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CONVERGENCE AND ERROR ESTIMATES FOR THE COMPRESSIBLE NAVIER-STOKES EQUATIONS

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Abstract. We are interested in the paper by the discretization of the (unsteady and stationary) compressible (isentropic) Navier-Stokes Equations with the Marker-And- Cell scheme. We present recent results for the convergence (as the discretization parameter goes to zero) of the approximate solutions to a weak solution of the continuous equations and error estimates when the solution of the continuous equations is regular enough.

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§1. Introduction

I present in this paper some results obtained with R. Eymard, R. Herbin, J. C. Latché, D. Maltese and A. Novotny.

Let Ω be a bounded open connected set of $\mathbb{R}^3$ with a Lipschitz continuous boundary, $T > 0$, $\gamma > 3/2$, $\mathbf{u_0} \in L^2(\Omega)$, $\rho_0 \in L^\gamma(\Omega)$ and $\mathbf{f} \in L^2(]0,T[, L^2(\Omega)^3)$. The compressible Navier-Stokes equations read

$$\partial_t\rho + \mathrm{div}(\rho\mathbf{u}) = 0 \text{ in } \Omega\times]0,T[, \qquad \text{(mass equation)} \quad (1)$$

$$\partial_t(\rho\mathbf{u}) + \mathrm{div}(\rho\mathbf{u}\otimes\mathbf{u}) - \Delta\mathbf{u} + \mathrm{grad}\, p = \mathbf{f} \text{ in } \Omega\times]0,T[, \qquad \text{(momentum equation)} \quad (2)$$

$$p = \rho^\gamma \text{ in } \Omega\times]0,T[. \qquad \text{(Equation Of State)} \quad (3)$$

To this system, we add a Dirichlet boundary condition,

$$\mathbf{u} = 0 \text{ on } \partial\Omega\times]0,T[, \quad (4)$$

and an initial condition

$$\mathbf{u}(\cdot,\mathbf{0}) = \mathbf{u_0},\ \rho(\cdot,0) = \rho_0 \text{ on } \partial\Omega. \quad (5)$$

The main unknowns of Problem (1)-(5) are $\mathbf{u}$ and ρ (then, p is given with (3)). Under the assumption $\rho_0 > 0$ a.e. on Ω and $\int_\Omega(\frac{1}{2}\rho_0|\mathbf{u_0}|^2 + \rho_0^\gamma/(\gamma-1))dx < +\infty$, existence of a weak solution $(\mathbf{u},\rho)$ to (1)-(5) is known (but no uniqueness in general) since the works of P.-L. Lions [18] and E. Feireisl and coauthors [5], [6]. This weak solution sastifies $\rho \in L^\infty(]0,T[, L^\gamma(\Omega))$, $\rho \geq 0$ a.e., $\mathbf{u} \in L^2([0,T[, H_0^1(\Omega)^3)$ and $\rho|\mathbf{u}|^2 \in L^\infty(]0,T[, L^1(\Omega))$. Futhermore, $\int_\Omega \rho(x,t)dx = \int_\Omega \rho_0(x)dx$ a.e.. In particular, such a weak solution has a finite energy. More precisely, for a.e. t in $]0,T[$, if $\mathbf{f} = \mathbf{0}$,

$$\int_\Omega (\frac{1}{2}\rho|\mathbf{u}|^2 + \frac{\rho^\gamma}{\gamma-1})(t)\,dx + \int_0^t\int_\Omega |\,\mathrm{grad}\,\mathbf{u}|^2\,dxd\tau \leq \int_\Omega(\frac{1}{2}\rho_0|\mathbf{u_0}|^2 + \frac{\rho_0^\gamma}{\gamma-1})dx. \quad (6)$$

It is said that this weak solution is a "suitable" solution.

We are also interested by the stationary compressible Navier Stokes equations. In this case, Ω is a bounded open set of $\mathbb{R}^3$, with a Lipschitz continuous boundary, $\gamma > 3/2$, $\mathbf{f} \in L^2(\Omega)^3$ and $M > 0$. The equations read

$$\operatorname{div}(\rho\mathbf{u}) = 0 \text{ in } \Omega, \tag{7}$$

$$\operatorname{div}(\rho\mathbf{u} \otimes \mathbf{u}) - \Delta\mathbf{u} + \operatorname{grad} p = \mathbf{f} \text{ in } \Omega, \tag{8}$$

$$p = \rho^\gamma \text{ in } \Omega. \tag{9}$$

To this system, we add a Dirichlet boundary condition,

$$\mathbf{u} = 0 \text{ on } \partial\Omega\times]0,T[, \tag{10}$$

and

$$\rho \geq 0 \text{ a.e. } \int_\Omega \rho(x)dx = M. \tag{11}$$

Here also, the main unknowns of Problem (7)-(11) are $\mathbf{u}$ and ρ (and p is given with (9)). Existence of a weak solution $(\mathbf{u}, \rho)$ to (7)-(11) is known (but no uniqueness) with $\mathbf{u} \in H_0^1(\Omega)^3$ and $\rho \in L^\gamma(\Omega)$, at least for $\gamma > 5/3$, see for instance [19], [20]. Indeed, the "optimal" space for this weak solution depends on γ (except for $\mathbf{u}$ which always belongs to $H_0^1(\Omega)^3$). If $\gamma > 3$, $\rho \in L^{2\gamma}(\Omega)$ and then $p \in L^2(\Omega)$. If $\gamma < 3$, $\rho \in L^{\gamma\delta}(\Omega)$, with $\delta = 3(\gamma-1)/\gamma$, and then $p \in L^\delta(\Omega)$. In particular, the function ρ belongs to $L^2(\Omega)$ for $\gamma \geq 5/3$.

Remark 1. For $\gamma = 3/2$, one has $\bar{q} = 3(\gamma - 1)/\gamma = 1$, and $\gamma\delta = 3(\gamma - 1) = 3/2$, so that the natural spaces for $p, \rho, \mathbf{u}$ seem to be $p \in L^1(\Omega)$, $\rho \in L^{\frac{3}{2}}(\Omega)$, $\mathbf{u} \in H_0^1(\Omega)^3$. Using the Sobolev embedding $H_0^1(\Omega) \subset L^6(\Omega)$ these natural spaces gives $\rho\mathbf{u} \otimes \mathbf{u} \in L^1(\Omega)^3$. This is a reason for the limitation $\gamma > 3/2$. However, in the case of the stationary compressible Stokes equations (that is without this term $\rho\mathbf{u} \otimes \mathbf{u}$ in (8)), one has a weak solution with $p \in L^2(\Omega)$ (and $\rho \in L^{2\gamma}(\Omega)$) and there is no restriction on γ in the sense that we can take $\gamma \geq 1$ (see for instance [4, 3] for $\gamma > 1$ and [9] for $\gamma = 1$).

For this two problems (Compressible Navier-Stokes Equations and Stationary Compressible Navier-Stokes Equations, namely Problem (1)-(5) and Problem (7)-(11)) we are interested by the discretized models obtained with the Marker-And-Cell scheme (MAC in short) and, for the unsteady problem, with an implicit discretization in time. The reason of this choice is that the MAC scheme is widely used in computational fluid dynamics. It was introduced in [16] and considered (since the beginning) as a suitable space discretization for both incompressible and compressible flow problems (see [14, 15] for the seminal papers and [23] for a review). We refer to [3], [10], [12] for a description of the MAC scheme. Of course, we have to consider a domain Ω adapted to the discretization by the MAC scheme.

Admitting the existence of an approximate solution, that is a solution of the discretized problem (this existence can be proven), two questions ares interesting:

1. Is it possible to prove convergence (up to the subsequence) of the approximate solution to the weak solution of the continuous problem as the mesh size goes to 0 (and also the time step in the evolution case) ?
2. In case of uniqueness of the solution of the continuous problem, is it possible to obtain error estimates and what are they ?

The answer for this two questions are partially known, it remains some open questions (and the known results are completely different between the unsteady case and the steady case) :

1. For the stationary compressible Navier-Stokes problem (namely (7)-(11)), we prove, for $\gamma > 3$, convergence (up to the subsequence) of the approximate solution to the (weak) solution of (7)-(11) as the mesh size goes to 0, see [10]. But it is an open problem for $3/2 < \gamma \leq 3$. Note that for $\gamma > 3$ the proof of convergence given in [10] also gives existence of a weak solution to (7)-(11) since the existence of an approximate solution is also proven in [10].

2. For the compressible Navier-Stokes problem (namely (1)-(5)), the convergence of the approximate solution, up to a subsequence, to the solution of the continuous problem is probably true, but we do not have a complete proof.

3. For the compressible Navier-Stokes problem (namely (1)-(5)), if the solution of the continuous problem is regular enough (then we call it a "strong solution"), we obtain, for $\gamma > 3/2$ an error estimate, cf. [12] for the case $\mathbf{f} = \mathbf{0}$. The rate of convergence obtained in [12] depends on γ and is probably not optimal.

4. For the stationary compressible Navier-Stokes problem, even when the solution of the continuous problem is regular, we are not able to obtain error estimates.

Remark 2. It is possible to obtain some convergence results or some error estimates with other schemes than the MAC scheme. For instance, a convergence result is given for the unsteady compressible Navier-Stokes equations in [17] with a FV-FE scheme, albeit only in the case $\gamma > 3$ (the difficulty in the realistic case $\gamma \leq 3$ arise from the treatment of the non linear convection term). Some error estimates (when the solution of these unsteady compressible Navier-Stokes equations is regular enough) have been derived for this FV-FE scheme in [11] if $\gamma > 3/2$.

§2. Error estimates

2.1. For the compressible Navier-Stokes problem

For the compressible Navier-Stokes problem the proof of an error estimate, that is the comparison of a "strong" solution of Problem (1)-(5) and an approximate solution (that is a solution given by the MAC-scheme in space and an Euler-backward scheme in time) is very close to the so called "weak-strong uniqueness principle", which is the comparison of a "strong" solution and a weak solution of Problem (1)-(5). Indeed, the weak-strong uniqueness principle states that if Problem (1)-(5) has a regular enough solution (the main hypothesis on the solution is the fact that $\operatorname{div} \mathbf{u} \in L^1(]0,T[, L^\infty(\Omega))$ and $\operatorname{grad} p \in L^1(]0,T[, L^\infty(\Omega)))$ then Problem (1)-(5) has a unique weak solution (and this solution is equal to the strong solution).

This idea of the weak-strong uniqueness principle comes back to G. Prodi [21] (1959) and J. Serrin [22] (1963) for the case of Incompressible Navier-Stokes Equations. For the compressible isentropic Navier-Stokes equations, the first result is probably in [13]. More general Equation Of State are considered in [6].

For the compressible Navier-Stokes equations, the proof of this weak-strong uniqueness principle uses the so-called “relative entropy” introduced by C. M. Dafermos for Euler Equations [2]. In other papers, the “relative entropy” is called “modulated energy”. We will use below the term “relative energy” which seems to be more adapted to our system of equations.

We first describe in Sec. 2.2 this weak-strong uniqueness principle in a very simple case containing the main idea of the method.

2.2. Weak-strong uniqueness principle, simple case

We present in this section the weak-strong uniqueness principle in the case of the compressible Stokes equations with $\gamma = 2$ and $\mathbf{f} = \mathbf{0}$. The set Ω is still a bounded open connected set of $\mathbb{R}^3$, with a Lipschitz continuous boundary and $T > 0$. The problem read

$$\partial_t \rho + \operatorname{div}(\rho \mathbf{u}) = 0 \text{ in } \Omega\times]0,T[, \tag{12}$$

$$\partial_t \mathbf{u} - \Delta \mathbf{u} + \operatorname{grad} p = \mathbf{0} \text{ in } \Omega\times]0,T[, \tag{13}$$

$$p = \rho^2 \text{ in } \Omega\times]0,T[. \tag{14}$$

with a Dirichlet boundary condition,

$$\mathbf{u} = 0 \text{ on } \partial\Omega\times]0,T[, \tag{15}$$

and an initial condition

$$\mathbf{u}(\cdot,\mathbf{0}) = \mathbf{u_0},\ \rho(\cdot,0) = \rho_0 \text{ on } \partial\Omega. \tag{16}$$

Let $(\bar{\mathbf{u}}, \bar\rho, \bar p)$ be a regular solution of (12)-(16) (we call it ”strong solution”) and let $(\mathbf{u}, \rho, p)$ be a suitable weak solution of (12)-(16).

The idea of the proof is to use a Gronwall inequality on the “relative energy" between $(\mathbf{u}, \rho)$ and $(\bar{\mathbf{u}}, \bar\rho)$ which reads in this case (Stokes Equations, $\gamma = 2$), for $t \in [0,T]$,

$$E_t(\mathbf{u},\rho \,|\, \bar{\mathbf{u}},\bar\rho) = \int_\Omega (\frac{1}{2}|\mathbf{u}(t) - \bar{\mathbf{u}}(t)|^2 + |\rho(t) - \bar\rho(t)|^2)\, dx.$$

Note that this quantity is indeed well defined for any t, thanks to some continuity which can be proven for $\mathbf{u}$ and ρ. We now transform formally the quantity $E_t(\mathbf{u},\rho \,|\, \bar{\mathbf{u}},\bar\rho)$ in three steps, using (12)-(16).

Step 1 Energy Inequalities for the suitable weak solution and for the strong solution

We formally take $\mathbf{u}$ as test function in the momentum equation for $\mathbf{u}$ (Equation (13)) to obtain, for $t \in [0,T]$,

$$\frac{1}{2}\int_\Omega |\mathbf{u}|^2(t)\, dx + \int_0^t \int_\Omega (|\operatorname{grad}\mathbf{u}|^2 - p \operatorname{div}\mathbf{u})\, dx d\tau = \frac{1}{2}\int_\Omega |\mathbf{u_0}|^2 dx. \tag{17}$$

We formally take ρ as test function in the mass equation (Equation (12)) to obtain

$$\frac{1}{2}\int_\Omega \rho^2(t)\, dx - \frac{1}{2}\int_\Omega \rho_0^2\, dx - \int_0^t \int_\Omega \rho\mathbf{u} \cdot \operatorname{grad}\rho\, dx d\tau = 0.$$

But, since $\rho^2 = p$,

$$\int_0^t \int_\Omega \rho \mathbf{u} \cdot \operatorname{grad} \rho \, dx d\tau = \frac{1}{2} \int_0^t \int_\Omega \mathbf{u} \cdot \operatorname{grad}(\rho^2) \, dx d\tau = -\frac{1}{2} \int_0^t \int_\Omega p \operatorname{div} \mathbf{u} \, dx d\tau$$

and then

$$\int_\Omega \rho^2(t) \, dx + \int_0^t \int_\Omega p \operatorname{div} \mathbf{u} \, dx d\tau = \int_\Omega \rho_0^2 \, dx. \tag{18}$$

Then, adding Equations (17) and (18) gives for all $t \in [0, T]$,

$$\frac{1}{2} \int_\Omega |\mathbf{u}|^2(t) \, dx + \int_\Omega \rho^2(t) \, dx + \int_0^t \int_\Omega (|\operatorname{grad} \mathbf{u}|^2 \, dx d\tau = \frac{1}{2} \int_\Omega |\mathbf{u_0}|^2 dx + \int_\Omega \rho_0^2 \, dx.$$

Indeed, this is Inequality (6) with an equality instead of an inequality, but the computation here is formal. For the suitable weak solution, one has Inequality (6) which is here

$$\frac{1}{2} \int_\Omega |\mathbf{u}|^2(t) \, dx + \int_\Omega \rho^2(t) \, dx + \int_0^t \int_\Omega (|\operatorname{grad} \mathbf{u}|^2 \, dx d\tau \leq \frac{1}{2} \int_\Omega |\mathbf{u_0}|^2 dx + \int_\Omega \rho_0^2 \, dx. \tag{19}$$

For the the strong solution (which is "more" than a suitable weak solution), one has also

$$\frac{1}{2} \int_\Omega |\bar{\mathbf{u}}|^2(t) \, dx + \int_\Omega \bar{\rho}^2(t) \, dx + \int_0^t \int_\Omega (|\operatorname{grad} \bar{\mathbf{u}}|^2 \, dx d\tau \leq \frac{1}{2} \int_\Omega |\mathbf{u_0}|^2 dx + \int_\Omega \rho_0^2 \, dx. \tag{20}$$

Using (19) and (20), for all t,

$$\begin{aligned} E_t(\mathbf{u}, \rho \,|\, \bar{\mathbf{u}}, \bar{\rho}) = \int_\Omega (\frac{1}{2} |\mathbf{u}(t) - \bar{\mathbf{u}}(t)|^2 + |\rho - \bar{\rho}|^2) \, dx \leq \\ - \int_\Omega \mathbf{u}(t) \cdot \bar{\mathbf{u}}(t) \, dx - 2 \int_\Omega \rho(t) \bar{\rho}(t) \, dx - \int_0^t \int_\Omega (|\operatorname{grad} \mathbf{u}|^2 + |\operatorname{grad} \bar{\mathbf{u}}|^2) \, dx d\tau \\ + \int_\Omega |\mathbf{u_0}|^2 \, dx + 2 \int_\Omega |\rho_0|^2 \, dx, \end{aligned} \tag{21}$$

We have now to transform the two first terms of the right hand side of (21).

Step 2 Transformation of $\int_\Omega \rho(t) \bar{\rho}(t) \, dx$.

Using the regularity of the strong solution, we can take $\bar{\rho}$ as test function in the mass equation for the weak solution (Equation (12)) and ρ as test function in the mass equation for the strong solution. This gives

$$\int_0^t \int_\Omega (\partial_t \rho) \bar{\rho} \, dx d\tau - \int_0^t \int_\Omega \rho u \cdot \operatorname{grad} \bar{\rho} \, dx d\tau = 0,$$

$$\int_0^t \int_\Omega (\partial_t \bar{\rho}) \rho \, dx d\tau + \int_0^t \int_\Omega \operatorname{div}(\bar{\rho} \bar{u}) \rho \, dx d\tau = 0.$$

The non-symmetry between these two equalities is due to fact that $(\mathbf{u}, \rho)$ is only a weak solution. Adding the two equations leads to

$$\int_\Omega \bar{\rho}(t) \rho(t) \, dx - \int_\Omega \rho_0^2 \, dx = \int_0^t \int_\Omega \rho \mathbf{u} \cdot \operatorname{grad} \bar{\rho} \, dx d\tau - \int_0^t \int_\Omega \operatorname{div}(\bar{\rho} \bar{\mathbf{u}}) \rho \, dx d\tau. \tag{22}$$

Step 3 Transformation of $\int_\Omega \mathbf{u}(t)\cdot\bar{\mathbf{u}}(t)\,dx$.

Using, here also, the regularity of the strong solution, we can take $\bar{\mathbf{u}}$ as test function in the momentum equation for the weak solution (Equation (13)) and $\mathbf{u}$ as test function in the momentum equation for the strong solution. This gives

$$\int_0^t\int_\Omega(\partial_t\mathbf{u})\bar{\mathbf{u}}\,dxd\tau+\int_0^t\int_\Omega(\operatorname{grad}\mathbf{u}:\operatorname{grad}\bar{\mathbf{u}}-p\operatorname{div}(\bar{\mathbf{u}}))\,dxd\tau=0,$$

$$\int_0^t\int_\Omega(\partial_t\bar{\mathbf{u}})\mathbf{u}\,dxd\tau+\int_0^T\int_\Omega(\operatorname{grad}\mathbf{u}:\operatorname{grad}\bar{\mathbf{u}}+\mathbf{u}\cdot\operatorname{grad}\bar{p})\,dxd\tau=0.$$

Adding the two equations leads to

$$\int_\Omega\bar{\mathbf{u}}(t)\cdot\mathbf{u}(t)\,dx-\int_\Omega|\mathbf{u_0}|^2\,dx=\int_0^t\int_\Omega(-2\operatorname{grad}\mathbf{u}:\operatorname{grad}\bar{\mathbf{u}}+p\operatorname{div}(\bar{\mathbf{u}})-\mathbf{u}\cdot\operatorname{grad}\bar{p})\,dxd\tau.\quad(23)$$

Step 4 End of the proof of the weak strong uniqueness principle

We use (22) and (23) to transform (21). We obtain

$$E_t(\mathbf{u},\rho\,|\,\bar{\mathbf{u}},\bar{\rho})\le-\int_0^t\int_\Omega|\operatorname{grad}\mathbf{u}-\operatorname{grad}\bar{\mathbf{u}}|^2\,dxd\tau-\int_0^t\int_\Omega(p\operatorname{div}(\bar{\mathbf{u}})-\mathbf{u}\cdot\operatorname{grad}\bar{p})\,dxd\tau$$
$$-2\int_0^t\int_\Omega\rho\mathbf{u}\cdot\operatorname{grad}\bar{\rho}\,dxd\tau+2\int_0^t\int_\Omega\operatorname{div}(\bar{\rho}\bar{\mathbf{u}})\rho\,dxd\tau.$$

Using $p=\rho^2$, $\bar{p}=\bar{\rho}^2$, $\int_0^t\int_\Omega\operatorname{div}(\bar{\rho}\bar{\mathbf{u}})\rho\,dxd\tau=\int_0^t\int_\Omega(\bar{\rho}\rho\operatorname{div}(\bar{\mathbf{u}})+\rho\bar{\mathbf{u}}\cdot\operatorname{grad}\bar{\rho})\,dxd\tau$ and $\int_0^t\int_\Omega(\operatorname{div}(\bar{\mathbf{u}})\bar{\rho}^2+2\bar{\mathbf{u}}\bar{\rho}\cdot\operatorname{grad}\bar{\rho})dxd\tau=0$, this inequality can be rewritten as

$$E_t(\mathbf{u},\rho\,|\,\bar{\mathbf{u}},\bar{\rho})\le\int_0^t\int_\Omega(-|\operatorname{grad}\mathbf{u}-\operatorname{grad}\bar{\mathbf{u}}|^2-(\rho-\bar{\rho})^2\operatorname{div}(\bar{\mathbf{u}})-2(\bar{\rho}-\rho)(\bar{\mathbf{u}}-\mathbf{u})\cdot\operatorname{grad}\bar{\rho})\,dxd\tau.$$

and then

$$E_t(\mathbf{u},\rho\,|\,\bar{\mathbf{u}},\bar{\rho})\le\int_0^t\int_\Omega(-(\rho-\bar{\rho})^2\operatorname{div}(\bar{\mathbf{u}})-2(\bar{\rho}-\rho)(\bar{\mathbf{u}}-\mathbf{u})\cdot\operatorname{grad}\bar{\rho})\,dxd\tau.\qquad(24)$$

Setting $\varphi(t)=E_t(\rho,\mathbf{u}|\bar{\rho},\bar{\mathbf{u}})=\frac{1}{2}\int_\Omega|\mathbf{u}(t)-\bar{\mathbf{u}}(t)|^2\,dx+\int_\Omega(\rho(t)-\bar{\rho}(t))^2\,dx$, using Cauchy-Schwarz Inequality for the last term, we obtain from (24), since $\operatorname{div}\bar{\mathbf{u}}\in L^1(]0,T[,L^\infty(\Omega))$ and $\operatorname{grad}\bar{\rho}\in L^1(]0,T[,L^\infty(\Omega))$,

$$\varphi(t)\le C\int_0^t a(\tau)\varphi(\tau)d\tau\text{ for all }t\in[0,T],$$

with some $a\in L^1(]0,T[)$. This gives, by Gronwall Inequality, $\varphi(t)\le\varphi(0)e^{\int_0^t a(\tau)d\tau}$ and then, since $\varphi(0)=0$, $\varphi(t)=0$ for all $t\in[0,T]$. The weak-strong uniqueness principle is then proven for this simple case (compressible Stokes equations with $\gamma=2$ and $\mathbf{f}=\mathbf{0}$).

2.3. Error estimate for the compressible Navier Stokes equations

We consider here the compressible Navier Stokes system (1)-(5) with $\mathbf{f} = \mathbf{0}$, $\gamma > 3/2$ and a domain Ω adapted to the MAC scheme (for instance, $\Omega =]0,1[^3$). Mimicking the previous proof of uniqueness (given in Sec. 2.2) at the discrete level it is possible to obtain error estimates, that is an estimate between a strong solution (we assume existence of such a solution) and the approximate solution given by a numerical scheme (roughly speaking it is not so far of a weak solution with some errors due to the discretization). Instead of a suitable weak solution (ρ, u), we use now the solution of the scheme (that is the solution obtained with a space discretization using the MAC scheme and an Euler-backward discretiztation in time). This numerical solution is denoted by $(\mathbf{u}, \rho)$ and the strong solution is denoted by $(\mathbf{u}, \bar{\rho})$. The energy is now

$$E_t(\mathbf{u}, \rho \,|\, \bar{\mathbf{u}}, \bar{\rho}) = \int_\Omega (\frac{1}{2}\rho|u(t) - \bar{u}(t)|^2 + e(\rho(t)|\bar{\rho}(t)))\, dx,$$

with $e(\rho|\bar{\rho}) = \rho^\gamma - \bar{\rho}^{\gamma-1}\gamma(\rho - \bar{\rho}) - \bar{\rho}^\gamma$. Note that $e(\rho|\bar{\rho}) = 0$ if and only if $\rho = \bar{\rho}$.

If h is the mesh size and k the time step, the error estimate given in [12] is

$$E_t(\rho, u|\bar{\rho}, \bar{u}) \leq C(h^\alpha + k^{1/2}) \text{ for all } 0 \leq t \leq T,$$

where C depends only on the strong solution and on the regularity of the mesh and $\alpha = \min(\frac{2\gamma-3}{\gamma}, \frac{1}{2})$. For $\gamma = 2$, one has $\alpha = 1/2$ and E_t is the L^2-norm of $(\rho - \bar{\rho})$ plus the L^2-norm of $(u - \bar{u})$ weighted by ρ (and we have $\rho > 0$ a.e.).

2.4. For the stationary compressible Navier-Stokes problem

We are not able to give error estimates for the stationary compressible Navier-Stokes problem (that is problem (7)-(11)) as we did for the compressible Navier-Stokes problem in Sec. 2.3. The proof in Sec. 2.3 follows closely the proof of the weak-strong uniqueness principle. A crucial tool in the proof of weak-strong uniqueness principle is the use of the Gronwall inequality. Then a natural question is "What can play the role of Gronwall Inequality for stationary problems" ?

We present below a very simple example where uniqueness in the unsteady case follows easily from the Gronwall inequality and uniqueness is also true in the stationary case, with a trick which has some similarity with the Gronwall inequality. Unfortunately, we are not able to adapt the same trick in the case of the stationary compressible Navier-Stokes problem.

Let Ω be a bounded open set of $\mathbb{R}^3$, $T > 0$, $\mathbf{w} \in L^\infty(\Omega)^3$, $f \in L^2(]0,T[, L^2(\Omega))$, $u_0 \in L^2(\Omega)$ and φ be Lipschitz continuous function from $\mathbb{R}$ to $\mathbb{R}$. We consider the following problem,

$$\begin{aligned} \partial_t u + \operatorname{div}(\mathbf{w}\varphi(u)) - \Delta u = f \text{ in } \Omega\times]0,T[, \\ u(\cdot, t) = 0 \text{ on } \partial\Omega \text{ for all } t \in]0,T[, \\ u(\cdot, 0) = u_0 \text{ on } \partial\Omega. \end{aligned}$$

For this problem, one has existence of the solution in the space $L^2(]0,T[, H_0^1(\Omega))$ and the solution is continuous with value in $L^2(\Omega)$. Uniqueness easily follows from a Gronwall inequality.

We now consider the stationary case, that is $f \in L^2(\Omega)$ (and still $\mathbf{w} \in L^\infty(\Omega)^3$, φ Lipschitz continuous) and the stationary problem reads

$$\begin{aligned} \operatorname{div}(\mathbf{w}\varphi(u)) - \Delta u &= f \text{ in } \Omega, \\ u(\cdot, t) &= 0 \text{ on } \partial\Omega. \end{aligned}$$

Note that we do not have any hypothesis on $\operatorname{div}(\mathbf{w})$. Then, we may have a non-coercive differential operator.

For this problem, it is possible to prove existence in the space $H_0^1(\Omega)$ (for instance cf. [8], Exercice 3.5). But, for this problem, it is also possible to prove uniqueness. If u and $\bar{u}$ are two solutions, the idea is to take $T_\varepsilon(u - \bar{u})$ ($\varepsilon > 0$) as test function, where $T_\varepsilon(s) = \max(-\varepsilon, \min(s, \varepsilon))$ for $s \in \mathbb{R}$.

Using in particular Sobolev Injection of $W_0^{1,1}(\Omega)$ in $L^{1^\star}(\Omega)$ (with $1^\star = 3/2$ since $\Omega \subset \mathbb{R}^3$) and letting $\varepsilon \to 0$ allows us to conclude $u = \bar{u}$ a.e.. (for instance, cf. [1] or [8] Exercice 3.6.)

§3. Convergence results

3.1. For the stationary compressible Navier-Stokes problem

For the stationary compressible Navier-Stokes equations (7)-(11) discretized with a MAC scheme (of course, we assume that Ω is adapted to the MAC scheme), we prove (cf. [10]) convergence of the approximate solution (up to a subsequence) to a weak solution, in the case $\gamma > 3$ (and $f \in L^2(\Omega)^3$, $M > 0$) following the idea of P.L. Lions (cf. [18]) for proving existence of a solution.

Let $(\mathbf{u}_n, p_n, \rho_n)_{n\in\mathbb{N}}$ be a sequence a approximate solutions obtained with the MAC scheme (existence of such an approximate solution is proven, cf. [10]). We assume $\lim_{n\to+\infty} h_n = 0$, where h_n is the mesh size. The steps for proving the convergence result are

1. Estimates on the approximate solution $(\mathbf{u}_n, p_n, \rho_n)$;
2. Compactness result (convergence of the approximate solution, up to a subsequence);
3. Passage to the limit in the approximate equations.

The main difficulty is in the passage to the limit in the EOS ($p = \rho^\gamma$) since the EOS is a non linear function and Step 2 only leads to weak convergences of p_n and ρ_n.

The estimate on $\mathbf{u}_n$ is with a norm which mimic (at the discrete level) the $H_0^1(\Omega)^3$-norm. The estimate on p_n is in $L^2(\Omega)$-norm (thanks to $\gamma \geq 3$) and the estimate on ρ_n is in $L^{2\gamma}(\Omega)$-norm. Thanks to these estimates on $\mathbf{u}_n$, p_n, ρ_n, it is possible to assume (up to a subsequence) that, as $n \to +\infty$,

$$\begin{gathered} \mathbf{u}_n \to \mathbf{u} \text{ in } L^q(\Omega)^3 \text{ for } q < 6 \text{ and weakly in } L^6(\Omega)^3,\ \mathbf{u} \in H_0^1(\Omega)^3, \\ p_n \to p \text{ weakly in } L^2(\Omega), \quad \rho_n \to \rho \text{ weakly in } L^{2\gamma}(\Omega). \end{gathered}$$

We show now how to pass to the limit in the equations. For simplicity we will asume that $(\mathbf{u}_n, p_n, \rho_n)$ is a weak solution of (7)-(11) with $\mathbf{f}_n$ instead of $\mathbf{f}$, and $\mathbf{f}_n \to \mathbf{f}$ weakly in $L^2(\Omega)^3$ as $n \to +\infty$. The passage to the limit in the equation when $(\mathbf{u}_n, p_n, \rho_n)$ is an approximate solution given by MAC scheme follows the same lines, with some modifications that we indicate when there are interesting.

For the mass equation, let $v \in C_c^\infty(\mathbb{R}^3)$, one has

$$\int_\Omega \rho_n \mathbf{u}_n \cdot \operatorname{grad} v = 0, \tag{25}$$

Since $\rho_n \to \rho$ weakly in $L^{2\gamma}(\Omega)$, with $2\gamma > 6/5$, and $\mathbf{u}_n \to \mathbf{u}$ in $L^q(\Omega)^3$ for all $q < 6$. Then $\rho_n \mathbf{u}_n \to \rho\mathbf{u}$ weakly in $L^1(\Omega)^3$. This gives $\int_\Omega \rho\mathbf{u} \cdot \operatorname{grad} v = 0$. Indeed, at the discrete level, in Equation (25), there is an additional term which allows us to prove $\int_\Omega \rho_n \, dx = M$. This term vanishes as $n \to +\infty$ since it is of ordre h_n^α, where $\alpha \in]0,1[$ is a given parameter (cf. [10]).

The L^1-weak convergence of ρ_n gives non negativity of ρ and convergence of the total mass, that is $\rho \geq 0$ a.e. in Ω, $\int_\Omega \rho(x)dx = M$. For the momentum equation, let $\mathbf{v} \in C_c^\infty(\Omega)^3$,

$$\int_\Omega \operatorname{grad} \mathbf{u}_n : \operatorname{grad} \mathbf{v} \, dx - \int_\Omega \rho_n \mathbf{u}_n \otimes \mathbf{u}_n : \operatorname{grad} \mathbf{v} \, dx - \int_\Omega p_n \operatorname{div}(\mathbf{v}) dx = \int_\Omega \mathbf{f}_n \cdot \mathbf{u} \, dx \tag{26}$$

This is also true at the discrete level with an error term (vanishing as $n \to +\infty$) and a discret operator grad_n (acting on $\mathbf{u}_n$) mimicking grad. One has, as $n \to +\infty$, $\operatorname{grad} \mathbf{u}_n \to \operatorname{grad} \mathbf{u}$ weakly in $L^2(\Omega)^3$ (this is also true at the discrete level with grad_n isntead of grad). Furthemore, using $\rho_n \to \rho$ weakly in $L^{2\gamma}(\Omega)$, with $2\gamma > 3/2$, and $\mathbf{u}_n \to \mathbf{u}$ in $L^q(\Omega)^3$ for all $q < 6$ (and $\frac{2}{3} + \frac{1}{6} + \frac{1}{6} = 1$), $\rho_n \mathbf{u}_n \otimes \mathbf{u}_n \to \rho\mathbf{u} \otimes \mathbf{u}$ weakly in $L^1(\Omega)^{3\times 3}$. It remains to remark that $p_n \to p$ weakly in $L^2(\Omega)$ and $\mathbf{f}_n \to \mathbf{f}$ weakly in $L^2(\Omega)^3$. Then, we can pass to the limit in (26), it gives

$$\int_\Omega \operatorname{grad} \mathbf{u} : \operatorname{grad} \mathbf{v} \, dx - \int_\Omega \rho\mathbf{u} \otimes \mathbf{u} : \operatorname{grad} \mathbf{v} \, dx - \int_\Omega p \operatorname{div}(\mathbf{v}) \, dx = \int_\Omega \mathbf{f} \cdot \mathbf{v} \, dx.$$

For the moment, we proved that $(\mathbf{u}_n, p_n, \rho_n)$ is solution of the momentum equation and of the mass equation. We also proved non negativity of ρ and $\int_\Omega \rho dx = M$. It remains to prove $p = \rho^\gamma$. This is not easy since p_n and ρ_n converge only weakly... and $\gamma > 1$.

In order to prove $p = \rho^\gamma$ a.e. in Ω, the main step is to prove that

$$\liminf_{n \to +\infty} \int_\Omega p_n \rho_n \, dx \leq \int_\Omega p\rho \, dx. \tag{27}$$

(Then, we deduce the a.e. convergence of p_n and ρ_n and $p = \rho^\gamma$ using the fact that the function $y \mapsto y^\gamma$ is increasing and a variant of the Minty trick.) Note that for $\gamma < 3$ the natural spaces given in Sec. 1 are $L^{3(\gamma-1)}$ for p and $L^{3(\gamma-1)/\gamma}$ fro ρ. Then, we need here $\gamma \geq 2$, in order to have $p\rho \in L^1(\Omega)$.

In order to prove (27), we first remark that, for all $\bar{\mathbf{u}}, \bar{\mathbf{v}}$ in $H_0^1(\Omega)^3$,

$$\int_\Omega \operatorname{grad} \bar{\mathbf{u}} : \operatorname{grad} \bar{\mathbf{v}} \, dx = \int_\Omega \operatorname{div}(\bar{\mathbf{u}}) \operatorname{div}(\bar{\mathbf{v}}) \, dx + \int_\Omega \operatorname{curl}(\bar{\mathbf{u}}) \cdot \operatorname{curl}(\bar{\mathbf{v}}) \, dx. \tag{28}$$

A similar equality is true at the discrete level with the MAC scheme and the natural discrete operators grad_n and div_n (acting on discrete functions), cf. [3] (this is the first "miracle" with the Mac scheme). With other schemes, it seems that there is not a similar equality and this introduces an additional difficulty, needing, for instance, a "regularization" term for proving the convergence of the scheme, cf. [4].

Using (28), the momentum equation is, for all $\bar{\mathbf{v}}$ in $H_0^1(\Omega)^3$,

$$\int_\Omega \mathrm{div}(\mathbf{u}_n)\,\mathrm{div}(\bar{\mathbf{v}})\,dx + \int_\Omega \mathrm{curl}(\mathbf{u}_n)\cdot\mathrm{curl}(\bar{\mathbf{v}})\,dx - \int_\Omega (\rho_n\mathbf{u}_n\otimes\mathbf{u}_n) : \mathrm{grad}\,\bar{\mathbf{v}}\,dx \\ - \int_\Omega p_n\,\mathrm{div}(\bar{\mathbf{v}})\,dx = \int_\Omega \mathbf{f}_n\cdot\bar{\mathbf{v}}\,dx \quad (29)$$

Our aim is now to choose $\bar{\mathbf{v}} = \bar{\mathbf{v}}_n$ with $\mathrm{curl}(\bar{\mathbf{v}}_n) = 0$, $\mathrm{div}(\bar{\mathbf{v}}_n) = \rho_n$ and $(\bar{\mathbf{v}}_n)_{n\in\mathbb{N}}$ bounded in $H_0^1(\Omega)^3$. Unfortunately, it is possible to choose such a $\bar{\mathbf{v}}_n$ in $H^1(\Omega)^3$ (as we will below) but not in $H_0^1(\Omega)^3$. Assuming anyway that we can have such a $\bar{\mathbf{v}}_n$ in $H_0^1(\Omega)^3$, then, up to a subsequence,

$$\bar{\mathbf{v}}_n \to v \text{ in } L^2(\Omega)^3 \text{ and weakly in } H_0^1(\Omega)^3,\ \mathrm{curl}(\mathbf{v}) = 0,\ \mathrm{div}(\mathbf{v}) = \rho,$$

and (29) becomes

$$\int_\Omega (\mathrm{div}(\mathbf{u}_n) - p_n)\rho_n\,dx = \int_\Omega \rho_n\mathbf{u}_n\otimes\mathbf{u}_n : \mathrm{grad}\,\bar{v}_n\,dx + \int_\Omega f_n\cdot\bar{\mathbf{v}}_n\,dx.$$

If we prove that $\int_\Omega \rho_n\mathbf{u}_n\otimes\mathbf{u}_n : \mathrm{grad}\,\bar{\mathbf{v}}_n\,dx \to \int_\Omega \rho\mathbf{u}\otimes\mathbf{u} : \mathrm{grad}\,\mathbf{v}\,dx$ then

$$\lim_{n\to+\infty}\int_\Omega (\mathrm{div}(\mathbf{u}_n) - p_n)\rho_n\,dx = \int_\Omega \rho\mathbf{u}\otimes\mathbf{u} : \mathrm{grad}\,\mathbf{v}\,dx + \int_\Omega \mathbf{f}\cdot\mathbf{v}\,dx.$$

But, since we already know that $-\Delta\mathbf{u} + \mathrm{div}(\rho\mathbf{u}\otimes\mathbf{u}) + \mathrm{grad}\,p = \mathbf{f}$,

$$\int_\Omega \mathrm{div}(\mathbf{u})\,\mathrm{div}(\mathbf{v})\,dx + \int_\Omega \mathrm{curl}(\mathbf{u})\cdot\mathrm{curl}(\mathbf{v})\,dx - \int_\Omega p\,\mathrm{div}(\mathbf{v}) = \int_\Omega \rho\mathbf{u}\otimes\mathbf{u} : \mathrm{grad}\,\mathbf{v}\,dx + \int_\Omega \mathbf{f}\cdot\mathbf{v}\,dx,$$

which gives (using $\mathrm{div}\,\mathbf{v} = \rho$ and $\mathrm{curl}\,\mathbf{v} = 0$)

$$\int_\Omega (\mathrm{div}(\mathbf{u}) - p)\rho\,dx = \int_\Omega \rho\mathbf{u}\otimes\mathbf{u} : \mathrm{grad}\,\mathbf{v}\,dx + \int_\Omega \mathbf{f}\cdot\mathbf{v}\,dx,$$

Then, $\lim_{n\to+\infty}\int_\Omega (p_n - \mathrm{div}(\mathbf{u}_n))\rho_n\,dx = \int_\Omega (p - \mathrm{div}(\mathbf{u}))\rho\,dx$.

Finally, thanks to the mass equations, we can prove $\int_\Omega \rho_n\,\mathrm{div}(\mathbf{u}_n)\,dx = 0$ and $\int_\Omega \rho\,\mathrm{div}(\mathbf{u})\,dx = 0$. Then, $\lim_{n\to+\infty}\int_\Omega p_n\rho_n\,dx = \int_\Omega p\rho\,dx$.

Indeed, at the discrete level, one has only $\int_\Omega \rho_n\,\mathrm{div}(\mathbf{u}_n)\,dx \le 0$ and (27) is proven (even with lim sup instead of lim inf). It remains to prove

$$\int_\Omega \rho_n\mathbf{u}_n\otimes\mathbf{u}_n : \mathrm{grad}\,\bar{\mathbf{v}}_n\,dx \to \int_\Omega \rho\mathbf{u}\otimes\mathbf{u} : \mathrm{grad}\,\mathbf{v}\,dx. \quad (30)$$

We remark that (since $\mathrm{div}(\rho_n\mathbf{u}_n) = 0$)

$$\int_\Omega \rho_n\mathbf{u}_n\otimes\mathbf{u}_n : \mathrm{grad}\,\bar{\mathbf{v}}_n\,dx = \int_\Omega (\rho_n\mathbf{u}_n\cdot\mathrm{grad})\mathbf{u}_n\cdot\bar{\mathbf{v}}_n\,dx,$$

and the sequence $((\rho_n \mathbf{u}_n \cdot \mathrm{grad})\mathbf{u}_n)_{n\in\mathbb{N}}$ is bounded in $L^r(\Omega)^3$ with $\frac{1}{r} = \frac{1}{2} + \frac{1}{6} + \frac{1}{2\gamma}$, and $r > \frac{6}{5}$ since $\gamma > 3$. Then, up to a subsequence, $(\rho_n \mathbf{u}_n \cdot \mathrm{grad})\mathbf{u}_n \to G$ weakly in $L^r(\Omega)^3$. and (since $\bar{v}_n \to \bar{v}$ in $L^r(\Omega)^3$ for all $r < 6$),

$$\int_\Omega (\rho_n \mathbf{u}_n \cdot \mathrm{grad})\mathbf{u}_n \cdot \bar{v}_n \, dx \to \int_\Omega G \cdot \bar{v} \, dx.$$

But, $G = (\rho \mathbf{u} \cdot \mathrm{grad})\mathbf{u}$, since for a fixed $\mathbf{w} \in H_0^1(\Omega)^3$,

$$\int_\Omega (\rho_n \mathbf{u}_n \cdot \mathrm{grad})\mathbf{u}_n \cdot \mathbf{w} \, dx = \int_\Omega \rho_n \mathbf{u}_n \otimes \mathbf{u}_n : \mathrm{grad}\, \mathbf{w} \, dx \to \int_\Omega \rho \mathbf{u} \otimes \mathbf{u} : \mathrm{grad}\, \mathbf{w} \, dx.$$

Then, (30) is proven and this gives (27) except that there is a mistake in the previous proof since it is not possible to have such a $\bar{\mathbf{v}}_n$ in $H_0^1(\Omega)^3$ such that $\mathrm{curl}\, \bar{\mathbf{v}}_n = 0$, $\mathrm{div}\, \bar{\mathbf{v}}_n = \rho_n$ and $(\bar{\mathbf{v}}_n)_{n\in\mathbb{N}}$ bounded in $H_0^1(\Omega)^3$. In order to correct to proof, we will use such a $\bar{\mathbf{v}}_n$ in $H^1(\Omega)^3$ but not in $H_0^1(\Omega)^3$.

Let $w_n \in H_0^1(\Omega)$, $-\Delta w_n = \rho_n$, It is well known that $w_n \in H^2_{loc}(\Omega)$ (equivalent to say here, since $w_n \in H^1(\Omega)$, $\Delta(w_n \varphi) \in L^2(\Omega)$ for all $\varphi \in C_c^\infty(\Omega)$). An easy way to prove this regularity result is to remark that, for $\varphi \in C_c^\infty(\Omega)$, with C_φ depending only on φ and of the bound of the L^2-norm of ρ_n,

$$\sum_{i,j=1}^{3} \int_\Omega \partial_i \partial_j (w_n \varphi)\, \partial_i \partial_j (w_n \varphi) \, dx = \sum_{i,j=1}^{3} \int_\Omega \partial_i \partial_i (w_n \varphi)\, \partial_j \partial_j (w_n \varphi) \, dx$$
$$= \int_\Omega (\Delta(w_n \varphi))^2 \, dx = C_\varphi < +\infty.$$

The main interest of this way to prove he H^2_{loc}-regularity of w_n is that it is possible to prove a discrete version of this result with the corresponding discrete problem obtained on the primal mesh of the MAC discretization. Namely, we obtain an H^2_{loc}-discrete estimate on w_n in term of the L^2-norm of ρ_n when w_n is the solution of the discrete problem (it is the second miracle for the MAC scheme).

To continue our proof of (27), we take $\mathbf{v}_n = \mathrm{grad}\, w_n$ so that $\mathrm{div}\, \mathbf{v}_n = \rho_n$ and $\mathrm{curl}\, \mathbf{v}_n = 0$ a.e. in Ω. Furthermore, thanks to the H^2_{loc}-discrete estimate, the sequence $(\mathbf{v}_n)_{n\in\mathbb{N}}$ is bounded in $(H^1_{loc}(\Omega))^3$. Then, up to a subsequence, as $n \to +\infty$, $\mathbf{v}_n \to \mathbf{v}$ in $L^2_{loc}(\Omega)$ and weakly in $H^1_{loc}(\Omega)$, $\mathrm{curl}(\mathbf{v}) = 0$, $\mathrm{div}(\mathbf{v}) = \rho$.

Let $\varphi \in C_c^\infty(\Omega)$ (so that $\mathbf{v}_n \varphi \in H_0^1(\Omega)^3)$). Taking $\bar{\mathbf{v}} = \mathbf{v}_n \varphi$ in (29) gives

$$\int_\Omega \mathrm{div}(\mathbf{u}_n)\, \mathrm{div}(\mathbf{v}_n \varphi) \, dx + \int_\Omega \mathrm{curl}(\mathbf{u}_n) \cdot \mathrm{curl}(\mathbf{v}_n \varphi) \, dx - \int_\Omega p_n\, \mathrm{div}(\mathbf{v}_n \varphi) \, dx$$
$$= \int_\Omega \rho_n \mathbf{u}_n \otimes \mathbf{u}_n : \mathrm{grad}(\mathbf{v}_n \varphi) \, dx + \int_\Omega \mathbf{f}_n \cdot (\mathbf{v}_n \varphi) \, dx.$$

Using a proof similar to that given if $\varphi = 1$ (with additional terms involving φ), we obtain, as $n \to +\infty$,

$$\lim_{n\to+\infty} \int_\Omega (p_n - \mathrm{div}(\mathbf{u}_n))\rho_n \varphi \, dx = \int_\Omega (p - \mathrm{div}(\mathbf{u}))\rho \varphi \, dx \text{ for all } \varphi \in C_c^\infty(\Omega),$$

that is $F_n = (p_n - \mathrm{div}(\mathbf{u}_n))\rho_n) \to F = (p - \mathrm{div}(\mathbf{u}))\rho$ in the distribution sense. But since $(F_n)_{n\in\mathbb{N}}$ bounded in L^q for some $q > 1$ (this is due to the fact that $p_n - \mathrm{div}(\mathbf{u}_n)$ is bounded in $L^2(\Omega)$ and ρ_n is bounded in $L^r(\Omega)$ with some $r > 2$, here we use $\gamma > 5/3$), one has also $F_n \to F$ weakly in $L^1(\Omega)$ and therefore

$$\int_\Omega (p_n - \mathrm{div}(\mathbf{u}_n))\rho_n \, dx \to \int_\Omega (p - \mathrm{div}(\mathbf{u}))\rho \, dx.$$

Finally, thanks to the mass equations, $\int_\Omega \mathrm{div}(\mathbf{u})\rho \, dx = 0$ and $\int_\Omega \mathrm{div}(\mathbf{u}_n)\rho_n \, dx = 0$ (or $\leq$ in the case of the discrete setting) and one obtains (27), that is $\liminf_{n\to+\infty} \int_\Omega p_n\rho_n \, dx \leq \int_\Omega p\rho \, dx$.

We prove now the a.e. convergence of ρ_n and p_n. Let $G_n = (\rho_n^\gamma - \rho^\gamma)(\rho_n - \rho)$ so that $G_n \in L^1(\Omega)$ and $G_n \geq 0$ a.e. in Ω. Futhermore $G_n = (p_n - \rho^\gamma)(\rho_n - \rho) = p_n\rho_n - p_n\rho - \rho^\gamma\rho_n + \rho^\gamma\rho$ and:

$$\int_\Omega G_n \, dx = \int_\Omega p_n\rho_n \, dx - \int_\Omega p_n\rho \, dx - \int_\Omega \rho^\gamma\rho_n \, dx + \int_\Omega \rho^\gamma\rho \, dx.$$

Using the weak convergence in $L^2(\Omega)$ of p_n and ρ_n and (27), $\liminf_{n\to+\infty} \int_\Omega G_n = 0$. Then (up to a subsequence), $G_n \to 0$ a.e. and then $\rho_n \to \rho$ a.e. (since $y \mapsto y^\gamma$ is an increasing function on $\mathbb{R}_+$). Finally, $\rho_n \to \rho$ in $L^q(\Omega)$ for all $1 \leq q < 2\gamma$, $p_n = \rho_n^\gamma \to \rho^\gamma$ in $L^q(\Omega)$ for all $1 \leq q < 2$ and $p = \rho^\gamma$ a.e. in Ω.

It is possible to adapt this proof of convergence when $(\mathbf{u}_n, \rho_n, p_n)$ is the approximate solution given by the MAC scheme as it is done in [10]. As we said before, two main tools are interesting with the MAC scheme:

1. There exists a discrete counterpart of

$$\int_\Omega \mathrm{grad}\,\mathbf{u} : \mathrm{grad}\,\mathbf{v} \, dx = \int_\Omega (\mathrm{div}\,\mathbf{u}\,\mathrm{div}\,\mathbf{v} + \mathrm{curl}\,\mathbf{u} \cdot \mathrm{curl}\,\mathbf{v}) \, dx.$$

2. If w_n, belonging to a discrete equivalent of the $H_0^1(\Omega)$-space, is the solution of $-\Delta_n w_n = \rho_n$ where $-\Delta_n$ is the natural discretization of $-\Delta$ on the primal mesh of the MAC-discretization, then one has an estimate on w_n in the "discrete local H^2-norm" of w_n in term of the L^2-norm of ρ_n.

   If $\gamma < 3$, a new difficulty appears since we have to work with the local L^p-norm of the second dicrete derivatives of w_n for some $p > 2$.

In order to conclude this section, we recall that the convergence of approximate solutions (given by the MAC scheme) if $3/2 < \gamma \leq 3$ is, to our knowledge, still an open problem.

3.2. For the compressible Navier-Stokes problem

We consider in the section the compressible Navier-Stokes problem discretized with the MAC scheme and the Euler backward discretization in time, as in Sec. 2.3 (with $T > 0$, $\gamma > 3/2$ and $f \in L^2(]0, T[, L^2(\Omega)))$. For $n \in \mathbb{N}$, the approximate solution $(\mathbf{u_n}, \rho_n, p_n)$ is solution of the discretization of Problem (1)-(5). We assume that $\lim_{n\to+\infty} h_n = \lim_{n\to+\infty} k_n = 0$, where h_n and k_n are the mesh size and the time step of the discretization. Our objective is to prove that the approximate solution converges, in an appropriate sense, up to a subsequence, to a weak solution of (1)-(5).

As usual, the first step, for proving such a convergence result, is to obtain estimates on the approximate solution. A quite easy estimate is in $L^\infty(]0,T[,L^\gamma(\Omega))$ for ρ_n and in $L^2(]0,T[,H_n)$ for $\mathbf{u}_n$ where the norm in H_n is a discrete counterpart of the $H^1_0(\Omega)$-norm (this gives also an $L^2(]0,T[,L^6(\Omega))$ estimate on $\mathbf{u}_n$).

Then, in order to pass to the limit in the equations (as $n \to +\infty$), a new difficulty appears (with respect to the stationary case) for passing to the limit on the non linear terms, namely $\rho_n\mathbf{u}_n$ and $\rho_n\mathbf{u}_n\otimes\mathbf{u}_n$. For instance, in the stationary case (Sec. 3.1), we pass to the limit on $\rho_n\mathbf{u}_n$ (up to a subsequence) using the (strong) convergence of $\mathbf{u}_n$ in a Lebesgue space $L^q(\Omega)^3$ for some $q < 6$ and the weak convergence of ρ_n in the dual space $L^{q'}(\Omega)$, $q' = q/(q-1) > 6/5$. It gives convergence of $\rho_n\mathbf{u}_n$ in $L^1(\Omega)$. This method does not work in the unsteady case since we do not have relative (strong) compactness of the sequence $(\mathbf{u}_n)_n$ in a Lebesgue space. However, we can also conclude in the stationary case by changing the roles of $\mathbf{u}_n$ and ρ_n. Assuming, for simplicity that $(\mathbf{u}_n)_{n\in\mathbb{N}}$ is bounded in $H^1_0(\Omega)^3$, one has, up to subsequence, $u_n \to u$ weakly in $H^1_0(\Omega)^3$, $\rho_n \to \rho$ in $H^{-1}(\Omega)$ (thanks to the compact embedding of $L^{q'}(\Omega)$ in $H^{-1}(\Omega)$) and then, for all $\psi \in C^\infty_c(\mathbb{R})^3$,

$$\int_\Omega \rho_n\mathbf{u}_n\cdot\psi\,dx = \langle\rho_n,\mathbf{u}_n\cdot\psi\rangle_{H^{-1},H^1_0} \to \langle\rho,\mathbf{u}\cdot\psi\rangle_{H^{-1},H^1_0} = \int_\Omega \rho\mathbf{u}\cdot\psi\,dx.$$

For the discrete setting, we also have to replace the $H^1_0(\Omega)$-norm by the so-called discrete-H^1_0-norm (which depends on n), cf. [7] for a complete proof.

The main interest of this new proof for passing to the limit on $\rho_n\mathbf{u}_n$ is that it works also for the unsteady case. Assuming also for simplicity that $(\mathbf{u}_n)_{n\in\mathbb{N}}$ is bounded in $L^2(]0,T[,H^1_0(\Omega)^3)$ (cf. [7] for the discrete case), one has (up to a subsequence) $\mathbf{u}_n \to \mathbf{u}$ weakly in $L^2(]0,T[,H^1_0(\Omega)^3)$. We also know that $(\rho_n)_n$ in $L^2(]0,T[,L^{q'}(\Omega))$ for some $q' > 6/5$ and the mass equation (1) (together with the fact that $\mathbf{u}_n$ is bounded in $L^6(\Omega)$) gives that the sequence $(\partial_t\rho_n)_n$ is bounded in $L^2(]0,T[,W^{-1,1}(\Omega))$. Then $(\rho_n)_{n\in\mathbb{N}}$ is relatively compact in $L^2(]0,T[,H^{-1}(\Omega))$ (thanks to Aubin-Lions-Simon compactness results, since $L^{q'}(\Omega)$ is compactly embedded in $H^{-1}(\Omega)$). Then, up to a subsequence $\rho_n \to \rho$ in $L^2(]0,T[,H^{-1}(\Omega))$ and finally, for all $\psi \in C^\infty_c(\mathbb{R}\times\mathbb{R}^3)^3$,

$$\int_0^T\int_\Omega \rho_n\mathbf{u}_n\cdot\psi\,dxdt = \int_0^T \langle\rho_n,\mathbf{u}_n\cdot\psi\rangle_{H^{-1},H^1_0} \to= \int_0^T\int_\Omega \rho\mathbf{u}\cdot\psi\,dxdt.$$

The difficulty is similar for the term $\rho\mathbf{u}\otimes\mathbf{u}$. In Sec. 3.1 we pass to the limit on this term using $\mathbf{u}_n \to \mathbf{u}$ in $L^q(\Omega)^3$ for all $q < 6$ and $\rho_n\mathbf{u}_n \to \rho\mathbf{u}$ weakly in $L^{q'}(\Omega)^3$, with some $q' > \frac{6}{5}$. It gives $\rho_n\mathbf{u}_n\otimes\mathbf{u}_n \to \rho\mathbf{u}\otimes\mathbf{u}$ weakly in $L^1(\Omega)^{3\times3}$. But an other method is possible. One can use $\mathbf{u}_n \to \mathbf{u}$ weakly in $H^1_0(\Omega)^3$ and $\rho_n\mathbf{u}_n \to \rho\mathbf{u}$ in $H^{-1}(\Omega)^3$ (thanks to the compact embedding of $L^{q'}(\Omega)$ in $H^{-1}(\Omega)$). It also gives convergence of $\rho_n\mathbf{u}_n\otimes\mathbf{u}_n$ to $\rho\mathbf{u}\otimes\mathbf{u}$, that is, for all $\psi \in C^\infty_c(\mathbb{R})^{3\times3}$, $\int_\Omega \rho_n\mathbf{u}_n\otimes\mathbf{u}_n : \psi\,dx \to \int_\Omega \rho\mathbf{u}\otimes\mathbf{u} : \psi\,dx$. Here also, the generalization of this second method is possible for the unsteady case cf. [7].

This does not conclude the convergence (as $n \to +\infty$, up to a subsequence) of the approximate solution to a weak solution of Problem (1)-(5). It remains to pass to the limit on p_n and on the EOS $p_n = \rho_n^\gamma$. It is an ongoing work.

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