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RT_k mixed finite elements for some nonlinear problems

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Abstract. *We show that the family of gradient schemes includes the Raviart–Thomas RT_k mixed finite elements. This result can be used to obtain convergence results for a large number of linear and nonlinear problems.*

1 Introduction

The numerical solution of environmental underground studies often involves models which require the approximation of linear and nonlinear heterogeneous and anisotropic diffusion operators for general piecewise regular coefficients and on general meshes [4, 2, 1, 3]. A wide number of numerical schemes based on several different approaches have been developed in the last fifteen years to this purpose. An illustration of the variety of these approaches may be found in the two benchmarks which were held in 2008 (two-dimensional case) and in 2011 (3D case) [13, 12]. The family of gradient schemes was introduced to synthesize some of these approaches and was proven to converge for a large number of nonlinear problems [10, 5]. This family contains several wellknown schemes, such as conforming and lumped conforming schemes, mimetic schemes, discrete duality finite volume schemes.

The aim of this paper is to show that it also contains the RT_k mixed finite element method and apply it to two phase flow problems.

2 Gradient schemes for diffusion problems

Let Ω be an open bounded subset of $\mathbb{R}^d$, where d is the space dimension. A **gradient discretization** $\mathcal{D}$ for a space-dependent second order elliptic problem posed on the domain Ω, with homogeneous Dirichlet boundary conditions on the boundary $\partial\Omega$, is defined by $\mathcal{D} = (X_{\mathcal{D},0}, \Pi_{\mathcal{D}}, \nabla_{\mathcal{D}})$, where:
• the set of discrete unknowns $X_{\mathcal{D},0}$ is a finite dimensional vector space on $\mathbb{R}$, corresponding to the approximation of the homogeneous Dirichlet elliptic problem,
• the linear mapping $\Pi_{\mathcal{D}} : X_{\mathcal{D},0} \to L^2(\Omega)$ is the reconstruction of an approximate function from the discrete unknowns (also often called "lifting operator").
• the linear mapping $\nabla_{\mathcal{D}} : X_{\mathcal{D},0} \to L^2(\Omega)^d$ is the discrete gradient operator. It must be chosen such that $\|\cdot\|_{\mathcal{D}} := \|\nabla_{\mathcal{D}} \cdot\|_{L^2(\Omega)^d}$ is a norm on $X_{\mathcal{D},0}$.

Let us now give the fundamental properties that we seek when designing a gradient discretization (or when recognizing a gradient discretization in an existing scheme) in order to be able to prove its convergence.

• **Coercivity.** Let $C_{\mathcal{D}}$ be the norm of the linear mapping $\Pi_{\mathcal{D}}$, defined by

$$C_{\mathcal{D}} = \max_{v \in X_{\mathcal{D},0} \setminus \{0\}} \frac{\|\Pi_{\mathcal{D}} v\|_{L^2(\Omega)}}{\|v\|_{\mathcal{D}}}. \tag{2.1}$$

A sequence $(\mathcal{D}_m)_{m\in\mathbb{N}}$ of gradient discretizations is said to be coercive if there exists $C_P \in \mathbb{R}_+$ such that $C_{\mathcal{D}_m} \le C_P$ for all $m \in \mathbb{N}$.

• **Consistency.** Let $S_{\mathcal{D}}$ be defined by: $\varphi \in H^1(\Omega) \mapsto S_{\mathcal{D}}(\varphi) \in [0, +\infty)$ with

$$S_{\mathcal{D}}(\varphi) = \min_{v \in X_{\mathcal{D},0}} \Big(\|\Pi_{\mathcal{D}} v - \varphi\|_{L^2(\Omega)} + \|\nabla_{\mathcal{D}} v - \nabla\varphi\|_{L^2(\Omega)^d} \Big). \tag{2.2}$$

A sequence $(\mathcal{D}_m)_{m\in\mathbb{N}}$ of gradient discretizations is said to be consistent if, for all $\varphi \in H_0^1(\Omega)$, $S_{\mathcal{D}_m}(\varphi)$ tends to 0 as $m \to \infty$.

• **Limit–conformity.** Let $H_{\rm div}(\Omega) = \{\boldsymbol{\varphi} \in L^2(\Omega)^d, {\rm div}\boldsymbol{\varphi} \in L^2(\Omega)\}$ and let $W_{\mathcal{D}}$: $H_{\rm div}(\Omega) \times X_{\mathcal{D},0} \to [0, +\infty)$ be defined by

$$\begin{aligned} &\forall(\boldsymbol{\varphi}, u) \in H_{\rm div}(\Omega) \times X_{\mathcal{D},0}, \\ &W_{\mathcal{D}}(\boldsymbol{\varphi}, u) = \int_\Omega (\nabla_{\mathcal{D}} u(\boldsymbol{x}) \cdot \boldsymbol{\varphi}(\boldsymbol{x}) + \Pi_{\mathcal{D}} u(\boldsymbol{x}) {\rm div}\boldsymbol{\varphi}(\boldsymbol{x}))\, {\rm d}\boldsymbol{x}. \end{aligned} \tag{2.3}$$

Note that for a conforming finite element method, we have $W_{\mathcal{D}}(\boldsymbol{\varphi}, u) = 0$.

A sequence $(\mathcal{D}_m)_{m\in\mathbb{N}}$ of gradient discretizations is said to be limit-conforming if, for all sequence $u_m \in X_{\mathcal{D}_m,0}$ such that $\|u_m\|_{\mathcal{D}_m}$ is bounded, and for all $\boldsymbol{\varphi} \in H_{\rm div}(\Omega)$, $W_{\mathcal{D}_m}(\boldsymbol{\varphi}, u_m)$ tends to 0 as $m \to \infty$.

- **Compactness.** A sequence $(\mathcal{D}_m)_{m\in\mathbb{N}}$ of gradient discretizations is said to be compact if, for all sequence $u_m \in X_{\mathcal{D}_m,0}$ such that $\|u_m\|_{\mathcal{D}_m}$ is bounded, the sequence $(\Pi_{\mathcal{D}_m} u_m)_{m\in\mathbb{N}}$ is relatively compact in $L^2(\Omega)$.

The linear case. Let $\underline{\lambda}$ and $\overline{\lambda} \in \mathbb{R}$, such that $0 < \underline{\lambda} \le \overline{\lambda}$ and let $\mathcal{M}_d(\underline{\lambda}, \overline{\lambda})$ denote the set of $d \times d$ symmetric matrices with eigenvalues in $(\underline{\lambda}, \overline{\lambda})$. Assuming that Λ is a measurable function from Ω to $\mathcal{M}_d(\underline{\lambda}, \overline{\lambda})$, and $f \in L^2(\Omega)$, we seek an approximation of $\bar{u} \in H^1_0(\Omega)$ satisfying

$$\int_\Omega \Lambda(\boldsymbol{x})\nabla\bar{u}(\boldsymbol{x}) \cdot \nabla v(\boldsymbol{x})\mathrm{d}\boldsymbol{x} = \int_\Omega f(\boldsymbol{x})v(\boldsymbol{x})\mathrm{d}\boldsymbol{x}, \forall v \in H^1_0(\Omega). \tag{2.4}$$

If $\mathcal{D} = (X_{\mathcal{D},0}, \Pi_\mathcal{D}, \nabla_\mathcal{D})$ is a gradient discretization, the related gradient scheme for the discretization of this problem is to look for $u \in X_{\mathcal{D},0}$ such that

$$\int_\Omega \Lambda(\boldsymbol{x})\nabla_\mathcal{D} u(\boldsymbol{x}) \cdot \nabla_\mathcal{D} v(\boldsymbol{x})\mathrm{d}\boldsymbol{x} = \int_\Omega f(\boldsymbol{x})\Pi_\mathcal{D} v(\boldsymbol{x})\mathrm{d}\boldsymbol{x}, \forall v \in X_{\mathcal{D},0}. \tag{2.5}$$

The coercivity, consistency and limit-conformity properties for a family of gradient discretizations are sufficient to ensure the convergence of $\Pi_\mathcal{D} u$ to $\bar{u}$ in $L^2(\Omega)$ and that of $\nabla_\mathcal{D} u$ to $\nabla\bar{u}$ in $L^2(\Omega)^d$. The compactness property is only needed for the convergence of gradient schemes in the case of nonlinear problems.

3 RT_k mixed finite element schemes are gradient schemes

Let Ω be an open bounded connected subset of $\mathbb{R}^d$, $d \in \mathbb{N}^\star$. Let Q be a finite dimensional subspace of $L^2(\Omega)$. Let V be a finite dimensional subset of $H_{\rm div}(\Omega)$; the divergence operator is thus well defined on V whereas the gradient of an element of Q is not defined. It is however natural to define a discrete gradient on Q by a duality formula; let U be a finite dimensional space of $L^2(\Omega)^d$ with same dimension as V and such that

$$\forall \boldsymbol{v} \in U, \int_\Omega \boldsymbol{v}(\boldsymbol{x}) \cdot \boldsymbol{w}(\boldsymbol{x})\mathrm{d}\boldsymbol{x} = 0 \text{ for all } \boldsymbol{w} \in V \text{ implies } \boldsymbol{v} = 0,$$

then the discrete gradient of $u \in Q$, denoted by $\nabla_{\rm a} u$, is the unique solution of :

$$\nabla_{\rm a} u \in U,$$

$$\int_\Omega \boldsymbol{w}(\boldsymbol{x}) \cdot \nabla_{\rm a} u(\boldsymbol{x}) {\rm d}\boldsymbol{x} + \int_\Omega u(\boldsymbol{x}) {\rm div} \boldsymbol{w}(\boldsymbol{x}) {\rm d}\boldsymbol{x} = 0, \qquad \forall \boldsymbol{w} \in V. \tag{3.1}$$

In the case of the linear problem (2.4), a natural (non conforming) scheme is then

$$\begin{aligned} & u \in Q, \\ & \int_\Omega \Lambda(\boldsymbol{x}) \nabla_{\rm a} u(\boldsymbol{x}) \cdot \nabla_{\rm a} v(\boldsymbol{x}) {\rm d}\boldsymbol{x} = \int_\Omega f(\boldsymbol{x}) v(\boldsymbol{x}) {\rm d}\boldsymbol{x}, \forall v \in Q. \end{aligned} \tag{3.2}$$

In order to define the scheme completely, it remains to choose the subspace U. Since the solution u of the continuous problem (2.4) satisfies $\Lambda \nabla u \in H_{\rm div}(\Omega)$, it is natural to take

$$U = \{\boldsymbol{\varphi} \in L^2(\Omega)^d; \Lambda \boldsymbol{\varphi} \in V\}. \tag{3.3}$$

With this choice for U, a function $u \in Q$ is solution to (3.2) with $\nabla_{\rm a}$ defined by (3.1), if and only if the pair $(-\Lambda \nabla_{\rm a} u, u) = (\boldsymbol{v}, q)$ where $(\boldsymbol{v}, q) \in V \times Q$ is a solution to

$$\int_\Omega \boldsymbol{w}(\boldsymbol{x}) \cdot \Lambda^{-1}(\boldsymbol{x}) \boldsymbol{v}(\boldsymbol{x}) {\rm d}\boldsymbol{x} - \int_\Omega q(\boldsymbol{x}) {\rm div} \boldsymbol{w}(\boldsymbol{x}) {\rm d}\boldsymbol{x} = 0, \quad \forall \boldsymbol{w} \in V, \tag{3.4a}$$

$$\int_\Omega \psi(\boldsymbol{x}) {\rm div} \boldsymbol{v}(\boldsymbol{x}) {\rm d}\boldsymbol{x} = \int_\Omega \psi(\boldsymbol{x}) f(\boldsymbol{x}) {\rm d}\boldsymbol{x}, \qquad \forall \psi \in Q. \tag{3.4b}$$

Indeed, Equation (3.4a) corresponds to (3.1) with $\boldsymbol{v} = -\Lambda \nabla_{\rm a} u$. Letting $u = \psi \in Q$ in (3.1), we have

$$\int_\Omega \psi(\boldsymbol{x}) {\rm div} \boldsymbol{v}(\boldsymbol{x}) {\rm d}\boldsymbol{x} = -\int_\Omega \boldsymbol{v}(x) \cdot \nabla_{\rm a} \psi(\boldsymbol{x}) {\rm d}\boldsymbol{x} = \int_\Omega \Lambda(\boldsymbol{x}) \nabla_{\rm a} u(\boldsymbol{x}) \cdot \nabla_{\rm a} \psi(\boldsymbol{x}) {\rm d}\boldsymbol{x},$$

and therefore (3.4b) corresponds to (3.2), thanks to the choice (3.3) of U. Note that at this point, the problems (3.2) and (3.4) are not necessarily well posed. The well-posedness is obtained for adequate choices of (V, Q).
Here we choose V spanned by the corresponding RT_k basis functions on a regular simplicial mesh $\mathcal{T}$ [6], and Q spanned by the family $(\chi_i)_{i \in I}$ of piecewise polynomial basis functions of degree k on each cell of the mesh. It is wellknown that in this case, Problem (3.4) (and therefore (3.2)) is well posed.

We now wish to compare the mixed finite element for a general, possibly non linear problem, to a gradient scheme discretization. So we again consider a regular simplicial mesh, and V and Q as above, but we now generalize the space U as follows:

$$U = \{\boldsymbol{\varphi} \in L^2(\Omega)^d; A\boldsymbol{\varphi} \in V\}, \tag{3.5}$$

where A is an arbitrary measurable function from Ω to $\mathcal{M}_d(\underline{\lambda}, \overline{\lambda})$. Let us define the gradient discretization $\mathcal{D} = (X_{\mathcal{D},0}, \Pi_{\mathcal{D}}, \nabla_{\mathcal{D}})$ by: $X_{\mathcal{D},0} = \mathbb{R}^I$, $\Pi_{\mathcal{D}} u = \sum_{i\in I} u_i \chi_i$ and $\nabla_{\mathcal{D}} u = \nabla_{\text{a}}(\Pi_{\mathcal{D}} u)$. Then (3.1) with U defined by (3.5) may be written as

$$\begin{aligned} & A\nabla_{\mathcal{D}} u \in V, \\ & \int_\Omega \boldsymbol{v}(\boldsymbol{x}) \cdot \nabla_{\mathcal{D}} u(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} + \int_\Omega \Pi_{\mathcal{D}} u(\boldsymbol{x}) \mathrm{div} \boldsymbol{v}(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} = 0, \quad \forall \boldsymbol{v} \in V. \end{aligned} \tag{3.6}$$

Note that for most problems, we shall choose A to be the identity matrix. However we can also choose $A = \Lambda$, the absolute permeability matrix of an anisotropic heterogeneous porous media, for instance in the case modelled by the linear problem (2.4).

Let us now show that the resulting gradient scheme is coercive, consistent, limit conforming and compact, as stated in Theorem 3.1 below. In order to do so, we first recall some known results on the RT_k mixed finite element schemes. Let us introduce the broken Sobolev space $H^1(\mathcal{T})$ of functions whose restriction to each simplex K of the mesh belongs to H^1. First recall that, for (V, Q) defined by the $(RT_k, \mathcal{P}^k)$ mixed finite element approximation, there exists [6, Lemma 3.5 page 17] an interpolation operator $P_k : \boldsymbol{H}_{\mathcal{T}} = H_{\text{div}}(\Omega) \cap (H^1(\mathcal{T}))^d \to \mathcal{R}T_k$ such that

$$\forall p \in \mathcal{P}^k,\ \forall \boldsymbol{v} \in \boldsymbol{H}_{\mathcal{T}},\ \int_\Omega p(\boldsymbol{x}) \mathrm{div}(\boldsymbol{v} - P_k \boldsymbol{v})(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} = 0, \tag{3.7}$$

and there exists $\alpha > 0$, only depending on the regularity of the mesh [6, Theorem 3.1], such that

$$\forall \boldsymbol{v} \in \boldsymbol{H}_{\mathcal{T}},\ \|\boldsymbol{v} - P_k \boldsymbol{v}\|_{L^2(\Omega)^d} \le \alpha h \Big(\sum_{T\in\mathcal{T}} \|\boldsymbol{v}\|^2_{H^1(T)}\Big)^{1/2}, \tag{3.8}$$

where h denotes the size of the mesh $\mathcal{T}$. Let us recall how we deduce from the above properties the standard "inf-sup" condition: let $p \in Q$, let us prolong p by 0 on a ball B with radius R containing Ω. Then there exists $w \in H^1_0(B)$ such that

$$\forall q \in H^1_0(B), \int_B \nabla w(\boldsymbol{x}) \cdot \nabla q(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} = \int_B p(\boldsymbol{x}) q(\boldsymbol{x}) \mathrm{d}\boldsymbol{x}. \tag{3.9}$$

Moreover $w \in H^2(B)$ and there exists β only depending on d and R such that

$$\|w\|_{H^2(B)} \le \beta \|p\|_{L^2(\Omega)}.$$

Therefore, since $\nabla w \in \boldsymbol{H}_{\mathcal{T}}$, we have from (3.8)

$$\|\nabla w - P_k \nabla w\|_{L^2(\Omega)^d} \leq \alpha h \beta \|p\|_{L^2(\Omega)},$$

which shows that

$$\|P_k \nabla w\|_{L^2(\Omega)^d} \leq (2R\alpha + 1)\beta \|p\|_{L^2(\Omega)}, \tag{3.10}$$

which immediately leads to the inf-sup condition. This enables to apply the general result on mixed approximations [6, Theorem 5.3 p. 39] (due to Brezzi): for a given $f \in L^2(\Omega)$ and discretization spaces V, Q defined by the k-th order Raviart–Thomas mixed approximation $(\mathcal{RT}_k, \mathcal{P}^k)$, there exists one and only one $(\boldsymbol{v}, q) \in V \times Q$ solution to (3.4) and there exists δ, only depending on $\underline{\lambda}$, $\overline{\lambda}$, on the regularity of the mesh and on Ω such that

$$\|q - u\|_{L^2(\Omega)} + \|\boldsymbol{v} + A\nabla u\|_{H_{\mathrm{div}}(\Omega)} \leq \\ \delta(\inf_{\psi \in Q} \|\psi - u\|_{L^2(\Omega)} + \inf_{\boldsymbol{w} \in V} \|\boldsymbol{w} - \Lambda \nabla u\|_{H_{\mathrm{div}}(\Omega)}), \tag{3.11}$$

where $u \in H_0^1(\Omega)$ is the unique solution of

$$\forall v \in H_0^1(\Omega), \int_\Omega A(\boldsymbol{x}) \nabla u(\boldsymbol{x}) \cdot \nabla v(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} = \int_\Omega f(\boldsymbol{x}) v(\boldsymbol{x}) \mathrm{d}\boldsymbol{x}. \tag{3.12}$$

Let us now state that the RT_k mixed finite element scheme is a gradient scheme.

Theorem 3.1 *Let $(\mathcal{T}_m)_{m\in\mathbb{N}}$ be a sequence of regular simplicial meshes in the sense of [6, Theorem 3.1 p.14] such that the size h_m of the mesh $\mathcal{T}_m$, tends to 0 as $m \to \infty$. For $k \in \mathbb{N}$, let $(V_m, Q_m)_{m\in\mathbb{N}}$ be the corresponding sequence of RT_k finite element spaces. Let $\mathcal{D}_m = (X_{\mathcal{D}_m,0}, \Pi_{\mathcal{D}_m}, \nabla_{\mathcal{D}_m})$ be defined from (V_m, Q_m) by* (3.6)*. Then $\mathcal{D}_m$ is a gradient discretization and the family $(\mathcal{D}_m)_{m\in\mathbb{N}}$ is coercive, consistent, limit-conforming and compact in the sense of the definitions of Section 2.*

Proof

• **Coercivity.** Let $u \in X_{\mathcal{D}_m,0}$ (which means that $p = \Pi_{\mathcal{D}_m} u \in Q_m$). Using (3.10), let $w \in H_0^1(B)$ be defined by (3.9), and let $\boldsymbol{v} = P_k \nabla w \in V_m$. Thanks to (3.7), we get that

$$\|p\|^2_{L^2(\Omega)} = -\int_\Omega p(\boldsymbol{x}) \mathrm{div} \boldsymbol{v}(\boldsymbol{x}) \mathrm{d}\boldsymbol{x}.$$

From (3.6), we get

$$\|\Pi_{\mathcal{D}_m} u\|^2_{L^2(\Omega)} = \int_\Omega \boldsymbol{v}(\boldsymbol{x}) \cdot \nabla_{\mathcal{D}_m} u(\boldsymbol{x}) \mathrm{d}\boldsymbol{x}.$$

Thanks to (3.10), we then get

$$\|\Pi_{\mathcal{D}_m} u\|_{L^2(\Omega)} \le (\alpha D + 1)\beta \|\nabla_{\mathcal{D}_m} u\|_{L^2(\Omega)^d},$$

which proves the coercivity property.

• **Consistency.** Let us check the consistency property on the set $\mathcal{R} = \{\varphi \in H^1_0(\Omega)$; there exists $f \in C^\infty_c(\Omega)$ such that φ is solution to (3.12)$\}$. Let $\varphi \in \mathcal{R}$. Considering Problem (3.4) with $f = -\mathrm{div}(A\nabla\varphi)$, we define $u \in X_{\mathcal{D}_m,0}$ by $p = \Pi_{\mathcal{D}_m} u$ and $\boldsymbol{v} = -A\nabla_{\mathcal{D}_m} u$. Then, we get from (3.11)

$$\|\Pi_{\mathcal{D}_m} u - \varphi\|_{L^2(\Omega)} + \|A\nabla_{\mathcal{D}_m} u - A\nabla\varphi\|_{H_{\mathrm{div}}(\Omega)} \le \delta\big(\inf_{\psi\in Q_m} \|\psi - \varphi\|_{L^2(\Omega)} + \inf_{\boldsymbol{w}\in V_m} \|\boldsymbol{w} - A\nabla\varphi\|_{H_{\mathrm{div}}(\Omega)}\big).$$

Since the right hand side of the above inequality tends to 0 as $m \to \infty$, we obtain that $S_{\mathcal{D}_m}(\varphi)$ tends to 0 as $m \to \infty$. The proof of consistency is then concluded by density of $\mathcal{R}$ in $H^1_0(\Omega)$ (see Lemma 3.2 below).

• **Limit-conformity.** Let $(u_m)_{m\in\mathbb{N}}$ such that $u_m \in X_{\mathcal{D}_m,0}$ and $\nabla_{\mathcal{D}_m} u_m$ remains bounded in $L^2(\Omega)^d$ as $m \to \infty$. Let $\boldsymbol{\varphi} \in H_{\mathrm{div}}(\Omega)$, and $\boldsymbol{\varphi}_m \in V_m$ be an interpolation of $\boldsymbol{\varphi}$ such that $\|\boldsymbol{\varphi} - \boldsymbol{\varphi}_m\|_{H_{\mathrm{div}}(\Omega)}$ tends to 0 as $m \to \infty$. Then

$$W_{\mathcal{D}_m}(\boldsymbol{\varphi}, u_m) = \int_\Omega (\nabla_{\mathcal{D}_m} u_m(\boldsymbol{x}) \cdot \boldsymbol{\varphi}(\boldsymbol{x}) + \Pi_{\mathcal{D}_m} u_m(\boldsymbol{x}) \mathrm{div}\boldsymbol{\varphi}(\boldsymbol{x}))\,\mathrm{d}\boldsymbol{x} =$$
$$\int_\Omega \Big(\nabla_{\mathcal{D}_m} u_m(\boldsymbol{x})\cdot(\boldsymbol{\varphi}(\boldsymbol{x}) - \boldsymbol{\varphi}_m(\boldsymbol{x})) + \Pi_{\mathcal{D}_m} u_m(\boldsymbol{x})(\mathrm{div}\boldsymbol{\varphi}(\boldsymbol{x}) - \mathrm{div}\boldsymbol{\varphi}_m(\boldsymbol{x}))\Big)\mathrm{d}\boldsymbol{x},$$

thanks to (3.6). Applying the coercivity inequality, we get that the right term of the preceding inequality tends to 0 as $m \to \infty$, which shows the limit conformity of the sequence.

• **Compactness.** We consider a sequence $(u_m)_{m\in\mathbb{N}}$ such that $u_m \in X_{\mathcal{D}_m,0}$ and $\nabla_{\mathcal{D}_m} u_m$ remains bounded in $L^2(\Omega)^d$ as $m \to \infty$. Then, thanks to the coercivity property, we first extract a subsequence (samely denoted), such that $\Pi_{\mathcal{D}_m} u_m$ weakly converges in $L^2(\mathbb{R}^d)$ to some $u \in L^2(\mathbb{R}^d)$ (prolonging by 0 outside Ω). Using the limit-conformity, we get that $\nabla_{\mathcal{D}_m} u_m$ (prolonging by 0 outside Ω) weakly converges in $L^2(\mathbb{R}^d)^d$ to ∇u, which shows that $u \in H^1_0(\Omega)$. Let $w_m \in H^1_0(B) \cap H^2(B)$ (resp. $w \in H^1_0(B) \cap H^2(B)$) be defined by (3.9) for $p = \Pi_{\mathcal{D}_m} u_m$ (resp. $p = u$). A classical result is that w_m converges in $H^1_0(B)$ to w (from the weak convergence of the gradient and the convergence of its norm).

Letting $\boldsymbol{v} = P_k \nabla w_m$ in (3.6), we get

$$\int_\Omega P_k \nabla w_m(\boldsymbol{x}) \cdot \nabla_{\mathcal{D}_m} u_m(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} + \int_\Omega \Pi_{\mathcal{D}_m} u_m(\boldsymbol{x}) \mathrm{div} P_k \nabla w_m(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} = 0,$$

which provides, thanks to (3.7),

$$\int_\Omega P_k \nabla w_m(\boldsymbol{x}) \cdot \nabla_{\mathcal{D}_m} u_m(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} - \int_\Omega (\Pi_{\mathcal{D}_m} u_m(\boldsymbol{x}))^2 \mathrm{d}\boldsymbol{x} = 0.$$

Thanks to (3.8) and to the convergence of ∇w_m to ∇w in $L^2(\Omega)^d$, we get that $P_k \nabla w_m$ converges in $L^2(\Omega)^d$ to ∇w. By strong/weak convergence on the first term, we get

$$\begin{aligned} \lim_{m\to\infty} \int_\Omega (\Pi_{\mathcal{D}_m} u_m(\boldsymbol{x}))^2 \mathrm{d}\boldsymbol{x} &= \int_\Omega \nabla w(\boldsymbol{x}) \cdot \nabla u(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} \\ &= -\int_\Omega \mathrm{div}(\nabla w(\boldsymbol{x})) u(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} = \int_\Omega u(\boldsymbol{x})^2 \mathrm{d}\boldsymbol{x}. \end{aligned}$$

This shows the convergence of $\Pi_{\mathcal{D}_m} u_m$ to u in $L^2(\Omega)$, hence concluding the proof of the compactness of the discretization. □

Lemma 3.2 (A density result) *Let Ω be an open bounded subset of $\mathbb{R}^d$, let A be an arbitrary measurable function from Ω to the set of $d \times d$ matrices, and, for a.e. $\boldsymbol{x} \in \Omega$, $A(\boldsymbol{x})$ is symmetric with eigenvalues in $(\underline{\lambda}, \overline{\lambda}) \subset (0, +\infty)$, and let $\mathcal{R} = \{\varphi \in H_0^1(\Omega)$; there exists $f \in C_c^\infty(\Omega)$ such that φ solution of* (3.12)$\}$. *Then $\mathcal{R}$ is dense in $H_0^1(\Omega)$.*

Proof The mapping $T : H_0^1(\Omega) \to H^{-1}(\Omega)$ defined by $u \mapsto T(u) = -\mathrm{div}\Lambda\nabla u$ is continuous and one-to-one thanks to the Lax-Milgram lemma. Therefore the inverse mapping T^{-1} is also continuous. Since $C_c^\infty(\Omega)$ is dense in $H^{-1}(\Omega)$ and $\mathcal{R} = T^{-1}(C_c^\infty(\Omega))$, the conclusion follows. □

Convergence of the schemes. Let us recall that for a coercive, consistent, limit-conforming and compact gradient discretization, we are able to prove the convergence of the associated gradient scheme for a number of linear or non linear problems [8, 11, 7, 5] In particular, Theorem 3.1 proves that the gradient discretization defined by (3.5)-(3.6) is coercive, consistent, limit-conforming, and compact, whatever the choice of the matrix A in $\mathcal{M}_d(\underline{\lambda}, \overline{\lambda})$. In particular, for $A = \Lambda$, it yields the convergeequation*nce of the classical mixed finite element scheme for the diffusion problem 2.4, with no regularity assumption on the solution (see [9, Lemma 2.2] or [5, Lemma 3.1]). In the following section, we recall the convergence result that was proven for two phase flow in [10] for gradient discretizations and which naturally includes the RT_0 scheme.

4 Application to two phase flow in porous media and Richards' equation

We are interested here in the approximation of (u, v), solution to the incompressible two-phase flow problem in the space domain Ω during the time period $(0, T)$:

$$\Phi(\boldsymbol{x})\partial_t S(\boldsymbol{x}, p) - \operatorname{div}(k_1(\boldsymbol{x}, S(\boldsymbol{x}, p))\Lambda(\boldsymbol{x})\nabla u) = f_1, \tag{4.1a}$$

$$\Phi(\boldsymbol{x})\partial_t (1 - S(\boldsymbol{x}, p)) - \operatorname{div}(k_2(\boldsymbol{x}, S(\boldsymbol{x}, p))\Lambda(\boldsymbol{x})\nabla v) = f_2, \tag{4.1b}$$

$$p = u - v, \text{ for } \in \Omega \times (0, T), \tag{4.1c}$$

where u (resp. v) denotes the pressure of the phase 1, called the wetting phase (resp. of the phase 2, which is the nonwetting phase), p is the difference between the two pressures, called the capillary pressure, the saturation of the phase 1 is denoted by $S(\boldsymbol{x}, p)$ (it is called the "water content" in the framework of Richards' equation), and where Φ, Λ, k_i, $\boldsymbol{g}_i$ and f_i $(i = 1, 2)$ respectively denote the porosity, the absolute permeability, the relative permeabilities, the gravity and the source terms.

Remark *Alternately, we also consider a generalized Richards equation obtained from* (4.1) *either by replacing Equation* (4.1a) *by*

$$u(\boldsymbol{x}, t) = \bar{u}(\boldsymbol{x}) \textit{ for } (\boldsymbol{x}, t) \in \Omega \times (0, T), \tag{4.2}$$

or replacing (4.1b) *by*

$$v(\boldsymbol{x}, t) = \bar{v}(\boldsymbol{x}) \textit{ for } (\boldsymbol{x}, t) \in \Omega \times (0, T). \tag{4.3}$$

Problem (4.1) is considered with the following initial condition:

$$S(\boldsymbol{x}, p(\boldsymbol{x}, 0)) = S(\boldsymbol{x}, p_{\text{ini}}(\boldsymbol{x})), \text{ for a.e. } x \in \Omega, \tag{4.4}$$

together with the non-homogeneous Dirichlet boundary conditions:

$$u(\boldsymbol{x}, t) = \bar{u}(\boldsymbol{x}) \text{ and } v(\boldsymbol{x}, t) = \bar{v}(\boldsymbol{x}) \text{ on } \partial\Omega \times (0, T). \tag{4.5}$$

The detailed assumptions are provided in [10] and include that the functions k_i, $i = 1, 2$, are bounded by below by some value $k_{\min} > 0$. This latter assumption is needed for the convergence proof (it is classical for the Richards problem). Problem (4.1)-(4.4)-(4.5) is considered under an appropriate weak sense, and it is approximated by the tools provided here, in the following way.

We consider a time interval $(0, T)$ and $(t^{(n)})_{n=0,\dots,N}$ such that $t^{(0)} = 0 < t^{(1)} \dots < t^{(N)} = T$. We then set $\delta t^{(n+\frac{1}{2})} = t^{(n+1)} - t^{(n)}$, for $n = 0, \dots, N-1$.

Let us consider the following mixed finite element scheme for the approximation of Problem (4.1),(4.4),(4.5) (let us emphasize that this discretization is not based on the global pressure formulation of the problem). We consider a simplicial mesh of Ω, which is sufficiently regular, and we define $Q \subset L^2(\Omega)$ as the set of the piecewise constant functions on the elements of the mesh, and $V \subset H_{\rm div}(\Omega)$ be defined by the RT_0 basis. The scheme is then given (dropping the time indices $(n+1)$ for the unknowns) by

$$
\begin{aligned}
&\text{find } u, v \in Q,\ \boldsymbol{u}, \boldsymbol{v} \in V,\ p = u - v,\\
&\forall \boldsymbol{w} \in V, \int_\Omega (k_1(S(\boldsymbol{x}, p))\Lambda)^{(-1)} \boldsymbol{u} \cdot \boldsymbol{w} + \int_\Omega u \operatorname{div} \boldsymbol{w} = 0,\\
&\forall \boldsymbol{w} \in V, \int_\Omega (k_2(S(\boldsymbol{x}, p))\Lambda)^{(-1)} \boldsymbol{v} \cdot \boldsymbol{w} + \int_\Omega v \operatorname{div} \boldsymbol{w} = 0,\\
&\forall q \in Q, \int_\Omega \Big(\Phi(\boldsymbol{x}) \frac{S(\boldsymbol{x}, p) - S(\boldsymbol{x}, p^{(n)})}{\delta t^{(n+\frac12)}} + \operatorname{div} \boldsymbol{u}\Big) q(\boldsymbol{x}) \mathrm{d}\boldsymbol{x}\\
&= \frac{1}{\delta t^{(n+\frac12)}} \int_{t^{(n)}}^{t^{(n+1)}} \int_\Omega f_1(\boldsymbol{x}, t) q(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} \mathrm{d}t.\\
&\forall q \in Q, \int_\Omega \Big(-\Phi(\boldsymbol{x}) \frac{S(\boldsymbol{x}, p) - S(\boldsymbol{x}, p^{(n)})}{\delta t^{(n+\frac12)}} + \operatorname{div} \boldsymbol{v}\Big) q(\boldsymbol{x}) \mathrm{d}\boldsymbol{x}\\
&= \frac{1}{\delta t^{(n+\frac12)}} \int_{t^{(n)}}^{t^{(n+1)}} \int_\Omega f_2(\boldsymbol{x}, t) q(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} \mathrm{d}t.
\end{aligned}
$$

We can then rewrite the above discretization, for example in the case of (4.1a), under the form of a gradient scheme:

$$
\begin{aligned}
&\text{find } u, v \in X_{\mathcal{D},0},\ p = u - v,\\
&\qquad \forall w \in X_{\mathcal{D},0}, \int_\Omega \Big(\Phi(\boldsymbol{x}) \frac{S(\boldsymbol{x}, \Pi_\mathcal{D} p) - S(\boldsymbol{x}, \Pi_\mathcal{D} p^{(n)})}{\delta t^{(n+\frac12)}} \Pi_\mathcal{D} w\\
&\qquad\qquad + k_1(S(\boldsymbol{x}, \Pi_\mathcal{D} p)\Lambda \nabla_\mathcal{D} u \cdot \nabla_\mathcal{D} w\Big) \mathrm{d}\boldsymbol{x}\\
&\qquad\qquad\qquad = \frac{1}{\delta t^{(n+\frac12)}} \int_{t^{(n)}}^{t^{(n+1)}} \int_\Omega f_1(\boldsymbol{x}, t) \Pi_\mathcal{D} w(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} \mathrm{d}t,
\end{aligned}
$$

defining $\Pi_\mathcal{D}$ and $\nabla_\mathcal{D}$ by (3.6) with $A = k_1(S_\mathcal{D}(\boldsymbol{x}, p))\Lambda$. Writing a similar equation for (4.1b), the resulting scheme is then very close to that given in [10], and a similar study to that given in [10] can be done for proving its convergence to the weak sense of Problem (4.1),(4.4),(4.5), based on the four properties (coercivity, consistency, limit-conformity and compactness) introduced in this paper. This proof also requires that the reconstruction operator be piecewise constant, and therefore it applies to the case of the RT_0 scheme.

In the case of the Richards equation, the scheme is obtained by replacing one of the discrete conservation equations by the imposed value for the pressure. A possible extension of this work is the generalization of this result to the RT_k scheme, which could be obtained by a comparison between the reconstruction operator with a piecewise constant reconstruction operator.

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