$$m \le |\det \mathrm{D}H(y)| \le M, \text{ a.e. on } B_+$$

and

$$m|\xi|^2 \le |{}^{\mathrm{t}}\mathrm{D}J(H(y))\xi|^2 \le M|\xi|^2, \forall\xi \in \mathbb{R}^N, \text{ a.e. on } B_+. \tag{7}$$

Then, for all $\xi \in \mathbb{R}^N$ and almost everywhere on B_+, because

$$\begin{aligned}\Lambda(y)\xi.\xi &= |\det \mathrm{D}H(y)|\ \mathrm{D}J(H(y))A(H(y))^{\mathrm{t}}\mathrm{D}J(H(y))\xi.\xi \\ &= |\det \mathrm{D}H(y)|\ A(H(y))^{\mathrm{t}}\mathrm{D}J(H(y))\xi.{}^{\mathrm{t}}\mathrm{D}J(H(y))\xi,\end{aligned}$$

there exist α' and β', only depended on α, β, m and M, such that Λ satisfies

$$\forall\xi \in \mathbb{R}^N,\ \alpha'|\xi| \le \Lambda(y)\xi.\xi \text{ and } ||\Lambda||_\infty \le \beta'.$$

The operator g is carried out as an operator h of $(W^{1,q}(B_+))'$. One can describe that operator thanks to g and the function H. Indeed, if g is an element of $(W^{1,q}(\Omega\cap U))'$, there exist function g_0 in $L^p(\Omega\cap U)$ and G in $(L^p(\Omega\cap U))^N$ such that, for all φ in $W^{1,q}(\Omega\cap U)$,

$$\langle g,\varphi\rangle_{(W^{1,q}(U\cap\Omega))',W^{1,p}(U\cap\Omega)} = \int_{\Omega\cap U} g_0(x)\varphi(x)\ dx + \int_{\Omega\cap U} G(x).\nabla\varphi(x)\ dx.$$

Hence, for all $\psi \in H^1(B_+)$, $\varphi = \psi\circ J$,

$$\begin{aligned}\langle g,\varphi\rangle &= \int_{\Omega\cap U} g_0(x)\varphi(x)\ dx + \int_{\Omega\cap U} G(x).{}^{\mathrm{t}}\mathrm{D}J(x)\nabla\psi(J(x))\ dx \\ &= \int_{B+} |\det \mathrm{D}H|\ g_0(H(y))\psi(y)dy + \int_{B+} |\det \mathrm{D}H|\ \mathrm{D}J(H(y))G(H(y)).\nabla\psi(y)\ dy \\ &= \langle h,\psi\rangle.\end{aligned}$$

The function $|\det \mathrm{D}H|\ g_0(H(y))$ belongs to $L^p(B_+)$ and $|\det \mathrm{D}H|\ \mathrm{D}J(H(y))G(H(y))$ to $(L^p(B_+))^N$. Thus $h \in (W^{1,q}(B_+))'$ and it is easy to see that $||h||_{(W^{1,q})'} \le C||g||_{(W^{1,q})'}$, with $C > 0$. Finally, the function w is the solution to the new problem

$$\left\{\begin{array}{l} w \in H^1(B_+), \\ \displaystyle\int_{B+} \Lambda(y)\nabla w(y).\nabla\psi(y)\ dy = \langle h,\psi\rangle_{(H^1)',H^1}, \forall\psi \in H^1(\Omega), \end{array}\right. \tag{8}$$

where Λ is defined by (6), and h belongs to $(W^{1,q}(B_+))'$.

Step 3 (reflection) Let us now extend the solution by reflection, to get the following general result (the notation in this lemma is independent of that used in the rest of the paper):

Lemma 1 *For a given $u \in W^{1,p}(B_+)$, define on B the function u^* extended by reflection, that is to say*

$$u^*(x', x_N) = \begin{cases} u(x', x_N) & \text{if} \quad x_N > 0 \\ u(x', -x_N) & \text{if} \quad x_N < 0. \end{cases}$$

Then, $u^* \in W^{1,p}(B)$ and

$$||u^*||_{W^{1,p}(B)} \leq 2||u||_{W^{1,p}(B+)}.$$

This is a classical lemma (cf H. Brézis' book, [5], p. 158, for instance). Note that, for $x_N < 0$, one has the formulae

$$\frac{\partial u^*}{\partial x_i}(x', x_N) = \frac{\partial u}{\partial x_i}(x', -x_N) \text{ for } 1 \leq i \leq N-1,$$
$$\frac{\partial u^*}{\partial x_N}(x', x_N) = -\frac{\partial u}{\partial x_N}(x', -x_N).$$

Let us apply this result to our problem. w can be extended to a function w^* which is defined on the whole of B and is an element of $H^1(B)$. But θ is a function with compact support of U, hence the same holds for $\theta \circ H$ on B; in particular, there is an r, $r < 1$, such that B_r contents the support of θ. It is then obvious that the support of our function w^*, extended by reflection, is also contained in that ball. Thus w^* is in $H_0^1(B)$.

We extend the operator h the following way:

$$\langle h^*, \phi\rangle_{(W^{1,q}(B))', W^{1,q}(B)} = \langle h, \phi\rangle_{(W^{1,q}(B_+))', W^{1,q}(B_+)} + \langle h, \phi(x', -x_N)\rangle_{(W^{1,q}(B_+))', W^{1,q}(B_+)}.$$

for all ϕ in $W^{1,q}(B)$. In particular, $h^* \in W^{-1,p}(B)$ and $||h^*||_{W^{-1,p}} \leq 2||h||_{(W^{1,q})'}$.

To extend Λ is not that easy. We proceed as follows (we note $\Lambda = (\alpha_{kl})_{k,\ l}$)

- for all k and l less or equal than $N-1$, let $\alpha^*_{kl}(x', x_N) = \alpha_{kl}(x', -x_N)$ if $x_N < 0$,
- if $k = N$ or $l = N$ (but $(k, l) \neq (N, N)$), let $\alpha^*_{kl}(x', x_N) = -\alpha_{kl}(x', -x_N)$ if $x_N < 0$,
- $\alpha^*_{NN}(x', x_N) = \alpha_{NN}(x', -x_N)$ if $x_N < 0$.

Of course, we leave the α_{kl} as they are if $x_N > 0$. We get

$$\int_{B_-} \Lambda^* \nabla w^* . \nabla\phi(x)\, dx = \int_{B_+} \Lambda \nabla w(y) . \nabla \phi(y', -y_N)\, dy,$$

where $B_- = \{x \in B \mid x_N \leq 0\}$. There also remains to check that this matrix is elliptic. The case of $x_N > 0$ was seen before; if $x_N < 0$, then

$$\Lambda^* \xi . \xi = \sum_{i,j \leq N-1} \alpha(x', -x_N)\xi_i \xi_j + \sum_{j=1}^{N-1} -\alpha_{N,j}(x', -x_N)\xi_N \xi_j + \sum_{i=1}^{N-1} -\alpha_{i,N}(x', -x_N)\xi_i \xi_N + \alpha_{NN}\xi_N^2.$$

If $\xi^* = (\xi', -\xi_N)$, the preceding expression can then be written as

$$\Lambda^* \xi . \xi = \Lambda \xi^* . \xi^*.$$

Now, $|\xi| = |\xi^*|$; Λ^* satisfies the ellipticity condition indeed.

We can check that w^* is the solution of the following problem:

$$\left\{ \begin{array}{l} w^* \in H_0^1(B) \\ \displaystyle\int_B \Lambda^* \nabla w^* . \nabla \phi = \langle h^*, \phi \rangle_{H^{-1}, H_0^1}, \text{ for all } \phi \in H_0^1. \end{array} \right. \tag{9}$$

Note that h^* is an element of $W^{-1,p}(B)$ and w^* is the solution of problem (9). Then Theorem 1 is applied. There is a real p_i, $2^* > p_i > 2$, such that, if $p \in [2, p_i[$, w^* is in $W_0^{1,p}(B)$ and a real number $C_i(p)$ positive such that

$$||w^*||_{W_0^{1,p}} \leq C_i(p) ||h^*||_{W^{-1,p}}.$$

Moreover p_i depends on α', β' and N, and $C_i(p)$ on α', β', p and N, not on h^*. In fact, they hence depend on A and functions H and J, that is, on the change of map. We then get the desired estimate for $v = u_i$ by restriction and with the help of the Remark 2 of Section 2.

Let $p_M = \min_{i=0,\ldots,k}(p_i)$. As soon as $2 \leq p < p_M$, u_i belongs to $W^{1,p}(\Omega)$ and so, $u = \sum_{i=0,\ldots,k} u_i$ too. Moreover, there exists a real positive $C(p)$ such that

$$||u||_{W_0^{1,p}} \leq C(p) ||f||_{W^{-1,p}},$$

where $C(p)$ depends on all the $C_i(p)$, M_i and the norm of the transport operator T_J and T_H (see remark 2, section 2).
So we are done with the proof of Theorem 2.

Remarks :

1. The condition $\langle f, 1 \rangle_{(H^1)', H^1} = 0$ is necessary to have all the functions of $H^1(\Omega)$ as test functions. That is an important fact for the rest of the proof.

2. The inequality (7) is true only because H is an homeomorphism. Indeed, if J is differentiable almost everywhere (due to Rademacher Theorem), it is not sure for $J \circ H$...

4 Some Other Boundary Conditions

4.1 Fourier's Condition

The purpuse of this section is to give some other generalization of Meyers' Theorem for different boundary conditions. First, we consider Fourier's Condition, *i.e.*

$$A \nabla u . n + \lambda u = 0, \text{ on } \partial\Omega$$

where n denotes the outward normal on the boundary of Ω and λ a function $L^\infty(\partial\Omega)$ satisfying the following condition:

$$\exists \gamma > 0 \text{ such that, } \lambda(x) \geq \gamma \text{ for almost all } x \in \partial\Omega.$$

The rest of the notation is exactly the same as in the preceding section. We still consider a uniform elliptic operator, with coefficient in L^∞ defined on an open set Ω with a Lipschitz continuous boundary. The weak formulation of our new problem is then expressed by

$$\left\{ \begin{array}{l} u \in H^1(\Omega), \\ \displaystyle\int_\Omega A(x) \nabla u(x) . \nabla \varphi(x) \; dx + \int_{\partial\Omega} \lambda(x) u \varphi \; ds = \langle f, \varphi \rangle_{(H^1)', H^1}, \; \forall \varphi \in H^1(\Omega). \end{array} \right. \tag{10}$$

Once again, we want information about the regularity of the solution. The existence of solution can be proved by using Lax-Milgram theorem again. So let us express the regularity result.

Theorem 3 (Meyers Fourier) *Let Ω be a bounded connected open set of $\mathbb{R}^N$, with a Lipschitz continuous boundary. Let A of $(L^\infty(\Omega))^{N\times N}$ satisfy (2). For $p \geq 2$ and $q = p/p-1$, let T_p be the operator defined by $T_p(f) = u$ for $f \in (W^{1,q}(\Omega))'$, where u is the unique solution to (10). Then, there is a real number p_0, $2^* > p_0 > 2$, such that, for all p, $2 < p < p_0$, the operator T_p is linear continuous from $(W^{1,q}(\Omega))'$ to $W^{1,p}(\Omega)$. Moreover, the norm of T_p only depends on p, α, β and Ω and p_0 on α, β and Ω, not on f.*

The proof of this theorem works exactly as the preceding section. So let us consider only the differences. Fix p greater or equal to 2 and less than 2^*. Get $q = p/(p-1)$. For f in $(W^{1,q}(\Omega))'$, we have existence and unicity of solution to (10). So, u belongs to $H^1(\Omega)$. Let us just consider the following mapping

$$\varphi \to \int_{\partial\Omega} \lambda(x) u \varphi \, ds.$$

The trace of a function in $H^1(\Omega)$ is in $H^{1/2}(\partial\Omega)$. So using Sobolev injection (see [1]), we find that the trace of u belongs to $L^r(\partial\Omega)$ for all $r < 2(N-1)/(N-2)$ (let $r < \infty$ if $N = 2$). So the idea is to consider that our mapping can be defined on $W^{1,q}(\Omega)$, for $q < 2$. Computation shows that q must be greater than $2N/(N+2)$, hence that p must be less than $2N/(N-2)$. So, the term $\int_{\partial\Omega} \lambda(x) u \varphi \, ds$ can be brought in the operator f. It is possible now to reproduce the proof of preceding section.

4.2 The Dirichlet Problem Revisited

We claim here that the Meyers theorem is true on an open set with a Lipschitz continuous boundary. The proof doesn't work as before in the step 3. Indeed, it is not possible to extended our solution to B and find a new problem satisfy by the extension. We use a different way.
Let us consider only the following problem:

$$\left\{ \begin{array}{l} u \in H_0^1(B_+), \\ \displaystyle\int_{B_+} A(x)\nabla u(x).\nabla\varphi(x) \, dx = \langle f, \varphi \rangle_{H^{-1},H_0^1}, \ \forall \varphi \in H_0^1(B_+), \end{array} \right. \tag{11}$$

where A belongs to $(L^\infty(\Omega))^{N\times N}$ which satisfies the condition (2) and f belongs to $W^{-1,p}(B_+)$, $p > 2$. So, there exists a function F, of $(L^p(B_+))^N$ such that, for all $\varphi \in W_0^{1,q}(B_+)$,

$$\langle f, \varphi \rangle_{W^{-1,p},W_0^{1,q}} = \int_{B_+} F(x).\nabla\varphi(x) \, dx.$$

We define the function G on B by (we get, for all x in $\mathbb{R}^N$, $x = (x', x_N)$, x' in $\mathbb{R}^{N-1}$)

- if $x_N > 0$, $G(x', x_N) = F(x', x_N)$,
- if $x_N < 0$, for $i = 1, ..., N-1$, $G_i(x', x_N) = -F_i(x', -x_N)$ and $G_N(x', x_N) = F_N(x', -x_N)$.

Then G belongs to $(L^p(B))^N$ and we set, for all $\varphi \in W_0^{1,q}(B)$,

$$\langle g, \varphi \rangle_{W^{-1,p},W_0^{1,q}} = \int_B G(x).\nabla\varphi(x) \, dx.$$

For $x_n < 0$ we denote by $\tilde{A}$ the extension of A onto B, defined as:

- for all k and l less or equal than $N-1$, let $a^*_{kl}(x', x_N) = a_{kl}(x', -x_N)$,
- if $k = N$ or $l = N$ (but $(k,l) \neq (N,N)$), let $a^*_{kl}(x', x_N) = -a_{kl}(x', -x_N)$,
- $a^*_{NN}(x', x_N) = a_{NN}(x', -x_N)$.

We can now consider the following problem

$$\left\{\begin{array}{l} v \in H_0^1(B), \\ \displaystyle\int_B \tilde{A}(x)\nabla v(x).\nabla\varphi(x)\ dx = \langle g, \varphi\rangle_{H^{-1},H_0^1},\ \forall\varphi \in H_0^1(B). \end{array}\right. \tag{12}$$

Because Theorem 1, there exists $p_0 > 2$ such that the solution v of (12) belongs to $W_0^{1,p}(B)$ if g belongs to $W^{-1,p}(B)$, for $2 < p < p_0$. We want to prove that the restriction of v to B_+, denoted $v_{|B+}$, is equal to u.

Let us prove first that the trace of v on B^{N-1} is null. We get $w(x', x_N) = -v(x', -x_N)$. Due to the construction of g and $\tilde{A}$, w is a solution to (12). Then by unicity, $w = v$ in $H_0^1(B)$. For the trace operator γ on B^{N-1}, we have so

$$\gamma(v)(x') = \gamma(w)(x') = -\gamma(v)(x'),$$

then $\gamma(v) = 0$ on B^{N-1}.

Let φ be a function of $H_0^1(B_+)$. We can extend φ on B by zero, denoted $\tilde{\varphi}$. We can take $\tilde{\varphi}$ for test function in (12). Then

$$\int_B \tilde{A}(x)\nabla v(x).\nabla\tilde{\varphi}(x)\ dx = \langle g, \tilde{\varphi}\rangle_{H^{-1},H_0^1}.$$

But we have

$$\int_B \tilde{A}(x)\nabla v(x).\nabla\tilde{\varphi}(x)\ dx = \int_{B_+} A(x)\nabla v_{|B+}(x).\nabla\varphi(x)dx$$

and

$$\begin{aligned} \langle g, \tilde{\varphi}\rangle_{H^{-1},H_0^1} &= \int_B G(x).\nabla\tilde{\varphi}(x)dx \\ &= \int_{B_+} F(x).\nabla\varphi(x)dx \\ &= \langle f, \varphi\rangle_{H^{-1},H_0^1}. \end{aligned}$$

As we have seen that $v_{|B+}$ belongs to $H_0^1(B_+)$, we find finally that $v_{|B+}$ satisfies (11). By unicity, $v_{|B+} = u$ in $H_0^1(B_+)$, and so there exists a real $p_0 > 2$, such that u belongs to $W_0^{1,p}(B_+)$ if f belongs to $W^{-1,p}(B_+)$, for $2 < p < p_0$.

4.3 The mixed value boundary problem

We are interested in the mixed boundary value problem, *i.e.* u satisfies Dirichlet's Condition on a part $\tilde{\Gamma}$ of $\partial\Omega$ (with a non-zero $(N-1)$-dimensional measure) and a natural (Neumann or Fourier) boundary condition on $\Gamma = \partial\Omega\backslash\tilde{\Gamma}$. We need first a regularity condition on Γ. Here, we use some notations of [9], but the regularity condition on Γ are different. We set $\tilde{\Omega} = \Omega \cup \Gamma$.

Definition 2 *Let Ω be an open set with a Lipschitz continuous boundary. A measurable part Γ of $\partial\Omega$ is called* **regular**, *if there exists a family $(U_0,\ U_1,\ \ldots,\ U_k)$ of open sets of $\mathbb{R}^N$ satisfying (3) and $(J_1,\ \ldots,\ J_k)$ functions such that, for $i = 1,\ \ldots,\ k$, $J_i : U_i \to B$ is one-to-one, J_i and J_i^{-1} are Lipschitz continuous and we have one of the following condition*

a. $U_i \cap \Gamma = U_i \cap \partial\Omega$, and J_i satisfies (4).

b. $U_i \cap \Gamma = \emptyset$, and J_i satisfies (4).

c. $J_i(U_i \cap \Omega) = \{x \in B \mid x_N > 0$ and $x_{N-1} > 0\} = B_{++}$,
$J_i(U_i \cap \tilde{\Gamma}) = \{x \in B \mid x_N = 0$ and $x_{N-1} \geq 0\}$,
and $J_i(U_i \cap \Gamma) = \{x \in B \mid x_N > 0$ and $x_{N-1} = 0\}$.

Remarks

1. For $1 \leq p \leq \infty$, we denote $W_0^{1,p}(\tilde{\Omega})$ the closure of $\{u \in C_c^\infty(\mathbb{R}^N) \mid \text{supp} u \cap \tilde{\Gamma} = \emptyset\}$ in $W^{1,p}(\Omega)$.
2. When Γ is regular, the functions of $W_0^{1,p}(\tilde{\Omega})$ are the functions of $W^{1,p}(\Omega)$, null on $\tilde{\Gamma}$. In particular, if $\tilde{\Omega} = \Omega$, then $W_0^{1,p}(\tilde{\Omega}) = W_0^{1,p}(\Omega)$, of course. If $\tilde{\Omega} = \overline{\Omega}$, then $W_0^{1,p}(\tilde{\Omega}) = W^{1,p}(\Omega)$.
3. We denote $W^{-1,p}(\tilde{\Omega})$, the dual space of $W_0^{1,q}(\tilde{\Omega})$

Theorem 4 *Let Ω be a bounded connected open set with a Lipschitz continuous boundary of $\mathbb{R}^N$. Let Γ be a regular part of $\partial\Omega$ and $\tilde{\Gamma} = \partial\Omega \backslash \Gamma$. Suppose $\tilde{\Gamma}$ has a non-null $(N-1)$-dimensional measure. There is a real number p_0, $2^* \geq p_0 > 2$, such that, if u is the weak solution of*

$$\begin{cases} u \in W_0^{1,2}(\tilde{\Omega}) \\ \displaystyle\int_\Omega A(x)\nabla u(x).\nabla\varphi(x)dx = \langle f, \varphi\rangle_{W^{-1,2}(\tilde{\Omega}), W_0^{1,2}(\tilde{\Omega})}, \ \varphi \in W_0^{1,2}(\tilde{\Omega}), \end{cases} \tag{13}$$

where f belongs to $W^{-1,p}(\tilde{\Omega})$, for $p \in [2, p_0)$, then u belongs to $W_0^{1,p}(\tilde{\Omega})$ and there exists a real number $C(p)$ such that

$$||u||_{W_0^{1,p}(\tilde{\Omega})} \leq C(p)||f||_{W^{-1,p}(\tilde{\Omega})}.$$

Moreover, p_0 only depends on A and $\tilde{\Omega}$ and $C(p)$ on A, Ω and p, not on f.

.

Sketch of proof. We give here only the idea of the proof (due to J.Droniou, [6]). We need to study three cases

a) First, U_i satisfies a of Definition 2. The proof works exactly as the proof of Theorem 3 in Section 3.

b) U_i satisfies b of Definition 2. The proof works exactly as the proof in Section 4.2.

c) We are in the third case, c of the definition 2. We extend the solution first to B_+ by using reflection argument with respect to x_{N-1}, as the proof of Theorem 3 in Section 3. Then, we works exactly as the proof in Section 4.2 : we consider a new Dirichlet problem on B, and the restriction of the solution to B_{++} is well the researched function. Then we obtain $W^{1,p}$-estimate on u.

5 Application: A uniqueness theorem

Meyers' Theorem can notably be used to prove uniqueness of the solution of Dirichlet's problem for a linear elliptic differential equation with a 2-dimensional measure as right-hand side (see [8]).
One can now generalize this result to other boundary conditions. Regarding Neumann's Problem, for instance,

Theorem 5 *Let Ω be a bounded regular open set of $\mathbb{R}^N$. Let $N = 2$ and $\mu \in M(\Omega)$, $\int_\Omega 1 d\mu = 0$, where $M(\Omega)$ is the set of bounded Radon measures. Let A be in $(L^\infty(\Omega))^{N\times N}$, satisfying (2). Then, there exists a unique function u such that:*

$$\begin{cases} u \in \displaystyle\bigcap_{p<2} W^{1,p}(\Omega), \quad \int_\Omega u = 0, \\ \displaystyle\int_\Omega A(x)\nabla u(x).\nabla\varphi(x)\ dx = \int_\Omega \varphi(x)\ d\mu, \forall\varphi \in \bigcup_{q>2} W^{1,q}(\Omega). \end{cases} \tag{14}$$

Proof. [13], for instance, provides a proof of the existence of u. To prove its uniqueness, we show that if v satisfies

$$\begin{cases} v \in \bigcap_{p<2} W^{1,p}(\Omega), \quad \int_\Omega v = 0, \\ \int_\Omega A(x)\nabla v(x).\nabla\varphi(x)\ dx = 0, \forall\varphi \in \bigcup_{q>2} W^{1,q}(\Omega), \end{cases} \tag{15}$$

then v is null.

Indeed, suppose that v satisfies (15), and is not the null function. Let $B = \{x|v(x) > 0\}$. B is a measurable part. λ denotes the Lebesgue measure. By hypothesis, $\lambda(B) \neq 0$ and $\lambda(B) \neq \lambda(\Omega)$. Let $A^* = (a_{ji})_{i,j=1,2}$. Let ψ_B be the solution of the following problem

$$\begin{cases} \psi_B \in H^1(\Omega), \ \int_\Omega \psi_B(x)dx = 0, \\ \int_\Omega A^*(x)\nabla\psi_B(x).\nabla\varphi(x)dx = \lambda(B)^{-1}\int_B \varphi(x)dx - \lambda(\Omega - B)^{-1}\int_{\Omega-B}\varphi(x)\ dx,\ \forall\varphi \in H^1(\Omega). \end{cases} \tag{16}$$

One can then apply Theorem 2; as $\lambda(B)^{-1}\chi_B - \lambda(\Omega - B)^{-1}\chi_{\Omega-B}$ is an element of $L^\infty(\Omega)$ and its mean is null, there is a $\bar{q} > 2$ (which depends on A and Ω only, not on B) such that $\psi_B \in W^{1,\bar{q}}(\Omega)$. $\varphi = \psi_B$ can hence be chosen in (15):

$$\int A(x)\nabla v(x).\nabla\psi_B(x)\ dx = 0. \tag{17}$$

As $\bar{q}' = \bar{q}/(\bar{q} - 1) < 2$, we have $v \in W^{1,\bar{q}'}(\Omega)$. There exists a sequence of functions $(\varphi_n)_{n\in\mathbb{N}}$ de $H^1(\Omega)$ such that $(\varphi_n)_{n\in\mathbb{N}}$ converges to v in $W^{1,\bar{q}'}(\Omega)$. Next, choose $\varphi = \varphi_n$ in (16):

$$\int_\Omega A^*(x)\nabla\psi_B(x).\nabla\varphi_n(x)\ dx = \lambda(B)^{-1}\int_B \varphi_n(x)\ dx - \lambda(\Omega - B)^{-1}\int_{\Omega-B}\varphi_n(x)\ dx.$$

If n becomes infinite, we get:

$$\int_\Omega A^*(x)\nabla\psi_B(x).\nabla v(x)\ dx = \lambda(B)^{-1}\int_B v(x)\ dx - \lambda(\Omega - B)^{-1}\int_{\Omega-B} v(x)\ dx.$$

Now $A^*\nabla\psi_B.\nabla v = A\nabla v.\nabla\psi_B$. Hence, using (17) and $\int_\Omega v = 0$, we obtain $\int_B v(x)\ dx = 0$, which is impossible.

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