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A UNIFIED APPROACH TO MIMETIC FINITE DIFFERENCE, HYBRID FINITE VOLUME AND MIXED FINITE VOLUME METHODS

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We investigate the connections between several recent methods for the discretization of anisotropic heterogeneous diffusion operators on general grids. We prove that the Mimetic Finite Difference scheme, the Hybrid Finite Volume scheme and the Mixed Finite Volume scheme are in fact identical up to some slight generalizations. As a consequence, some of the mathematical results obtained for each of the method (such as convergence properties or error estimates) may be extended to the unified common framework. We then focus on the relationships between this unified method and nonconforming Finite Element schemes or Mixed Finite Element schemes. We also show that for isotropic operators, on particular meshes such as triangular meshes with acute angles, the unified method boils down to the well-known efficient two-point flux Finite Volume scheme.

Keywords: Elliptic equation Hybrid finite volume scheme, mixed finite volume scheme, Mimetic method

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1. Introduction

A benchmark was organized at the last FVCA 5 conference²⁵ in June 2008 to test the recently developed methods for the numerical solution of heterogeneous anisotropic

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problems. In that benchmark and in this paper, we consider the Poisson equation with homogeneous boundary condition

$$-\operatorname{div}(\Lambda \nabla p) = f \quad \text{in } \Omega, \quad (1.1a)$$

$$p = 0 \quad \text{on } \partial\Omega, \quad (1.1b)$$

where Ω is a bounded open subset of \mathbf{R}^d ($d \geq 1$), $\Lambda : \Omega \rightarrow M_d(\mathbf{R})$ is bounded measurable symmetric and uniformly elliptic (i.e. there exists $\zeta > 0$ such that, for a.e. $x \in \Omega$ and all $\xi \in \mathbf{R}^d$, $\Lambda(x)\xi \cdot \xi \geq \zeta|\xi|^2$) and $f \in L^2(\Omega)$.

The results of the FVCA benchmark, in particular those of Refs. 10, 20, 28, 29, seem to demonstrate that the behavior of three of the submitted methods, namely the Hybrid Finite Volume method^{19,22,21}, the Mimetic Finite Difference method^{5,8}, and the Mixed Finite Volume method^{15,16} are quite similar in a number of cases (to keep notations light while retaining the legibility, in the following we call ‘‘Hybrid’’, ‘‘Mimetic’’ and ‘‘Mixed’’ these respective methods).

A straightforward common point between these methods is that they are written using a general partition of Ω into polygonal open subsets and that they introduce unknowns which approximate the solution p and the fluxes of its gradient on the edges of the partition. However, a comparison of the methods is still lacking, probably because their mathematical analysis relies on different tool boxes. The mathematical analysis of the Mimetic method⁵ is based on an error estimate technique (in the spirit of the mixed finite element methods). For the Hybrid and the Mixed methods^{21,15}, the convergence proofs rely on discrete functional analysis tools which have recently been adapted to the discontinuous Galerkin framework¹³. The aim of this paper is to point out the common points of these three methods. To this purpose, we first gather, in Table 1 of Section 2, some definitions and notations associated with each method, and we present the three methods as they are introduced in the literature; we also present a generalized or modified form of each of the methods. The three resulting methods are then shown to be identical (Section 3) and to inherit some of the mathematical properties of the initial methods (Section 4). Particular cases are then explored in Section 5, which show the thorough relations between this unified method and a few classical methods (nonconforming Finite Elements, Mixed Finite Elements and two-point flux Finite Volumes) and, as a by-product, provide a flux lifting operator for a particular choice of the stabilization parameters (but any kind of grid). Finally, a few technical results are provided in an appendix.

2. The methods

We first provide the definitions and notations associated with each method (in order for the readers who are familiar with one or the other theory to easily follow the subsequent analysis, we shall freely use one or the other notation (this of course yields some redundant notations)).

	<i>Mim.</i>	<i>FV</i>
Partition of Ω in polygonal sets	Ω_h	\mathcal{M}
Elements of the partition (grid cells) (“control volume” in the finite volume framework)	E	K
Set of edges/faces of a grid cell	∂E , or numbered from 1 to k_E	\mathcal{E}_K
Edges/faces of grid cell	e	σ
Space of discrete p unknowns (piecewise approximations of p on the partition)	Q^h	$H_{\mathcal{M}}$
Approximation of the solution p on the grid cell $E = K$	p_E	p_K
Discrete flux: approximation of $\frac{1}{ e } \int_e -\Lambda \nabla p \cdot \mathbf{n}_E^e$ (with $e = \sigma$ an edge of $E = K$ and $\mathbf{n}_E^e = \mathbf{n}_{K,\sigma}$ the unit normal to e outward E)	F_E^e	$F_{K,\sigma}$
Space of discrete fluxes (approximations of $\frac{1}{ e } \int_{e/\sigma} -\Lambda \nabla p \cdot \mathbf{n}$)	X^h	\mathcal{F}

Table 1. Usual notations and definitions in Mimetic (*Mim.*) and Finite Volume frameworks (*FV*).

Remark 2.1. In the usual Finite Volume literature, the quantity $F_{K,\sigma}$ is usually rather an approximation of $\int_{\sigma} -\Lambda \nabla p \cdot \mathbf{n}_{K,\sigma}$ than $\frac{1}{|\sigma|} \int_{\sigma} -\Lambda \nabla p \cdot \mathbf{n}_{K,\sigma}$; we choose here the latter normalization in order to simplify the comparison.

In all three methods, a natural condition is imposed on the gradient fluxes (called conservativity in Finite Volume methods and continuity condition in Mimetic methods): for any interior edge σ (or e) between two polygons K and L (or E_1 and E_2),

$$F_{K,\sigma} + F_{L,\sigma} = 0, \quad (\text{or } F_{E_1}^e = -F_{E_2}^e). \quad (2.1)$$

This condition is included in the definition of the discrete flux space \mathcal{F} (or X^h).

The Mimetic, Hybrid and Mixed methods for (1.1) all consist in seeking $p \in H_{\mathcal{M}}$ (or Q^h) and $F \in \mathcal{F}$ (or X^h), which approximate respectively the solution p and its gradient fluxes, writing equations on these unknowns which discretize the continuous equation (1.1). Each method is in fact a family of schemes rather than a unique scheme: indeed, there exists some freedom on the choice of some of the parameters of the scheme (for instance in the stabilization terms which ensure the coercivity of the methods).

2.1. The Mimetic method

The standard Mimetic method first consists in defining, from the Stokes formula, a discrete divergence operator on the space of the discrete fluxes: for $G \in X^h$,

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$\mathcal{D}\mathcal{I}\mathcal{V}^h G \in Q^h$ is defined by

$$(\mathcal{D}\mathcal{I}\mathcal{V}^h G)_E = \frac{1}{|E|} \sum_{i=1}^{k_E} |e_i| G_E^{e_i}. \quad (2.2)$$

The space Q^h of piecewise constant functions is endowed with the usual L^2 inner product

$$[p, q]_{Q^h} = \sum_{E \in \Omega_h} |E| p_E q_E$$

and a local inner product is defined on the space of fluxes unknowns of each element E :

$$[F_E, G_E]_E = F_E^t \mathbb{M}_E G_E = \sum_{s,r=1}^{k_E} \mathbb{M}_{E,s,r} F_E^{e_s} G_E^{e_r}, \quad (2.3)$$

where \mathbb{M}_E is a symmetric definite positive matrix of order k_E . Each local inner product is assumed to satisfy the following discrete Stokes formula (called Condition **(S2)** in Refs. 5, 8):

$\forall E \in \Omega_h, \forall q$ affine function, $\forall G \in X^h$:

$$[(\Lambda \nabla q)^t, G]_E + \int_E q (\mathcal{D}\mathcal{I}\mathcal{V}^h G)_E dV = \sum_{i=1}^{k_E} G_E^{e_i} \int_{e_i} q d\Sigma, \quad (2.4)$$

where $((\Lambda \nabla q)^t)_E^e = \frac{1}{|e|} \int_e \Lambda_E \nabla q \cdot \mathbf{n}_E^e d\Sigma$ and Λ_E is the value, assumed to be constant, of Λ on E (if Λ is not constant on E , one can take Λ_E equal to the mean value of Λ on E).

Remark 2.2. Note that Condition **(S1)** of Refs. 5, 8 is needed in the convergence study of the method, but not in its definition; therefore it is only recalled in Section 4, see (4.7).

The local inner products (2.3) allow us to construct a complete inner product $[F, G]_{X^h} = \sum_{E \in \Omega_h} [F, G]_E$, which in turn defines a discrete flux operator $\mathcal{G}^h : Q^h \rightarrow X^h$ as the adjoint operator of $\mathcal{D}\mathcal{I}\mathcal{V}^h$: for all $F \in X^h$ and $p \in Q^h$,

$$[F, \mathcal{G}^h p]_{X^h} = [p, \mathcal{D}\mathcal{I}\mathcal{V}^h F]_{Q^h}$$

(notice that this definition of $\mathcal{G}^h p$ takes into account the homogeneous boundary condition (1.1b)). Using these definitions and notations, the standard Mimetic method then reads: find $(p, F) \in Q^h \times X^h$ such that

$$\mathcal{D}\mathcal{I}\mathcal{V}^h F = f_h, \quad F = \mathcal{G}^h p \quad (2.5)$$

where f_h is the L^2 projection of f on Q^h , or in the equivalent weak form: find $(p, F) \in Q^h \times X^h$ such that

$$\forall G \in X^h : [F, G]_{X^h} - [p, \mathcal{D}\mathcal{I}\mathcal{V}^h G]_{Q^h} = 0, \quad (2.6)$$

$$\forall q \in Q^h : [\mathcal{D}\mathcal{I}\mathcal{V}^h F, q]_{Q^h} = [f_h, q]_{Q^h}. \quad (2.7)$$

The precise definition of the Mimetic method requires to choose the local matrices \mathbb{M}_E defining the local inner products $[\cdot, \cdot]_E$. It can be shown⁸ (see also Lemma 7.2 in the appendix) that this matrix defines an inner product satisfying (2.4) if and only if $\mathbb{M}_E \mathbb{N}_E = \bar{\mathbb{R}}_E$ where

$$\begin{aligned} \bar{\mathbb{R}}_E &\text{ is the } k_E \times d \text{ matrix with rows } (|e_i|(\bar{x}_{e_i} - \bar{x}_E)^T)_{i=1, k_E}, \\ &\text{ with } \bar{x}_e \text{ the center of gravity of the edge } e \\ &\text{ and } \bar{x}_E \text{ the center of gravity of the cell } E, \end{aligned} \quad (2.8)$$

and \mathbb{N}_E is the $k_E \times d$ matrix with columns:

$$\begin{aligned} (\mathbb{N}_E)_j &= \begin{pmatrix} (\Lambda_E)_j \cdot \mathbf{n}_E^{e_1} \\ \vdots \\ (\Lambda_E)_j \cdot \mathbf{n}_E^{e_{k_E}} \end{pmatrix} \quad \text{for } j = 1, \dots, d, \\ &(\Lambda_E)_j \text{ being the } j\text{-th column of } \Lambda_E, \end{aligned} \quad (2.9)$$

or equivalently

$$\mathbb{M}_E = \frac{1}{|E|} \bar{\mathbb{R}}_E \Lambda_E^{-1} \bar{\mathbb{R}}_E^T + \mathbb{C}_E \mathbb{U}_E \mathbb{C}_E^T \quad (2.10)$$

where

$$\mathbb{C}_E \text{ is a } k_E \times (k_E - d) \text{ matrix such that } \text{Im}(\mathbb{C}_E) = (\text{Im}(\mathbb{N}_E))^\perp, \quad (2.11)$$

$$\mathbb{U}_E \text{ is a } (k_E - d) \times (k_E - d) \text{ symmetric definite positive matrix.} \quad (2.12)$$

Here, we consider a slightly more general version of the Mimetic method, replacing \bar{x}_E by a point x_E which may be chosen different from the center of gravity of E . We therefore take

$$\mathbb{M}_E = \frac{1}{|E|} \mathbb{R}_E \Lambda_E^{-1} \mathbb{R}_E^T + \mathbb{C}_E \mathbb{U}_E \mathbb{C}_E^T \quad (2.13)$$

where

$$\begin{aligned} \mathbb{R}_E &\text{ is the } k_E \times d \text{ matrix with rows } (|e_i|(\bar{x}_{e_i} - x_E)^T)_{i=1, k_E}, \\ &\text{ with } \bar{x}_e \text{ the center of gravity of the edge } e \\ &\text{ and } x_E \text{ any point in the cell } E. \end{aligned} \quad (2.14)$$

The other matrices \mathbb{C}_E and \mathbb{U}_E remain given by (2.11) and (2.12). The choice [(2.13),(2.14)] of \mathbb{M}_E no longer gives, in general, an inner product $[\cdot, \cdot]_E$ which satisfies (2.4), but it yields a generalization of this assumption; indeed, choosing a weight function $w_E : E \rightarrow \mathbf{R}$ such that

$$\int_E w_E(x) dx = |E| \quad \text{and} \quad \int_E x w_E(x) dx = |E| x_E, \quad (2.15)$$

we prove in the appendix (Section 7.1) that the matrix \mathbb{M}_E can be written (2.13) with \mathbb{R}_E defined by (2.14) if and only if the corresponding inner product $[\cdot, \cdot]_E$

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satisfies

$$\forall E \in \Omega_h, \forall q \text{ affine function}, \forall G \in X^h : \quad (2.16)$$

$$[(\Lambda \nabla q)^I, G]_E + \int_E q (\mathcal{D}\mathcal{I}\mathcal{V}^h G)_E w_E dV = \sum_{i=1}^{k_E} G_E^{e_i} \int_{e_i} q d\Sigma.$$

Definition 2.1 (Generalized Mimetic method). The Generalized Mimetic scheme for (1.1) reads:

Find $(p, F) \in Q^h \times X^h$ which satisfies [(2.2),(2.3),(2.6),(2.7),(2.13),(2.14)].

Its parameters are the family of points $(x_E)_{E \in \Omega_h}$ (which are freely chosen inside each grid cell) and the family of stabilization matrices $(\mathbb{C}_E, \mathbb{U}_E)_{E \in \Omega_h}$ satisfying (2.11) and (2.12).

2.2. The Hybrid method

The standard Hybrid method is best defined using additional unknowns p_σ playing the role of approximations of p on the edges of the discretization of Ω ; if \mathcal{E} is the set of such edges, we let $\tilde{H}_{\mathcal{M}}$ be the extension of $H_{\mathcal{M}}$ consisting of vectors $\tilde{p} = ((p_K)_{K \in \mathcal{M}}, (p_\sigma)_{\sigma \in \mathcal{E}})$. It will also be useful to consider the space \tilde{H}_K of the restrictions $\tilde{p}_K = (p_K, (p_\sigma)_{\sigma \in \mathcal{E}_K})$ to a control volume K and its edge of the elements $\tilde{p} \in \tilde{H}_{\mathcal{M}}$. A discrete gradient inside K is defined for $\tilde{p}_K \in \tilde{H}_K$ by

$$\nabla_K \tilde{p}_K = \frac{1}{|K|} \sum_{\sigma \in \mathcal{E}_K} |\sigma| (p_\sigma - p_K) \mathbf{n}_{K,\sigma}. \quad (2.17)$$

The definition of this discrete gradient stems from the fact that, for any vector \mathbf{e} , any control volume K and any $x_K \in \mathbf{R}^d$, we have

$$|K| \mathbf{e} = \sum_{\sigma \in \mathcal{E}_K} |\sigma| \mathbf{e} \cdot (\bar{x}_\sigma - x_K) \mathbf{n}_{K,\sigma} \quad (2.18)$$

where \bar{x}_σ is the center of gravity of σ and x_K is any point of K . Hence the discrete gradient is consistent in the sense that if $p_\sigma = \psi(\bar{x}_\sigma)$ and $p_K = \psi(x_K)$ for an affine function ψ on K , then $\nabla_K \tilde{p}_K = \nabla \psi$ on K (note that this consistency is not sufficient to ensure strong convergence, and in fact, only weak convergence of the discrete gradient will be obtained). A stabilization needs to be added to the discrete gradient (2.17) in order to build a discrete coercive bilinear form expected to approximate the bilinear form $(u, v) \mapsto \int_\Omega \Lambda \nabla u \cdot \nabla v dx$ occurring in the weak formulation of (1.1). Choosing a point x_K for each control volume K (for instance the center of gravity, but this is not mandatory), and keeping in mind that p_K is expected to be an approximation of the solution p of (1.1) at this point, a second-order consistency error term $S_K(\tilde{p}_K) = (S_{K,\sigma}(\tilde{p}_K))_{\sigma \in \mathcal{E}_K}$ is defined by

$$S_{K,\sigma}(\tilde{p}_K) = p_\sigma - p_K - \nabla_K \tilde{p}_K \cdot (\bar{x}_\sigma - x_K). \quad (2.19)$$

Note that thanks to (2.18),

$$\sum_{\sigma \in \mathcal{E}_K} |\sigma| S_{K,\sigma}(\tilde{p}_K) \mathbf{n}_{K,\sigma} = 0, \quad (2.20)$$

and that $S_{K,\sigma}(\tilde{p}_K) = 0$ if $p_\sigma = \psi(\bar{x}_\sigma)$ and $p_K = \psi(x_K)$ for an affine function ψ on K .

The fluxes $(F_{K,\sigma})_{\sigma \in \mathcal{E}_K}$ on the boundary of a control volume K associated with some $\tilde{p} \in \tilde{H}_\mathcal{M}$ are then defined by imposing that

$$\begin{aligned} \forall K \in \mathcal{M}, \forall \tilde{q}_K = (q_K, (q_\sigma)_{\sigma \in \mathcal{E}_K}) \in \tilde{H}_K : \\ \sum_{\sigma \in \mathcal{E}_K} |\sigma| F_{K,\sigma}(q_K - q_\sigma) = |K| \Lambda_K \nabla_K \tilde{p}_K \cdot \nabla_K \tilde{q}_K + \\ \sum_{\sigma \in \mathcal{E}_K} \alpha_{K,\sigma} \frac{|\sigma|}{d_{K,\sigma}} S_{K,\sigma}(\tilde{p}_K) S_{K,\sigma}(\tilde{q}_K) \end{aligned} \quad (2.21)$$

where Λ_K is the mean value of Λ on K , $d_{K,\sigma}$ is the distance between x_K and the hyperplane containing σ and $\alpha_{K,\sigma} > 0$ is a stabilization parameter, which ensures that the resulting matrix is invertible. Note that $F_{K,\sigma}$ is uniquely defined by (2.21), since this equation is equivalent to

$$|\sigma| F_{K,\sigma} = |K| \Lambda_K \nabla_K \tilde{p}_K \cdot \nabla_K \tilde{q}_K + \sum_{\sigma' \in \mathcal{E}_K} \alpha_{K,\sigma'} \frac{|\sigma'|}{d_{K,\sigma'}} S_{K,\sigma'}(\tilde{p}_K) S_{K,\sigma'}(\tilde{q}_K) \quad (2.22)$$

where \tilde{q}_K satisfies $q_K - q_\sigma = 1$ and $q_K - q_{\sigma'} = 0$ for $\sigma' \neq \sigma$. To take into account the boundary condition (1.1b), we denote by $\tilde{H}_{\mathcal{M},0} = \{\tilde{p} \in \tilde{H}_\mathcal{M} \text{ such that } p_\sigma = 0 \text{ if } \sigma \subset \partial\Omega\}$ and the Hybrid method can then be written: find $\tilde{p} \in \tilde{H}_{\mathcal{M},0}$ such that, with $F_{K,\sigma}$ defined by (2.21),

$$\forall \tilde{q} \in \tilde{H}_{\mathcal{M},0} : \sum_{K \in \mathcal{M}} \sum_{\sigma \in \mathcal{E}_K} |\sigma| F_{K,\sigma}(q_K - q_\sigma) = \sum_{K \in \mathcal{M}} q_K \int_K f. \quad (2.23)$$

In particular, taking $q_K = 0$ for all K , and $q_\sigma = 1$ for one σ and 0 for the others in (2.23) yields that F satisfies (2.1); once this conservativity property is imposed by requiring that $F \in \mathcal{F}$, we may eliminate the q_σ from (2.23) and reduce the Hybrid method to: find $(\tilde{p}, F) \in \tilde{H}_{\mathcal{M},0} \times \mathcal{F}$ satisfying (2.21) and

$$\forall K \in \mathcal{M} : \sum_{\sigma \in \mathcal{E}_K} |\sigma| F_{K,\sigma} = \int_K f. \quad (2.24)$$

This last equation is the flux balance, one of the key ingredients of the finite volume methods.

Let us now introduce a generalization of the Hybrid method for the comparison with the other methods. As previously mentioned, the stabilization term S_K is added in (2.21) in order to obtain a coercive scheme (the sole discrete gradient $(\nabla_K \tilde{p}_K)_{K \in \mathcal{M}}$ does not allow to control p itself); the important feature of S_K is

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that it yields a coercive bilinear form, so that we may in fact replace (2.21) by the more general equation

$$\begin{aligned} \forall K \in \mathcal{M}, \forall \tilde{q}_K \in \tilde{H}_K : \sum_{\sigma \in \mathcal{E}_K} |\sigma| F_{K,\sigma} (q_K - q_\sigma) \\ = |K| \Lambda_K \nabla_K \tilde{p}_K \cdot \nabla_K \tilde{q}_K + \sum_{\sigma, \sigma' \in \mathcal{E}_K} \mathbb{B}_{K,\sigma,\sigma'}^H S_{K,\sigma}(\tilde{p}_K) S_{K,\sigma'}(\tilde{q}_K) \\ = |K| \Lambda_K \nabla_K \tilde{p}_K \cdot \nabla_K \tilde{q}_K + S_K(\tilde{q}_K)^T \mathbb{B}_K^H S_K(\tilde{p}_K), \end{aligned} \quad (2.25)$$

where $\mathbb{B}_{K,\sigma,\sigma'}^H$ are the entries of a symmetric definite positive matrix \mathbb{B}_K^H (the superscript H refers to the Hybrid method).

Definition 2.2 (Generalized Hybrid method). A Generalized Hybrid scheme for (1.1) reads:

$$\text{Find } (\tilde{p}, F) \in \tilde{H}_{\mathcal{M},0} \times \mathcal{F} \text{ which satisfies } [(2.17),(2.19),(2.24),(2.25)].$$

Its parameters are the family of points $(x_K)_{K \in \mathcal{M}}$ (which are freely chosen inside each grid cell) and the family $(\mathbb{B}_K^H)_{K \in \mathcal{M}}$ of symmetric definite positive matrices.

Remark 2.3. Another presentation of the Generalized Hybrid method is possible from (2.23) and (2.25) by eliminating the fluxes:

$$\begin{aligned} \text{Find } \tilde{p} \in \tilde{H}_{\mathcal{M},0} \text{ such that, } \forall \tilde{q} \in \tilde{H}_{\mathcal{M},0} : \\ \sum_{K \in \mathcal{M}} |K| \Lambda_K \nabla_K \tilde{p}_K \cdot \nabla_K \tilde{q}_K + \sum_{K \in \mathcal{M}} S_K(\tilde{q}_K)^T \mathbb{B}_K^H S_K(\tilde{p}_K) \\ = \sum_{K \in \mathcal{M}} q_K \int_K f. \end{aligned} \quad (2.26)$$

2.3. The Mixed method

As in the Hybrid method, we use the unknowns \tilde{p} in $\tilde{H}_{\mathcal{M}}$ for the Mixed method (that is to say approximate values of the solution to the equation inside the control volumes and on the edges), and fluxes unknowns in \mathcal{F} . However, contrary to the Hybrid method, the discrete gradient is no longer defined from p , but rather from the fluxes, using the dual version of (2.18), that is:

$$|K| \mathbf{e} = \sum_{\sigma \in \mathcal{E}_K} |\sigma| \mathbf{e} \cdot \mathbf{n}_{K,\sigma} (\bar{x}_\sigma - x_K). \quad (2.27)$$

For $F \in \mathcal{F}$, we denote $F_K = (F_{K,\sigma})_{\sigma \in \mathcal{E}_K}$ its restriction to the edges of the control volume K and \mathcal{F}_K is the set of such restrictions; if $F_K \in \mathcal{F}_K$, then $\mathbf{v}_K(F_K)$ is defined by

$$|K| \Lambda_K \mathbf{v}_K(F_K) = - \sum_{\sigma \in \mathcal{E}_K} |\sigma| F_{K,\sigma} (\bar{x}_\sigma - x_K) \quad (2.28)$$

where, again, Λ_K is the mean value of Λ on K , \bar{x}_σ is the center of gravity of σ and x_K a point chosen inside K . Recalling that $F_{K,\sigma}$ is an approximation of $\frac{1}{|\sigma|} \int_\sigma -\Lambda \nabla p \cdot \mathbf{n}_{K,\sigma}$, Formula (2.27) then shows that $\mathbf{v}_K(F_K)$ is expected to play the role of an approximation of ∇p in K .

The Mixed method then consists in seeking $(\tilde{p}, F) \in \tilde{H}_{\mathcal{M},0} \times \mathcal{F}$ (recall that $F \in \mathcal{F}$ satisfies (2.1), and we impose $p_\sigma = 0$ if $\sigma \subset \partial\Omega$) which satisfies the following natural discrete relation between p and this discrete gradient, with a stabilization term involving the fluxes and a positive parameter $\nu > 0$:

$$\forall K \in \mathcal{M}, \forall \sigma \in \mathcal{E}_K : p_\sigma - p_K = \mathbf{v}_K(F_K) \cdot (\bar{x}_\sigma - x_K) - \nu \text{diam}(K) F_{K,\sigma} \quad (2.29)$$

and the flux balance:

$$\forall K \in \mathcal{M} : \sum_{\sigma \in \mathcal{E}_K} |\sigma| F_{K,\sigma} = \int_K f. \quad (2.30)$$

Multiplying, for any $G_K \in \mathcal{F}_K$, each equation of (2.29) by $|\sigma| G_{K,\sigma}$ and summing on $\sigma \in \mathcal{E}_K$, we obtain

$$\begin{aligned} \forall K \in \mathcal{M}, \forall G_K \in \mathcal{F}_K : & \sum_{\sigma \in \mathcal{E}_K} (p_K - p_\sigma) |\sigma| G_{K,\sigma} \\ & = |K| \mathbf{v}_K(F_K) \cdot \Lambda_K \mathbf{v}_K(G_K) + \sum_{\sigma \in \mathcal{E}_K} \nu \text{diam}(K) |\sigma| F_{K,\sigma} G_{K,\sigma}. \end{aligned}$$

The term $\sum_{\sigma \in \mathcal{E}_K} \nu \text{diam}(K) |\sigma| F_{K,\sigma} G_{K,\sigma}$ in this equation can be considered as a strong stabilization term, in the sense that it vanishes (for all G_K) only if F_K vanishes. We modify here the Mixed method by replacing this strong stabilization by a weaker stabilization which, as in the Hybrid method, is expected to vanish on ‘‘appropriate fluxes’’. To achieve this, we use the quantity

$$T_{K,\sigma}(F_K) = F_{K,\sigma} + \Lambda_K \mathbf{v}_K(F_K) \cdot \mathbf{n}_{K,\sigma}, \quad (2.31)$$

which vanishes if $(F_{K,\sigma})_{\sigma \in \mathcal{E}_K}$ are the genuine fluxes of a given vector \mathbf{e} . Then, taking \mathbb{B}_K^M to be a symmetric positive definite matrix, we endow \mathcal{F}_K with the inner product

$$\begin{aligned} \langle F_K, G_K \rangle_K & = |K| \mathbf{v}_K(F_K) \cdot \Lambda_K \mathbf{v}_K(G_K) + \sum_{\sigma, \sigma' \in \mathcal{E}_K} \mathbb{B}_{K,\sigma,\sigma'}^M T_{K,\sigma}(F_K) T_{K,\sigma'}(G_K) \\ & = |K| \mathbf{v}_K(F_K) \cdot \Lambda_K \mathbf{v}_K(G_K) + T_K(G_K)^T \mathbb{B}_K^M T_K(F_K) \end{aligned} \quad (2.32)$$

and we replace the stabilized formula (2.29) linking p and F by

$$\forall K \in \mathcal{M}, \forall G_K \in \mathcal{F}_K : \langle F_K, G_K \rangle_K = \sum_{\sigma \in \mathcal{E}_K} (p_K - p_\sigma) |\sigma| G_{K,\sigma}. \quad (2.33)$$

We can get back a formulation more along the lines of (2.29) if we fix $\sigma \in \mathcal{E}_K$ and take $G_K(\sigma) \in \mathcal{F}_K$ defined by $G_K(\sigma)_\sigma = 1$ and $G_K(\sigma)_{\sigma'} = 0$ if $\sigma' \neq \sigma$: (2.33) then gives

$$p_\sigma - p_K = -\frac{1}{|\sigma|} |K| \mathbf{v}_K(F_K) \cdot \Lambda_K \mathbf{v}_K(G_K(\sigma)) - \frac{1}{|\sigma|} T_K(G_K(\sigma))^T \mathbb{B}_K^M T_K(F_K).$$

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But $|K|\Lambda_K \mathbf{v}_K(G_K(\sigma)) = -|\sigma|(\bar{x}_\sigma - x_K)$ and thus

$$p_\sigma - p_K = \mathbf{v}_K(F_K) \cdot (\bar{x}_\sigma - x_K) - \frac{1}{|\sigma|} T_K(G_K(\sigma))^T \mathbb{B}_K^M T_K(F_K). \quad (2.34)$$

This equation is precisely the natural discrete relation (2.29) between p and its gradient, in which the strong stabilization involving $F_{K,\sigma}$ has been replaced by a “weak” stabilization using $T_K(F_K)$.

Definition 2.3 (Modified Mixed method). A Modified Mixed scheme for (1.1) reads:

Find $(\tilde{p}, F) \in \tilde{H}_{\mathcal{M},0} \times \mathcal{F}$ which satisfies [(2.28),(2.30),(2.31),(2.32),(2.33)].

Its parameters are the family of points $(x_K)_{K \in \mathcal{M}}$ (which are freely chosen inside each grid cell) and the family $(\mathbb{B}_K^M)_{K \in \mathcal{M}}$ of symmetric definite positive matrices.

Remark 2.4. If F_K and \tilde{p}_K are linked through (2.33) then the discrete gradient $\mathbf{v}_K(F_K)$ and $\nabla_K \tilde{p}_K$ from the Modified Mixed and Generalized Hybrid methods are in fact identical. Indeed, we then have (2.34) and thus, by (2.18),

$$\begin{aligned} \nabla_K \tilde{p}_K &= \mathbf{v}_K(F_K) \\ &\quad - \frac{1}{|K|} \sum_{\sigma', \sigma'' \in \mathcal{E}_K} \mathbb{B}_{K,\sigma',\sigma''}^M T_{K,\sigma'}(F_K) \sum_{\sigma \in \mathcal{E}_K} T_{K,\sigma''}(G_K(\sigma)) \mathbf{n}_{K,\sigma}. \end{aligned} \quad (2.35)$$

But $\Lambda_K \mathbf{v}_K(G_K(\sigma)) = -\frac{|\sigma|}{|K|}(\bar{x}_\sigma - x_K)$ and thus, using again (2.18),

$$\begin{aligned} \sum_{\sigma \in \mathcal{E}_K} T_{K,\sigma''}(G_K(\sigma)) \mathbf{n}_{K,\sigma} &= \mathbf{n}_{K,\sigma''} - \frac{1}{|K|} \sum_{\sigma \in \mathcal{E}_K} |\sigma|(\bar{x}_\sigma - x_K) \cdot \mathbf{n}_{K,\sigma''} \mathbf{n}_{K,\sigma} \\ &= \mathbf{n}_{K,\sigma''} - \mathbf{n}_{K,\sigma''} = 0 \end{aligned}$$

so that (2.35) reduces to $\nabla_K \tilde{p}_K = \mathbf{v}_K(F_K)$.

Note that this gradient was also used in the MFD setting for error estimators² and for the reconstruction of a piecewise linear approximate solution⁹.

3. Algebraic correspondence between the three methods

We now focus on the main result of this paper, which is the following.

Theorem 3.1 (Equivalence of the methods). *The Generalized Mimetic, Generalized Hybrid and Modified Mixed methods (see Definitions 2.1, 2.2 and 2.3) are identical, in the sense that for any choice of parameters for one of these methods, there exists a choice of parameters for the other two methods which leads to the same scheme.*

The proof of this result is given in the remainder of this section, and decomposed as follows: for comparison purposes, the Generalized Mimetic method is first written under a hybridized form in Section 3.1; then, the correspondence between the Generalized Mimetic and the Modified Mixed methods is studied in Section 3.2;

finally, the correspondence between the Generalized Mimetic and the Generalized Hybrid methods is carried out in Section 3.3.

3.1. Hybridization of the Generalized Mimetic method

Although the edge unknowns introduced to define the Generalized Hybrid and Modified Mixed methods may be eliminated, they are in fact essential to these methods; indeed, both methods can be hybridized and reduced to a system with unknowns $(p_\sigma)_{\sigma \in \mathcal{E}}$ only. In order to compare the methods, we therefore also introduce such edge unknowns in the Generalized Mimetic method; this corresponds to the Lagrange multipliers procedure, quite classical for instance for the Raviart-Thomas mixed finite element, and already used in Ref. 27.

Let E be a grid cell and $e \in \partial E$ be an edge. If e is an interior edge, we denote by \tilde{E} the cell on the other side of e and define $G(e, E) \in X^h$ by: $G(e, E)|_E^e = 1$, $G(e, E)|_{\tilde{E}}^e = -1$ and $G(e, E)|_{E'}^{e'} = 0$ if $e' \neq e$. We notice that $G(e, E) = -G(e, \tilde{E})$ and, using $G(e, E)$ in (2.6), the definitions of \mathcal{DTV}^h and of the inner products on X^h and Q^h give $|e|(p_E - p_{\tilde{E}}) = [F, G(e, E)]_E + [F, G(e, E)]_{\tilde{E}} = [F, G(e, E)]_E - [F, G(e, \tilde{E})]_{\tilde{E}}$. It is therefore natural to define p_e (only depending on e and not on the grid cell E such that $e \subset \partial E$) by

$$\forall E \in \Omega_h, \forall e \in \partial E : |e|(p_E - p_e) = [F, G(e, E)]_E. \quad (3.1)$$

This definition can also be applied for boundary edges $e \subset \partial\Omega$, in which case it gives $p_e = 0$ (thanks to (2.6)).

We thus extend $p \in Q^h$ into an element $\tilde{p} \in \tilde{H}_{\mathcal{M},0}$ having values inside the cells and on the edges of the mesh. Denoting, as in Section 2.3, \mathcal{F}_E the space of restrictions G_E to the edges of E of elements $G \in X^h$ and writing any $G_E \in \mathcal{F}_E$ as a linear combination of $(G(E, e))_{e \in \partial E}$, it is easy to see from (3.1) that the Generalized Mimetic method [(2.6),(2.7)] is equivalent to: find $(\tilde{p}, F) \in \tilde{H}_{\mathcal{M},0} \times X^h$ such that

$$\forall E \in \Omega_h, \forall G_E \in \mathcal{F}_E : [F_E, G_E]_E = \sum_{e \in \partial E} |e|(p_E - p_e)G_E^e \quad (3.2)$$

and

$$\forall E \in \Omega_h : \sum_{e \in \partial E} |e|F_E^e = \int_E f. \quad (3.3)$$

3.2. Proof of the correspondence Generalized Mimetic \leftrightarrow Modified Mixed

The simplest comparison is probably to be found between the Modified Mixed and Generalized Mimetic methods. Indeed, we see from [(2.30),(2.33)] and [(3.2),(3.3)] that both methods are identical provided that, for any grid cell $K = E$, the local inner products defined by (2.32) and (2.13) are equal: $\langle \cdot, \cdot \rangle_K = [\cdot, \cdot]_E$. We therefore have to study these local inner products and understand whether they can be identical (recall that there is some latitude in the choice of both inner products).

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Let us start with the term $|K|\mathbf{v}_K(F_K) \cdot \Lambda_K \mathbf{v}_K(G_K)$ in the definition of $\langle \cdot, \cdot \rangle_K$. Thanks to (2.28),

$$|K|\mathbf{v}_K(F_K) \cdot \Lambda_K \mathbf{v}_K(G_K) = \frac{1}{|K|} \left(\sum_{\sigma \in \mathcal{E}_K} \Lambda_K^{-1} (|\sigma|(\bar{x}_\sigma - x_K)) F_{K,\sigma} \right) \times \left(\sum_{\sigma \in \mathcal{E}_K} |\sigma|(\bar{x}_\sigma - x_K) G_{K,\sigma} \right).$$

But, with the definition (2.14) of \mathbb{R}_E , $\sum_{\sigma \in \mathcal{E}_K} |\sigma|(\bar{x}_\sigma - x_K) F_{K,\sigma} = \mathbb{R}_E^T F_K$ and thus

$$|K|\mathbf{v}_K(F_K) \cdot \Lambda_K \mathbf{v}_K(G_K) = \frac{1}{|K|} (\Lambda_K^{-1} \mathbb{R}_E^T F_K) \cdot (\mathbb{R}_E^T G_K) = G_K^T \left(\frac{1}{|K|} \mathbb{R}_E \Lambda_K^{-1} \mathbb{R}_E^T \right) F_K.$$

Hence, the term $|K|\mathbf{v}_K(\cdot) \cdot \Lambda_K \mathbf{v}_K(\cdot)$ in the definition of $\langle \cdot, \cdot \rangle_K$ is identical to the first term $\frac{1}{|E|} \mathbb{R}_E \Lambda_E^{-1} \mathbb{R}_E^T$ in the definition of the matrix \mathbb{M}_E of $[\cdot, \cdot]_E$ (see (2.13)). Therefore, in order to prove that $\langle \cdot, \cdot \rangle_K = [\cdot, \cdot]_E$, it only remains to prove that, for appropriate choices of \mathbb{C}_E , \mathbb{U}_E and \mathbb{B}_K^M , we have for any $(F_K, G_K) \in \mathcal{F}_K^2$:

$$T_K(G_K)^T \mathbb{B}_K^M T_K(F_K) = G_K^T \mathbb{C}_E \mathbb{U}_E \mathbb{C}_E^T F_K \quad (3.4)$$

(see (2.32) and (2.13)); in fact, this is the consequence of Lemma 7.3 in the appendix and of the following lemma.

Lemma 3.1. *The mappings $T_K : \mathbf{R}^{k_E} \rightarrow \mathbf{R}^{k_E}$ and $\mathbb{C}_E^T : \mathbf{R}^{k_E} \rightarrow \mathbf{R}^{k_E-d}$ have the same kernel.*

Proof. We first prove that:

- i) $\text{Im}(\mathbb{N}_E) \subset \text{Ker}(T_K)$ i.e. $T_{K,\sigma}((\mathbb{N}_E)_j) = 0$ for all $\sigma \in \mathcal{E}_K$ and all $j = 1, \dots, d$ (which also amounts to the fact that the lines of T_K are orthogonal to the vectors $(\mathbb{N}_E)_j$).
- ii) $\dim(\text{Im}(T_K)) = k_E - d$, and therefore $\dim(\text{Ker}(T_K)) = d$.

Item i) follows from (2.28) and (2.27): we have $\Lambda_K \mathbf{v}_K((\mathbb{N}_E)_j) = -\frac{1}{|K|} \sum_{\sigma \in \mathcal{E}_K} |\sigma| (\Lambda_K)_j \cdot \mathbf{n}_{K,\sigma} (\bar{x}_\sigma - x_K) = -(\Lambda_K)_j$ and thus $T_{K,\sigma}((\mathbb{N}_E)_j) = (\Lambda_K)_j \cdot \mathbf{n}_{K,\sigma} - (\Lambda_K)_j \cdot \mathbf{n}_{K,\sigma} = 0$.

In order to obtain Item ii), we first remark that $k_E - d$ is an upper bound of the rank of T_K since the lines of T_K are in the orthogonal space of the independent vectors $((\mathbb{N}_E)_j)_{j=1,\dots,d}$ ^(a). Moreover, (2.28) shows that the rank of $\mathbf{v}_K : \mathbf{R}^{k_E} \rightarrow \mathbf{R}^{k_E}$ is the rank of the family $(\bar{x}_\sigma - x_K)_{\sigma \in \mathcal{E}_K}$, that is to say d , and the kernel of \mathbf{v}_K therefore has dimension $k_E - d$; since $T_K = \text{Id}$ on this kernel, we conclude that the rank of T_K is at least $k_E - d$, which proves ii).

^aLet us notice that the independence of $((\mathbb{N}_E)_j)_{j=1,\dots,d}$ can be deduced from the independence of the columns of Λ_K : thanks to (2.27), any non-trivial linear combination of the $(\mathbb{N}_E)_j$ gives a non-trivial combination of the columns of Λ_K .

These properties show that $\text{Ker}(T_K) = \text{Im}(\mathbb{N}_E) = (\text{Im}(\mathbb{C}_E))^\perp = \text{ker}(\mathbb{C}_E^T)$, and the proof is complete. \square

The comparison between $T_K(G_K)^T \mathbb{B}_K^M T_K(F_K)$ and $G_K^T \mathbb{C}_E \mathbb{U}_E \mathbb{C}_E^T F_K$ is now straightforward. Indeed, let $(\mathbb{C}_E, \mathbb{U}_E)$ be any pair chosen to construct the Generalized Mimetic method; applying Lemma 7.3, thanks to Lemma 3.1, with $A = \mathbb{C}_E^T$, $B = T_K$ and $\{\cdot, \cdot\}_{\mathbf{R}^{k_E-d}}$ the inner product on \mathbf{R}^{k_E-d} corresponding to \mathbb{U}_E , we obtain an inner product $\{\cdot, \cdot\}_{\mathbf{R}^{k_E}}$ on \mathbf{R}^{k_E} such that $\{T_K(F_K), T_K(G_K)\}_{\mathbf{R}^{k_E}} = \{\mathbb{C}_E^T F_K, \mathbb{C}_E^T G_K\}_{\mathbf{R}^{k_E-d}} = G_K^T \mathbb{C}_E \mathbb{U}_E \mathbb{C}_E^T F_K$; this exactly means that, if we define \mathbb{B}_K^M as the matrix of $\{\cdot, \cdot\}_{\mathbf{R}^{k_E}}$, (3.4) holds. Similarly, inverting the role of \mathbb{C}_E^T and T_K when applying Lemma 7.3, for any \mathbb{B}_K^M used in the Modified Mixed method we can find \mathbb{U}_E satisfying (3.4) and the proof of the correspondence between the Generalized Mimetic and Modified Mixed methods is concluded.

3.3. Proof of the correspondence Generalized Mimetic \leftrightarrow Generalized Hybrid

To compare the Generalized Mimetic and Generalized Hybrid methods, we use a result of Refs. 8, 30 which states that the inverse of the matrix \mathbb{M}_E in (2.13) can be written

$$\mathbb{M}_E^{-1} = \mathbb{W}_E = \frac{1}{|E|} \mathbb{N}_E \Lambda_E^{-1} \mathbb{N}_E^T + \mathbb{D}_E \tilde{\mathbb{U}}_E \mathbb{D}_E^T \quad (3.5)$$

where \mathbb{D}_E is a $k_E \times (k_E - d)$ matrix such that

$$\text{Im}(\mathbb{D}_E) = (\text{Im}(\mathbb{R}_E))^\perp \quad (3.6)$$

and $\tilde{\mathbb{U}}_E$ is a symmetric definite positive $(k_E - d) \times (k_E - d)$ matrix (note that the proof in Refs. 8, 30 assumes x_E to be the center of gravity of E , i.e. \mathbb{M}_E to satisfy (2.10), but that it remains valid for any choice of x_E , i.e. for any matrix \mathbb{M}_E satisfying (2.13)). This result is to be understood in the following sense: for any $(\mathbb{C}_E, \mathbb{U}_E)$ used to construct \mathbb{M}_E by (2.13), there exists $(\mathbb{D}_E, \tilde{\mathbb{U}}_E)$ such that \mathbb{W}_E defined by (3.5) satisfies $\mathbb{W}_E = \mathbb{M}_E^{-1}$, and *vice-versa*.

For $\tilde{p}_E = (p_E, (p_e)_{e \in \partial E}) \in \tilde{H}_E$, we denote by \mathcal{P}_E the vector in \mathbf{R}^{k_E} with components $(|e|(p_E - p_e))_{e \in \partial E}$. The relation (3.2) can be re-written $\mathbb{M}_E F_E = \mathcal{P}_E$, that is to say $F_E = \mathbb{W}_E \mathcal{P}_E$, which is also equivalent, taking the inner product in \mathbf{R}^{k_E} with an arbitrary \mathcal{Q}_E (built from a $\tilde{q}_E \in \tilde{H}_E$), to

$$\forall E \in \Omega_h, \forall \tilde{q}_E \in \tilde{H}_E : \sum_{e \in \partial E} |e|(q_E - q_e) F_E^e = \mathcal{Q}_E^T \mathbb{W}_E \mathcal{P}_E \quad (3.7)$$

The Generalized Mimetic method [(3.2),(3.3)] is therefore identical to the Generalized Hybrid method [(2.24),(2.25)] provided that, for all $E = K \in \Omega_h$,

$$\forall (\tilde{p}_E, \tilde{q}_E) \in \tilde{H}_E : \mathcal{Q}_E^T \mathbb{W}_E \mathcal{P}_E = |K| \Lambda_K \nabla_K \tilde{p}_K \cdot \nabla_K \tilde{q}_K + S_K(\tilde{q}_K)^T \mathbb{B}_K^H S_K(\tilde{p}_K). \quad (3.8)$$

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As in the comparison between the Generalized Mimetic and Modified Mixed methods, the proof of (3.8) is obtained from the separate study of each term of (3.5).

First, by the definition (2.9) of the matrix \mathbb{N}_E and the definition (2.17) of $\nabla_K \tilde{p}_K$, we have $(\mathbb{N}_E^T \mathcal{P}_E)_j = \sum_{e \in \partial E} |e| (\Lambda_E)_j \cdot \mathbf{n}_E^e (p_E - p_e) = -|K| (\Lambda_K)_j \cdot \nabla_K \tilde{p}_K$ for all $j = 1, \dots, d$, that is to say, by symmetry of Λ , $\mathbb{N}_E^T \mathcal{P}_E = -|K| \Lambda_K \nabla_K \tilde{p}_K$. Hence,

$$\mathcal{Q}_E^T \left(\frac{1}{|E|} \mathbb{N}_E \Lambda_K^{-1} \mathbb{N}_E^T \right) \mathcal{P}_E = |K| (\Lambda_K \nabla_K \tilde{q}_K)^T \Lambda_K^{-1} (\Lambda_K \nabla_K \tilde{p}_K) = |K| \Lambda_K \nabla_K \tilde{p}_K \cdot \nabla_K \tilde{q}_K.$$

The first part of the right-hand side in (3.8) thus corresponds to the first part of \mathbb{W}_E in (3.5), and it remains to prove that (with appropriate choices of \mathbb{D}_E , $\tilde{\mathbb{U}}_E$ and \mathbb{B}_K^H), for all $(\tilde{p}_E, \tilde{q}_E) \in \tilde{H}_E$, $\mathcal{Q}_E^T \mathbb{D}_E \tilde{\mathbb{U}}_E \mathbb{D}_E^T \mathcal{P}_E = S_K(\tilde{q}_K)^T \mathbb{B}_K^H S_K(\tilde{p}_K)$. Let us notice that, defining $L_K : \mathbf{R}^{k_E} \rightarrow \mathbf{R}^{k_E}$ by

$$L_K(V) = \left(\frac{1}{|\sigma|} V_\sigma - D_K V \cdot (\bar{x}_\sigma - x_K) \right)_{\sigma \in \mathcal{E}_K}^T, \quad (3.9)$$

with $D_K V = \frac{1}{|K|} \sum_{\sigma \in \mathcal{E}_K} V_\sigma \mathbf{n}_{K,\sigma}$,

this boils down (letting $V = \mathcal{P}_E$ and $V' = \mathcal{Q}_E$) to proving that

$$\forall (V, V') \in \mathbf{R}^{k_E} : (V')^T \mathbb{D}_E \tilde{\mathbb{U}}_E \mathbb{D}_E^T V = L_K(V')^T \mathbb{B}_K^H L_K(V). \quad (3.10)$$

As previously, this will be a consequence of Lemma 7.3 and of the following result.

Lemma 3.2. *The mappings $L_K : \mathbf{R}^{k_E} \rightarrow \mathbf{R}^{k_E}$ and $\mathbb{D}_E^T : \mathbf{R}^{k_E} \rightarrow \mathbf{R}^{k_E-d}$ have the same kernel.*

Proof. From (3.6), we get that $\text{Ker}(\mathbb{D}_E^T) = \text{Im}(\mathbb{R}_E)$. Hence it remains to prove that $\text{Im}(\mathbb{R}_E) = \text{Ker}(L_K)$

Let us first prove that $\text{Im}(\mathbb{R}_E) \subset \text{Ker}(L_K)$. The j -th column of \mathbb{R}_E is $(\mathbb{R}_E)_j = (|\sigma| (\bar{x}_\sigma^j - x_K^j))_{\sigma \in \mathcal{E}_K}^T$ (the superscript j denotes the j -th coordinate of points in \mathbf{R}^d). Thus, for any $\mathbf{e} \in \mathbf{R}^d$, by (2.27),

$$D_K(\mathbb{R}_E)_j \cdot \mathbf{e} = \frac{1}{|K|} \sum_{\sigma \in \mathcal{E}_K} |\sigma| \mathbf{e} \cdot \mathbf{n}_{K,\sigma} (\bar{x}_\sigma^j - x_K^j) = \mathbf{e}^j,$$

which means that $D_K(\mathbb{R}_E)_j$ is the j -th vector of the canonical basis of \mathbf{R}^d . We then have $D_K(\mathbb{R}_E)_j \cdot (\bar{x}_\sigma - x_K) = \bar{x}_\sigma^j - x_K^j$ and thus

$$(L_K((\mathbb{R}_E)_j))_\sigma = \bar{x}_\sigma^j - x_K^j - D_K(\mathbb{R}_E)_j \cdot (\bar{x}_\sigma - x_K) = 0;$$

this proves that the columns of \mathbb{R}_E are in the kernel of L_K , that is $\text{Im}(\mathbb{R}_E) \subset \text{Ker}(L_K)$.

Next we notice that the rank of the mapping $D_K: V \in \mathbf{R}^{k_E} \mapsto D_K V \in \mathbf{R}^d$ (i.e. the rank of the family $(\mathbf{n}_{K,\sigma})_{\sigma \in \mathcal{E}_K}$) is d and that the mapping L_K is one-to-one on $\text{Ker}(D_K)$. Hence $\dim(\text{Im}(L_K)) \geq \dim(\text{Ker}(D_K)) = k_E - d$, and therefore $\dim(\text{Ker}(L_K)) \leq d$. But since $\text{Im}(\mathbb{R}_E) \subset \text{Ker}(L_K)$ and $\dim(\text{Im}(\mathbb{R}_E)) = d$ (the rank of the rows of \mathbb{R}_E), we thus get that $\text{Im}(\mathbb{R}_E) = \text{Ker}(L_K)$. \square

From Lemmas 7.3 and 3.2, we deduce as in Section 3.2 that for any choice of $(\mathbb{D}_E, \tilde{\mathbb{U}}_E)$ there corresponds a choice of \mathbb{B}_K^H such that (3.10) holds, and *vice-versa*, which concludes the proof that the Generalized Mimetic method is identical to the Generalized Hybrid method.

Remark 3.1. These proofs and Remark 7.1 show that one can explicitly compute the parameters of one method which gives back the parameters of another method. Notice also that the algebraic computations required in Remark 7.1 to obtain these parameters are made in spaces with small dimensions; the cost of their practical computations is therefore very low. However, to implement the methods, it is not necessary to understand which $(\mathbb{C}_E, \mathbb{U}_E)$ or $(\mathbb{D}_E, \tilde{\mathbb{U}}_E)$ corresponds to which \mathbb{B}_K^M or \mathbb{B}_K^H , since the only useful quantities are $\mathbb{C}_E \mathbb{U}_E \mathbb{C}_E^T$, $\mathbb{D}_E \tilde{\mathbb{U}}_E \mathbb{D}_E^T$, $T_K(\cdot)^T \mathbb{B}_K^M T_K(\cdot)$ and $L_K(\cdot)^T \mathbb{B}_K^H L_K(\cdot)$, and the relations between these quantities are trivial (see (3.4) and (3.10)).

4. Convergence and error estimates

We showed in Section 3 that the three families of schemes which we called Generalized Mimetic, Generalized Hybrid FV and Modified Mixed FV are in fact one family of schemes, which we call the HMM (Hybrid Mimetic Mixed) family for short in the remainder of the paper. We now give convergence and error estimate results for the HMM method.

4.1. Convergence with no regularity

In this section, we are interested in convergence results which hold without any other regularity assumption on the data than those stated in Section 1. We therefore consider that Λ is only bounded and uniformly elliptic (not necessarily Lipschitz continuous or even piecewise continuous), that $f \in L^2(\Omega)$ and that the solution to (1.1) only belongs to $H_0^1(\Omega)$ (and not necessarily to $H^2(\Omega)$).

We study how existing results, previously established for the Hybrid scheme, can be extended to the HMM method. In Ref. 21, we proved the L^2 convergence of the “standard” Hybrid method for a family of partitions of Ω such that any cell K is star-shaped with respect to a point x_K and such that there exists $\theta > 0$ satisfying, for any partition of the family,

$$\max \left(\max_{\substack{\sigma \in \mathcal{E}_{\text{int}} \\ K, L \in \mathcal{M}_\sigma}} \frac{d_{K,\sigma}}{d_{L,\sigma}}, \max_{\substack{K \in \mathcal{M} \\ \sigma \in \mathcal{E}_K}} \frac{\text{diam}(K)}{d_{K,\sigma}} \right) \leq \theta, \quad (4.1)$$

where \mathcal{E}_{int} denotes the set of internal edges of the mesh and \mathcal{M}_σ the set of cells to which σ is an edge. We now show how the convergence of the HMM schemes may be deduced from an easy extension of Theorem 4.1 and Lemma 4.4 in Ref. 21 provided that:

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- in the Generalized Mimetic formulation [(2.2),(2.3),(2.6),(2.7),(2.13), (2.14)],

$$\left. \begin{array}{l} \text{there exist } s_* > 0 \text{ and } S_* > 0, \text{ independent of the mesh,} \\ \text{such that, for all cell } K \text{ and all } V = (V_\sigma)_{\sigma \in \mathcal{E}_K}, \\ s_* \sum_{\sigma \in \mathcal{E}_K} |\sigma| d_{K,\sigma} (V_\sigma)^2 \leq V^T \mathbb{M}_K V \leq S_* \sum_{\sigma \in \mathcal{E}_K} |\sigma| d_{K,\sigma} (V_\sigma)^2, \end{array} \right\} \quad (4.2)$$

- in the Generalized Hybrid formulation [(2.17),(2.19),(2.24),(2.25)], using the notation (3.9),

$$\left. \begin{array}{l} \text{there exist } \bar{s}_* > 0 \text{ and } \bar{S}_* > 0, \text{ independent of the mesh,} \\ \text{such that, for all cell } K \text{ and all } V = (V_\sigma)_{\sigma \in \mathcal{E}_K}, \\ \bar{s}_* \sum_{\sigma \in \mathcal{E}_K} \frac{|\sigma|}{d_{K,\sigma}} (L_K(V))_\sigma^2 \leq L_K(V)^T \mathbb{B}_K^H L_K(V) \leq \\ \bar{S}_* \sum_{\sigma \in \mathcal{E}_K} \frac{|\sigma|}{d_{K,\sigma}} (L_K(V))_\sigma^2, \end{array} \right\} \quad (4.3)$$

- in the Modified Mixed formulation [(2.28),(2.30),(2.31),(2.32),(2.33)],

$$\left. \begin{array}{l} \text{there exist } \tilde{s}_* > 0 \text{ and } \tilde{S}_* > 0, \text{ independent of the mesh,} \\ \text{such that, for all cell } K \text{ and all } V = (V_\sigma)_{\sigma \in \mathcal{E}_K}, \\ \tilde{s}_* \sum_{\sigma \in \mathcal{E}_K} |\sigma| d_{K,\sigma} (T_{K,\sigma}(V))^2 \leq T_K(V)^T \mathbb{B}_K^M T_K(V) \leq \\ \tilde{S}_* \sum_{\sigma \in \mathcal{E}_K} |\sigma| d_{K,\sigma} (T_{K,\sigma}(V))^2. \end{array} \right\} \quad (4.4)$$

The three conditions (4.2), (4.3) and (4.4) are in fact equivalent (this is stated in the next theorem), and one only has to check the condition corresponding to the chosen framework.

Theorem 4.1 (Convergence of the HMM method). *Assume that $\Lambda : \Omega \rightarrow M_d(\mathbf{R})$ is bounded measurable symmetric and uniformly elliptic, that $f \in L^2(\Omega)$ and that the solution to (1.1) belongs to $H_0^1(\Omega)$. Let $\theta > 0$ be given. Consider a family of polygonal meshes of Ω such that any cell K is star-shaped with respect to a point x_K , and satisfying (4.1). Then the three conditions (4.2), (4.3) and (4.4) are equivalent. Moreover, if for any mesh of the family we choose a HMM scheme such that one of the conditions (4.2), (4.3) or (4.4) holds, then, the family of corresponding approximate solutions converges in $L^2(\Omega)$ to the unique solution of (1.1) as the mesh size tends to 0.*

Proof. Let us first prove the equivalence between (4.2) and (4.3), assuming (4.2) to begin with. Using (3.10), we get that, for all V ,

$$V^T \mathbb{D}_K \tilde{\mathbb{U}}_K \mathbb{D}_K^T V = L_K(V)^T \mathbb{B}_K^H L_K(V).$$

Let us apply the above relation to $\tilde{L}_K(V)$ defined by $\tilde{L}_K(V)_\sigma = |\sigma| L_K(V)_\sigma$. From (2.27) (see also (2.18)) and recalling the operator D_K defined in (3.9), we easily get

that $D_K \tilde{L}_K(V) = 0$, which provides $L_K(\tilde{L}_K(V)) = L_K(V)$. Hence we get

$$(\tilde{L}_K(V))^T \mathbb{D}_K \tilde{\mathbb{U}}_K \mathbb{D}_K^T \tilde{L}_K(V) = L_K(V)^T \mathbb{B}_K^H L_K(V).$$

We then remark that $\mathbb{N}_K^T \tilde{L}_K(V) = 0$ (once again from (2.27)) and therefore, using (3.5),

$$L_K(V)^T \mathbb{B}_K^H L_K(V) = (\tilde{L}_K(V))^T \mathbb{W}_K \tilde{L}_K(V). \quad (4.5)$$

Let \mathbb{I}_K be the diagonal matrix $\mathbb{I}_K = \text{diag}((\sqrt{|\sigma|d_{K,\sigma}})_{\sigma \in \mathcal{E}_K})$. Condition (4.2) applied to $G_K = \mathbb{I}_K F_K$ gives, for all $G_K \in \mathcal{F}_K$,

$$s_* G_K^T G_K \leq G_K^T \mathbb{I}_K^{-1} \mathbb{M}_K \mathbb{I}_K^{-1} G_K \leq S_* G_K^T G_K.$$

This shows that the eigenvalues of $\mathbb{I}_K^{-1} \mathbb{M}_K \mathbb{I}_K^{-1}$ are in $[s_*, S_*]$, and thus that the eigenvalues of $\mathbb{I}_K \mathbb{M}_K^{-1} \mathbb{I}_K = \mathbb{I}_K \mathbb{W}_K \mathbb{I}_K$ belong to $[\frac{1}{S_*}, \frac{1}{s_*}]$. Translating this into bounds on $(\mathbb{I}_K^{-1} \tilde{L}_K(V))^T \mathbb{I}_K \mathbb{W}_K \mathbb{I}_K (\mathbb{I}_K^{-1} \tilde{L}_K(V))$, we deduce

$$\frac{1}{S_*} \sum_{\sigma \in \mathcal{E}_K} \frac{1}{|\sigma|d_{K,\sigma}} (\tilde{L}_K(V)_\sigma)^2 \leq (\tilde{L}_K(V))^T \mathbb{W}_K \tilde{L}_K(V) \leq \frac{1}{s_*} \sum_{\sigma \in \mathcal{E}_K} \frac{1}{|\sigma|d_{K,\sigma}} (\tilde{L}_K(V)_\sigma)^2,$$

and (4.3) follows from (4.5) (with $\bar{s}_* = \frac{1}{S_*}$ and $\bar{S}_* = \frac{1}{s_*}$). Reciprocally, from (4.3), following the proof of Lemma 4.4 in Ref. 21 and setting $V_\sigma = |\sigma|(p_K - p_\sigma)$ in that proof, we get the existence of $c_1 > 0$ and $c_2 > 0$, independent of the mesh, such that

$$c_1 \sum_{\sigma \in \mathcal{E}_K} \frac{1}{|\sigma|d_{K,\sigma}} V_\sigma^2 \leq V^T \mathbb{W}_K V \leq c_2 \sum_{\sigma \in \mathcal{E}_K} \frac{1}{|\sigma|d_{K,\sigma}} V_\sigma^2. \quad (4.6)$$

Using \mathbb{I}_K as before, we then get (4.2) with $s_* = \frac{1}{c_2}$ and $S_* = \frac{1}{c_1}$.

Let us turn to the proof of the equivalence between (4.2) and (4.4), beginning by assuming that (4.2) holds. Using (2.27), one has $T_K((\Lambda_K \mathbf{v}_K(F_K) \cdot \mathbf{n}_{K,\sigma})_{\sigma \in \mathcal{E}_K}) = 0$, and thus $T_K(T_K(F_K)) = T_K(F_K)$; hence, noting that $\mathbb{R}_K^T T_K(F_K) = 0$ (once again thanks to (2.27)) and remembering (2.13), (3.4), we see that (4.2) applied to $V = T_K(F_K)$ directly gives (4.4). The reciprocal property follows, in a similar way as (4.6), from a simple adaptation of classical Mixed FV manipulations (used for example in the proof of *a priori* estimates on the approximate solution), see Ref. 15.

Note also that obtaining (4.2) from (4.3) or (4.4) is very similar to Theorem 3.6 in Ref. 8.

We can now conclude the proof of the theorem, that is to say establish the convergence of the approximate solutions: using (4.3), this convergence is a direct consequence of Theorem 4.1 in Ref. 21 with a straightforward adaptation of the proof of Lemma 4.4 in the same reference. The convergence could also be obtained, using (4.4), by an easy adaptation of the techniques of proof used in the standard Mixed setting (see Ref. 15). \square

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Mimetic schemes are usually studied under a condition on the local inner products usually called **(S1)** and which reads⁵: there exist $s_* > 0$, $S_* > 0$ independent of the mesh in the chosen family such that

$$\forall E \in \Omega_h, \forall G \in X^h : s_* \sum_{i=1}^{k_E} |E|(G_E^{e_i})^2 \leq [G, G]_E \leq S_* \sum_{i=1}^{k_E} |E|(G_E^{e_i})^2. \quad (4.7)$$

The mesh regularity assumptions in Refs. 5, 8 also entail the existence of $C_1 > 0$, independent of the mesh, such that

$$\begin{aligned} C_1 \text{diam}(K)^{d-1} &\leq |\sigma|, \quad \forall \sigma \in \mathcal{E}_K, \\ C_1 \text{diam}(K) &\leq d_{K,\sigma}, \quad \forall \sigma \in \mathcal{E}_K. \end{aligned} \quad (4.8)$$

Under such mesh assumptions and since $|\sigma| \leq 2^{d-1} \text{diam}(L)^{d-1}$ whenever $\sigma \in \mathcal{E}_L$, it is easy to see that there exists θ only depending on C_1 such that (4.1) holds; still using (4.8), we see that the quantities $|\sigma|d_{K,\sigma}$ are of the same order as $|K|$ and thus that (4.7) implies (4.2) (with possibly different s_* and S_*). We can therefore apply Theorem 4.1 to deduce the following convergence result under the usual assumptions in the mimetic literature (except for the regularity assumptions on the data).

Corollary 4.1 (Convergence under the usual mimetic assumptions). *We assume that $\Lambda : \Omega \rightarrow M_d(\mathbf{R})$ is bounded measurable symmetric and uniformly elliptic, that $f \in L^2(\Omega)$ and that the solution to (1.1) is in $H_0^1(\Omega)$. Consider a family of polygonal meshes of Ω such that any cell E is star-shaped with respect to a point x_E , and such that (4.8) holds (with C_1 independent of the mesh in the family). We choose local inner products satisfying Condition **(S1)** of Ref. 5 (i.e. (4.7) with s_* , S_* not depending on the mesh) and (2.16). Then, the family of approximate solutions given by the corresponding Generalized Mimetic schemes converges in $L^2(\Omega)$ to the solution of (1.1) as the mesh size tends to 0.*

*In particular, taking x_E as the center of gravity \bar{x}_E of E , (2.16) reduces to Condition **(S2)** of Ref. 5 (that is to say (2.4)), and the family of approximate solutions given by the corresponding “standard” Mimetic schemes converges in $L^2(\Omega)$ to the solution of (1.1) as the mesh size tends to 0.*

Remark 4.1 (Compactness and convergence of a gradient). The proofs of the above convergence results rely on the use of the Kolmogorov compactness theorem on the family of approximate solutions, see Lemma 3.3 in 15 or Section 5.2 in 21. In fact, this latter study shows that for $p \in [1, +\infty)$, any family of piecewise constant functions that is bounded in an adequate discrete mesh-dependent $W^{1,p}$ norm converges in $L^p(\Omega)$ (up to a subsequence) to a function of $W_0^{1,p}(\Omega)$. The regularity of the limit is shown thanks to the weak convergence of a discrete gradient to the gradient of the limit (see Step 1 in the proof of Lemma 5.7 in Ref. 21). Note that this compactness result and the construction of this weakly converging gradient are completely independent of the numerical scheme, which is only used to obtain the discrete $W^{1,p}$ estimate and the fact that the limit is indeed the solution of (1.1).

Moreover, it is possible, from the approximate solutions given by the HMM schemes, to reconstruct a gradient (in a similar way as Eqs. (22)-(26) in Ref. 21) which strongly converges in $L^2(\Omega)$ to the gradient of the solution of (1.1), see Theorem 4.1 in Ref. 21. One can also directly prove¹⁵ that the gradient \mathbf{v}_K , defined from the fluxes by (2.28), is already a strongly convergent gradient in $L^2(\Omega)$; in fact, the strong convergence of this gradient is valid in a more general framework than the one of the methods presented here¹.

4.2. Order 1 error estimate

Let us first consider the original Mimetic Finite Difference method (2.2)–(2.12). This method is consistent in the sense that it satisfies the condition **(S2)** of Ref. 5. Under the assumptions that:

- Condition **(M1)** of Ref. 5 on the domain Ω (namely Ω is polyhedral and its boundary is Lipschitz continuous) and Conditions **(M2)**–**(M6)** of Ref. 5 (corresponding to **(M1)**–**(M4)** in Ref. 8) hold;
- The stability condition **(S1)** of Ref. 5, 8 holds; this condition concerns the eigenvalues of the matrix \mathbb{U}_E in (2.10), and are connected with the discretization,
- $\Lambda \in W^{1,\infty}(\Omega)^{d \times d}$ and $p \in H^2(\Omega)$,

then Theorem 5.2 in Ref. 5 gives an order 1 error estimate on the fluxes, for an adequate discrete norm; moreover, if Ω is convex and the right-hand side belongs to $H^1(\Omega)$, an order 1 error estimate on p in the L^2 norm is also established.

By the equivalence theorem (Theorem 3.1), this result yields similar error estimates for the Generalized Hybrid and Modified Mixed methods in the case where the point x_K is taken as the center of gravity \bar{x}_K of K .

In the case of the original Hybrid method (2.23), an order 1 error estimate is proved in Ref. 21 only under the assumption of an homogeneous isotropic medium, that is $\Lambda = \text{Id}$, and in the case where the solution to (1.1) also belongs to $C^2(\bar{\Omega})$ (but with no convexity assumption on the domain Ω). As stated in Remark 4.2 of Ref. 21, the proof is readily extended if the solution is piecewise H^2 (in fact, in Ref. 21 the situation is more complex because the Hybrid scheme is considered in the more general setting of the SUSHI scheme, which involves the elimination of some or all edge unknowns; see Section 5.4 below).

4.3. Order 2 error estimate

Under an additional assumption of existence of a specific lifting operator, compatible with the considered Mimetic Finite Difference method, a theoretical L^2 -error estimate of order 2 on p is proved in Ref. 5. A condition on the matrix \mathbb{M}_E which ensures the existence of such a lifting operator is given in Ref. 7: in particular, if the smallest eigenvalue of \mathbb{U}_E is large enough (with respect to the inverse of the

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smallest eigenvalue of $\mathbb{C}_E^T \mathbb{C}_E$ and to the largest eigenvalue of a local inner product involving a generic lifting operator), then the existence of a lifting operator compatible with \mathbb{M}_E , and thus the super-convergence of the associated Mimetic scheme, can be proved.

Regarding the consequence on the HMM method, this means that, if x_K is the center of gravity of K and if the symmetric positive definite matrices $(\mathbb{U}_E)_{E \in \Omega_h}$ or $(\mathbb{B}_K^H)_{K \in \mathcal{M}}$ or $(\mathbb{B}_K^M)_{K \in \mathcal{M}}$ are “large enough”, then the approximate solution p given by the corresponding HMM method converges in L^2 with order 2. It does not seem easy to give a practical lower bound on these stabilization terms which ensure that they are indeed “large enough” for the theoretical proof; however, several numerical tests (both with x_K the center of gravity of K , or x_K elsewhere inside K) suggest that the HMM methods enjoy a superconvergence property for a wider range of parameters than those satisfying the above theoretical assumptions.

5. Links with other methods

We now show that, under one of its three forms and according to the choice of its parameters, a scheme from the HMM family may be interpreted as a nonconforming Finite Element scheme, as a mixed Finite Element scheme, or as the classical two-point flux finite volume method.

5.1. A nonconforming Finite Element method

In this section, we aim at identifying a hybrid FV scheme of the HMM family with a nonconforming Finite Element method. Hence we use the notations of Definition 2.2 and Remark 2.3. For any $K \in \mathcal{M}$ and $\sigma \in \mathcal{E}_K$, we denote by $\Delta_{K,\sigma}$ the cone with vertex x_K and basis σ . For any given $\tilde{p} \in \tilde{H}_{\mathcal{M}}$ we define the piecewise linear function $\hat{p} : \Omega \rightarrow \mathbf{R}$ by:

$$\forall K \in \mathcal{M}, \forall \sigma \in \mathcal{E}_K, \text{ for a.e. } x \in \Delta_{K,\sigma} : \\ \hat{p}(x) = p_K + \left(\nabla_K \tilde{p}_K + \frac{\beta_K}{d_{K,\sigma}} S_{K,\sigma}(\tilde{p}_K) \mathbf{n}_{K,\sigma} \right) \cdot (x - x_K)$$

where $\beta_K > 0$ (note that, since $\mathbf{n}_{K,\sigma} \cdot (\bar{x}_\sigma - x_K) = d_{K,\sigma}$, in the particular case where $\beta_K = 1$ we have $\hat{p}(\bar{x}_\sigma) = p_\sigma$). We let $\hat{H}_{\mathcal{M}} = \{\hat{p}, \tilde{p} \in \tilde{H}_{\mathcal{M}}\}$ and we define the “broken gradient” of $\hat{p} \in \hat{H}_{\mathcal{M}}$ by

$$\forall K \in \mathcal{M}, \forall \sigma \in \mathcal{E}_K, \text{ for a.e. } x \in \Delta_{K,\sigma} : \\ \widehat{\nabla} \hat{p}(x) = \nabla \hat{p}(x) = \nabla_K \tilde{p}_K + \frac{\beta_K}{d_{K,\sigma}} S_{K,\sigma}(\tilde{p}_K) \mathbf{n}_{K,\sigma}.$$

We consider the following nonconforming finite element problem: find $\hat{p} \in \hat{H}_{\mathcal{M}}$ such that

$$\forall \hat{q} \in \hat{H}_{\mathcal{M}} : \int_{\Omega} \hat{\Lambda}(x) \widehat{\nabla} \hat{p}(x) \cdot \widehat{\nabla} \hat{q}(x) dx = \int_{\Omega} f(x) \hat{q}(x) dx, \quad (5.1)$$

where $\widehat{\Lambda}(x)$ is equal to Λ_K for a.e. $x \in K$. Then the above method leads to a matrix which belongs to the family of matrices corresponding to the Generalized Hybrid method (and thus also to the Generalized Mimetic and Modified Mixed methods). Therefore, since $|\Delta_{K,\sigma}| = \frac{|\sigma|d_{K,\sigma}}{d}$, we get from (2.20):

$$\int_{\Omega} \widehat{\Lambda}(x) \widehat{\nabla} \widehat{p}(x) \cdot \widehat{\nabla} \widehat{q}(x) dx = \sum_{K \in \mathcal{M}} |K| \Lambda_K \nabla_K \widetilde{p}_K \cdot \nabla_K \widetilde{q}_K + \sum_{K \in \mathcal{M}} S_K(\widetilde{q}_K)^T \mathbb{B}_K^H S_K(\widetilde{p}_K),$$

with

$$\mathbb{B}_{K,\sigma,\sigma}^H = \frac{|\sigma|(\beta_K)^2}{d d_{K,\sigma}} \Lambda_K \mathbf{n}_{K,\sigma} \cdot \mathbf{n}_{K,\sigma},$$

and, for $\sigma \neq \sigma'$,

$$\mathbb{B}_{K,\sigma,\sigma'}^H = 0.$$

Note that this definition for \mathbb{B}_K^H fulfills (4.3) under the regularity hypothesis (4.1), hence ensuring convergence properties which can easily be extended to this nonconforming Finite Element method (5.1) even though it is not completely identical to a HMM scheme, since the right-hand sides do not coincide: in general,

$$\int_{\Omega} f(x) \widehat{q}(x) dx \neq \sum_{K \in \mathcal{M}} q_K \int_K f.$$

Indeed, this does not prevent the study of convergence since the difference between the two right-hand-sides is of order h .

5.2. Relation with Mixed Finite Element methods

In this section, we aim at identifying a particular HMM scheme, under its mixed version with a Mixed Finite Element method. Let us first recall the remark provided in Section 5.1 in Ref. 5 and Example 1 in Ref. 9: on a simplicial mesh (triangular if $d = 2$, tetrahedral if $d = 3$), the Raviart-Thomas RT0 Mixed Finite Element method fulfills properties (4.7) and (2.4), and it is therefore possible to include this Mixed Finite Element scheme in the HMM framework. Our purpose is to show that our framework also provides a Mixed Finite Element method on a general mesh (note that, even in the case of simplices, this method differs from the Raviart-Thomas RT0 method). We use here the notations provided by Definition 2.3. Still denoting $\Delta_{K,\sigma}$ the cone with vertex x_K and basis σ , we define, for $F \in \mathcal{F}$,

$$\forall K \in \mathcal{M}, \forall \sigma \in \mathcal{E}_K, \forall x \in \Delta_{K,\sigma} : \widehat{F}_{K,\sigma}(x) := -\Lambda_K \mathbf{v}_K(F_K) + T_{K,\sigma}(F_K) \frac{x - x_K}{d_{K,\sigma}}. \quad (5.2)$$

If x belongs to the interface $\partial \Delta_{K,\sigma} \cap \partial \Delta_{K,\sigma'}$ between two cones of a same control volume, we have $(x - x_K) \cdot \mathbf{n}_{\partial \Delta_{K,\sigma} \cap \partial \Delta_{K,\sigma'}} = 0$ and thus the normal fluxes of $\widehat{F}_{K,\sigma}$

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are conservative through such interfaces; moreover, for all $\sigma \in \mathcal{E}_K$ and all $x \in \sigma$, we have $(x - x_K) \cdot \mathbf{n}_{K,\sigma} = d_{K,\sigma}$ and thus, by (2.31), $\widehat{F}_{K,\sigma}(x) \cdot \mathbf{n}_{K,\sigma} = F_{K,\sigma}$; since the elements of \mathcal{F} satisfy (2.1), these observations show that the function \widehat{F} , defined by

$$\forall K \in \mathcal{M}, \forall \sigma \in \mathcal{E}_K, \text{ for a.e. } x \in \Delta_{K,\sigma} : \widehat{F}(x) = \widehat{F}_{K,\sigma}(x), \quad (5.3)$$

satisfies $\widehat{F} \in H_{\text{div}}(\Omega)$. Noting that

$$\sum_{\sigma \in \mathcal{E}_K} T_{K,\sigma}(F_K) \int_{\Delta_{K,\sigma}} \frac{x - x_K}{d_{K,\sigma}} dx = \sum_{\sigma \in \mathcal{E}_K} T_{K,\sigma}(F_K) \frac{|\sigma|(\bar{x}_\sigma - x_K)}{d + 1} = 0 \quad (5.4)$$

(thanks to (2.28), (2.31) and (2.27)), we have, with $\widehat{\Lambda}$ again denoting the piecewise-constant function equal to Λ_K in K ,

$$\int_K \widehat{\Lambda}(x)^{-1} \widehat{F}(x) \cdot \widehat{G}(x) dx = |K| \mathbf{v}_K(F_K) \cdot \Lambda_K \mathbf{v}_K(G_K) + \sum_{\sigma \in \mathcal{E}_K} \gamma_{K,\sigma} T_{K,\sigma}(F_K) T_{K,\sigma}(G_K) \quad (5.5)$$

with

$$\gamma_{K,\sigma} = \int_{\Delta_{K,\sigma}} \widehat{\Lambda}(x)^{-1} \frac{x - x_K}{d_{K,\sigma}} \cdot \frac{x - x_K}{d_{K,\sigma}} dx > 0.$$

The right-hand side of (5.5) defines an inner product $\langle \cdot, \cdot \rangle_K$ which enters the framework defined by (2.32), setting

$$\mathbb{B}_{K,\sigma,\sigma}^M = \gamma_{K,\sigma}, \text{ and } \mathbb{B}_{K,\sigma,\sigma'}^M = 0 \text{ for } \sigma \neq \sigma', \quad (5.6)$$

which fulfills (4.4) under the regularity hypothesis (4.1) (hence ensuring convergence properties). Therefore the form [(2.6),(2.7)] of the resulting HMM scheme resumes to the following Mixed Finite Element formulation: find $(p, \widehat{F}) \in H_{\mathcal{M}} \times \widehat{\mathcal{F}}$ such that

$$\forall \widehat{G} \in \widehat{\mathcal{F}} : \int_{\Omega} \widehat{\Lambda}(x)^{-1} \widehat{F}(x) \cdot \widehat{G}(x) dx - \int_{\Omega} p(x) \text{div} \widehat{G}(x) dx = 0, \quad (5.7)$$

$$\forall q \in H_{\mathcal{M}} : \int_{\Omega} q(x) \text{div} \widehat{F}(x) dx = \int_{\Omega} q(x) f(x) dx, \quad (5.8)$$

where $\widehat{\mathcal{F}} = \{\widehat{F}, F \in \mathcal{F}\}$. Indeed, since $|\Delta_{K,\sigma}| = \frac{|\sigma| d_{K,\sigma}}{d}$ and $\sum_{\sigma \in \mathcal{E}_K} |\sigma| \mathbf{n}_{K,\sigma} = 0$, we have, for all $F \in \widehat{\mathcal{F}}$,

$$\int_K \text{div} \widehat{F}(x) dx = \sum_{\sigma \in \mathcal{E}_K} |\sigma| T_{K,\sigma}(F_K) = \sum_{\sigma \in \mathcal{E}_K} |\sigma| F_{K,\sigma} \quad (5.9)$$

and (5.8) written on the canonical basis of $H_{\mathcal{M}}$ is exactly (2.30). Summing (2.33) on K , the terms involving p_σ vanish thanks to the conservativity of G and we get (5.7). Reciprocally, to pass from (5.7) to (2.33), one has to get back the edge values p_σ , which can be done exactly as for the hybridization of the Generalized Mimetic method in Section 3.1 (using $G = G(K, \sigma)$ such that $G(K, \sigma)_{K,\sigma} = 1$, $G(K, \sigma)_{L,\sigma} = -1$ if L is the control volume on the other side of σ and $G(K, \sigma)_{Z,\theta} = 0$ for other control volumes Z and/or edges θ).

An important element of study of the standard Mimetic method seems to be the existence of a suitable lifting operator, re-constructing a flux unknown inside each grid cell from the fluxes unknowns on the boundary of the grid cell (see Ref. 5 and Section 4.3). We claim that, for the Generalized Mimetic method corresponding to the choice (5.6), the flux \widehat{F} given by [(5.2),(5.3)] provides a (nearly) suitable lifting operator : it does not completely satisfy the assumptions demanded in Theorem 5.1 of Ref. 5, but enough so that the conclusion of this theorem still holds.

Proposition 5.1. *For $F_K \in \mathcal{F}_K$, let \widehat{F}_K be the restriction to K of \widehat{F} defined by (5.3). Then the operator $F_K \in \mathcal{F}_K \mapsto \widehat{F}_K \in L^2(K)$ satisfies the following properties:*

$$\forall F_K \in \mathcal{F}_K, \forall \sigma \in \mathcal{E}_K, \forall x \in \sigma : \widehat{F}_K(x) \cdot \mathbf{n}_{K,\sigma} = F_{K,\sigma}, \quad (5.10)$$

$$\forall F_K \in \mathcal{F}_K, \forall q \text{ affine function} : \int_K q(x) \operatorname{div}(\widehat{F}_K)(x) dx = \int_E q(x) \mathcal{D}\mathcal{I}\mathcal{V}^h(F_K) w_E(x) dx, \quad (5.11)$$

$$\forall F \in \mathbf{R}^d, \text{ defining } F_K = (F \cdot \mathbf{n}_{K,\sigma})_{\sigma \in \mathcal{E}_K} : \widehat{F}_K = F, \quad (5.12)$$

for any w_E satisfying (2.15).

Remark 5.1. Properties (5.10) and (5.12) are the same as in Theorem 5.1 in Ref. 5, but Property (5.11) is replaced in this reference by the stronger form “ $\operatorname{div}(\widehat{F}_K) = \mathcal{D}\mathcal{I}\mathcal{V}^h(F_K)$ on K ” (x_K is also taken as the center of gravity, which corresponds to $w_E = 1$ in (5.11)).

Proof. We already noticed (5.10) (consequence of (2.31) and the fact that $(x - x_K) \cdot \mathbf{n}_{K,\sigma} = d_{K,\sigma}$ for all $x \in \sigma$). If $F \in \mathbf{R}^d$, and $F_K = (F \cdot \mathbf{n}_{K,\sigma})_{\sigma \in \mathcal{E}_K}$, then (2.27) and (2.28) show that $\Lambda_K \mathbf{v}_K(F_K) = -F$ and thus that $T_K(F_K) = 0$, in which case $\widehat{F}_K = -\Lambda_K \mathbf{v}_K(F_K) = F$ and (5.12) holds.

Let us now turn to (5.11). For $q \equiv 1$, this relation is simply (5.9). If $q(x) = x$, then

$$\int_K q(x) \operatorname{div}(\widehat{F}_K)(x) dx = \sum_{\sigma \in \mathcal{E}_K} T_{K,\sigma}(F_K) d \int_{\Delta_{K,\sigma}} \frac{x}{d_{K,\sigma}} dx$$

and (5.4) then gives

$$\begin{aligned} \int_K q(x) \operatorname{div}(\widehat{F}_K)(x) dx &= \sum_{\sigma \in \mathcal{E}_K} T_{K,\sigma}(F_K) d \frac{x_K}{d_{K,\sigma}} |\Delta_{K,\sigma}| \\ &= \sum_{\sigma \in \mathcal{E}_K} |\sigma| T_{K,\sigma}(F_K) x_K \\ &= \left(\sum_{\sigma \in \mathcal{E}_K} |\sigma| F_{K,\sigma} \right) x_K \\ &= \int_K q(x) \mathcal{D}\mathcal{I}\mathcal{V}^h(F_K) w_E(x) dx \end{aligned}$$

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by assumption on w_E . □

5.3. Two-point flux cases

We now consider isotropic diffusion tensors:

$$\Lambda = \lambda(x)\text{Id}, \quad (5.13)$$

with $\lambda(x) \in \mathbf{R}$ and exhibit cases in which the HMM method provides two-point fluxes, in the sense that the fluxes satisfy $F_E^e = \tau_{E,E'}(p_E - p_{E'})$, in the case where e is the common edge of two neighboring cells E and E' , with $\tau_{E,E'} \geq 0$ only depending on the grid (not on the unknowns). Recall that using a two-point flux scheme yields a matrix with positive inverse, and is the easiest way to ensure that, in the case of a linear scheme, monotony and local maximum principle hold. The two-point flux scheme is also probably the cheapest scheme in terms of implementation and computing cost. Moreover, although no theoretical proof is yet known, numerical evidence shows the order 2 convergence²⁶ of the two-point scheme on triangular meshes, taking for x_K the intersection of the orthogonal bisectors^{24,18}. Therefore, whenever possible, one should strive to recover this scheme when using admissible meshes (in the sense of Ref. 18 previously mentioned). In the HMM framework, two-point fluxes are obtained, using the notations provided by Definition 2.1, if the matrix of the bilinear form $[F, G]_{X^h}$ in (2.6) is diagonal. This implies, in the case of meshes such that two neighboring grid cells only have one edge in common, that the matrices \mathbb{M}_E defining the local inner product (2.3) are diagonal. If the matrix \mathbb{M}_E is diagonal, we get from the property $\mathbb{M}_E \mathbb{N}_E = \mathbb{R}_E$ and from (5.13) that there exists $\mu_E^e \in \mathbf{R}$ such that

$$\mu_E^e \mathbf{n}_E^e = \bar{x}_e - x_E. \quad (5.14)$$

This implies that x_E is, for any face e of E , a point of the orthogonal line to e passing through \bar{x}_e (x_E is then necessarily unique) and that μ_E^e is the orthogonal distance between x_E and e . In the case of a triangle, x_E is thus the intersection of the orthogonal bisectors of the sides of the triangle, which is the center of gravity only if the triangle is equilateral; hence, except in this restricted case, the original Mimetic method cannot yield a two-point flux method. Note also that there are meshes such that the orthogonal bisectors of the faces do not intersect, but for which nevertheless some “centers” in the cells exist and are such that the line joining the centers of two neighboring cells is orthogonal to their common face: these meshes are referred to “admissible” meshes in Definition 9.1 p. 762 of Ref. 18; a classical example of such admissible meshes are the general Voronoi meshes. On such admissible meshes, the HMM method does not provide a two-point flux scheme for isotropic diffusion operators, although a two-point flux Finite Volume scheme can be defined, with the desired convergence properties¹⁸.

In this section, we shall call “super-admissible discretizations” the discretizations which fulfill the property (5.14) for some choice of $(x_E)_{E \in \Omega_h}$. We wish to show that

for all super-admissible discretizations and in the isotropic case (5.13), the HMM method provides a two-point flux scheme. Using the notations of Definitions 2.2 and 2.3, (5.14) is written $\mathbf{n}_{K,\sigma} = (\bar{x}_\sigma - x_K)/d_{K,\sigma}$ and defines our choice of parameters $(x_K)_{K \in \mathcal{M}}$.

Let us take $\alpha_{K,\sigma} = \lambda_K$ in the Hybrid presentation (2.21), denoting by λ_K the mean value of the function $\lambda(x)$ in K . Using Definition (2.17) for $\nabla_K \tilde{p}_K$ and thanks to (2.18), a simple calculation shows that

$$\sum_{\sigma \in \mathcal{E}_K} \frac{|\sigma| \lambda_K}{d_{K,\sigma}} S_{K,\sigma}(\tilde{p}_K) S_{K,\sigma}(\tilde{q}_K) = \sum_{\sigma \in \mathcal{E}_K} \frac{|\sigma| \lambda_K}{d_{K,\sigma}} (p_K - p_\sigma)(q_K - q_\sigma) - |K| \lambda_K \nabla_K \tilde{p}_K \cdot \nabla_K \tilde{q}_K.$$

Hence we deduce from (2.21) that $F_{K,\sigma} = \frac{\lambda_K}{d_{K,\sigma}} (p_K - p_\sigma)$ and the conservativity (2.1) leads, for an internal edge between the control volumes K and L , to $p_\sigma = \frac{d_{K,\sigma} \lambda_L p_L + d_{L,\sigma} \lambda_K p_K}{d_{K,\sigma} \lambda_L + d_{L,\sigma} \lambda_K}$. The resulting expression for the flux becomes

$$F_{K,\sigma} = \frac{\lambda_K \lambda_L}{d_{K,\sigma} \lambda_L + d_{L,\sigma} \lambda_K} (p_K - p_L),$$

which is the expression of the flux for the standard 2-points Finite Volume scheme with harmonic averaging of the diffusion coefficient.

It is also easy to find back this expression from the Mixed presentation (2.32), taking

$$\mathbb{B}_{K,\sigma,\sigma}^M = |\sigma| \frac{d_{K,\sigma}}{\lambda_K},$$

and, for $\sigma \neq \sigma'$,

$$\mathbb{B}_{K,\sigma,\sigma'}^M = 0.$$

The property

$$\sum_{\sigma \in \mathcal{E}_K} |\sigma| \frac{d_{K,\sigma}}{\lambda_K} T_{K,\sigma}(F_K) T_{K,\sigma}(G_K) = \sum_{\sigma \in \mathcal{E}_K} |\sigma| \frac{d_{K,\sigma}}{\lambda_K} F_{K,\sigma} G_{K,\sigma} - |K| \mathbf{v}_K(F_K) \cdot \lambda_K \mathbf{v}_K(G_K),$$

which results from (2.28), (2.31), (2.18) and (5.14) (under the form $d_{K,\sigma} \mathbf{n}_{K,\sigma} = \bar{x}_\sigma - x_K$), shows that

$$\langle F_K, G_K \rangle_K = \sum_{\sigma \in \mathcal{E}_K} |\sigma| \frac{d_{K,\sigma}}{\lambda_K} F_{K,\sigma} G_{K,\sigma}.$$

Thanks to (2.33), this gives $F_{K,\sigma} = \frac{\lambda_K}{d_{K,\sigma}} (p_K - p_\sigma)$ and we conclude as above.

5.4. Elimination of some edge unknowns

In the study of the hybrid version (2.26) of the HMM scheme^{20,21}, it was suggested to replace the space $\tilde{H}_{\mathcal{M},0}$ by the space $\tilde{H}_{\mathcal{M},0}^{\mathcal{B}}$ defined by $\tilde{H}_{\mathcal{M},0}^{\mathcal{B}} = \{\tilde{p} \in \tilde{H}_{\mathcal{M},0} \text{ such that } p_{\sigma} = \sum_{K \in \mathcal{M}_{\sigma}} \beta_{\sigma}^K p_K \text{ if } \sigma \in \mathcal{B}\}$, where:

- i) \mathcal{M}_{σ} is a subset of \mathcal{M} , including a few cells (in general, less than $d + 1$, where we recall that d is the dimension of the space) “close” from σ ,
- ii) the coefficients β_{σ}^K are barycentric weights of the point \bar{x}_{σ} with respect to the points x_K , which means that $\sum_{K \in \mathcal{M}_{\sigma}} \beta_{\sigma}^K = 1$ and $\sum_{K \in \mathcal{M}_{\sigma}} \beta_{\sigma}^K x_K = \bar{x}_{\sigma}$,
- iii) \mathcal{B} is any subset of the set of all internal edges (the cases of the empty set or of the full set itself being not excluded).

Then the scheme is defined by: find $\tilde{p} \in \tilde{H}_{\mathcal{M},0}^{\mathcal{B}}$ such that

$$\forall \tilde{q} \in \tilde{H}_{\mathcal{M},0}^{\mathcal{B}} : \sum_{K \in \mathcal{M}} |K| \Lambda_K \nabla_K \tilde{p}_K \cdot \nabla_K \tilde{q}_K + \sum_{K \in \mathcal{M}} S_K(\tilde{q}_K)^T \mathbb{B}_K^H S_K(\tilde{p}_K) = \sum_{K \in \mathcal{M}} q_K \int_K f. \quad (5.15)$$

In the case where $\mathcal{B} = \emptyset$, then the method belongs to the HMM family and in the case where \mathcal{B} is the full set of the internal edges, then there is no more edge unknowns, and we get back a cell-centered scheme. In the intermediate cases, we get schemes where the unknowns are all the cell unknowns, and the edge unknowns p_{σ} with $\sigma \notin \mathcal{B}$.

This technique has been shown in Refs. 20, 21 to fulfill the convergence and error estimates requirements in the case of diagonal matrices \mathbb{B}_K^H . It can be applied, with the same convergence properties, to HMM methods with symmetric positive definite matrices \mathbb{B}_K^H as in Section 4.

6. Conclusion

In this article, we showed that the Mimetic method, the Hybrid Finite Volume scheme and the Mixed finite volume scheme may be, up to some slight generalizations, cast into a unified framework which we called the HMM family of schemes. This study was performed in the case of a linear single equation (1.1), but the three above methods have also been used on more complex situations.

The Finite Volume literature addressed nonlinear problems, such as the incompressible Navier-Stokes equations^{12,16}, fully non-linear equations of the p -Laplacian type^{14,17}, non-linear coupled problems¹¹, etc. All the ingredients needed to write the schemes for these nonlinear problems are already present in the linear setting which we studied here. Moreover, the techniques of proof for the convergence of the scheme developed for the linear diffusion equation (1.1) are based on compactness arguments which are naturally adapted to nonlinear problems, such as those studied in the above cited papers.

The Mimetic Finite Difference literature contains several extensions and fine studies of the scheme for the linear diffusion problem: generalization of the method to meshes made of cells with curved faces⁶, construction of higher-order schemes^{4,23} or study of optimal *a posteriori* error estimators for local refinement^{2,3}, for example. These tools are based on the same ideas that lead to the basic MFD scheme, and could therefore also be used in the HMM setting, modulo some slight adaptations. Another lead, probably a little less straightforward but nevertheless possible and certainly more interesting, would be to try and adapt the ideas of the scheme for meshes with curved faces or the *a posteriori* error estimators to non-linear problems such as the ones mentioned above: all these schemes and developments being based, as we showed in this paper, on the same ideas, their mixing seems at least like a reasonable possibility.

In conclusion, the unifying framework was proposed here to facilitate the transfer of ideas and techniques used for one method to another; of course, it has a larger field of application than (1.1) and should help in giving some new leads for each method.

7. Appendix

7.1. About the Generalized Mimetic definition

We prove here two results linked with the definition of the Generalized Mimetic method: the existence of a weight function satisfying (2.15) and the equivalence between (2.16) and (2.13).

Lemma 7.1. *If E is a bounded non-empty open subset of \mathbf{R}^d and $x_E \in \mathbf{R}^d$, then there exists an affine function $w_E : \mathbf{R}^d \rightarrow \mathbf{R}$ satisfying (2.15).*

Proof. We look for $\xi \in \mathbf{R}^d$ such that $w_E(x) = 1 + \xi \cdot (x - \bar{x}_E) = 1 + (x - \bar{x}_E)^T \xi$ satisfies the properties (where \bar{x}_E is the center of gravity of E). The first property of (2.15) is straightforward since $\int_E (x - \bar{x}_E) dx = 0$, and the second property is equivalent to $|E|\bar{x}_E + \int_E x(x - \bar{x}_E)^T \xi dx = |E|x_E$; since $\int_E \bar{x}_E(x - \bar{x}_E)^T dx = 0$, this boils down to

$$\left(\int_E (x - \bar{x}_E)(x - \bar{x}_E)^T dx \right) \xi = |E|(x_E - \bar{x}_E). \quad (7.1)$$

Let J_E be the $d \times d$ matrix $\int_E (x - \bar{x}_E)(x - \bar{x}_E)^T dx$: we have, for all $\eta \in \mathbf{R}^d \setminus \{0\}$, $J_E \eta \cdot \eta = \int_E ((x - \bar{x}_E) \cdot \eta)^2 dx$ and the function $x \rightarrow (x - \bar{x}_E) \cdot \eta$ vanishes only on an hyperplane of \mathbf{R}^d ; this proves that $J_E \eta \cdot \eta > 0$ for all $\eta \neq 0$. Hence, J_E is invertible and there exists (a unique) ξ satisfying (7.1), which concludes the proof. \square

Lemma 7.2. *Let $[\cdot, \cdot]_E$ be a local inner product on the space of the fluxes unknowns of a grid cell E , and let \mathbb{M}_E be its matrix. Then $[\cdot, \cdot]_E$ satisfies (2.16) (with w_E satisfying (2.15)) if and only if \mathbb{M}_E satisfies (2.13) (with \mathbb{R}_E defined by (2.14) and $(\mathbb{C}_E, \mathbb{U}_E)$ defined by (2.11) and (2.12)).*

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Proof. It is known⁸ that, for the standard Mimetic method, (2.4) is equivalent to $\mathbb{M}_E \mathbb{N}_E = \bar{\mathbb{R}}_E$ with $\bar{\mathbb{R}}_E$ defined by (2.8) and \mathbb{N}_E defined in (2.9); similarly, it is quite easy to see that (2.16) (with w_E satisfying (2.15)) is equivalent to

$$\mathbb{M}_E \mathbb{N}_E = \mathbb{R}_E \quad (7.2)$$

with \mathbb{R}_E defined by (2.14): indeed, (2.16) with $q = 1$ is simply the definition (2.2) of the discrete divergence operator (because $\int_E w_E(x) dx = |E|$) and, with $q(x) = x_j$, since $\int_E x_j w_E(x) dx = |E|(x_E)_j$, (2.16) boils down to

$$G_E^T \mathbb{M}_E (\mathbb{N}_E)_j + \sum_{i=1}^{k_E} G_E^{e_i} |e_i| (x_E)_j = \sum_{i=1}^{k_E} G_E^{e_i} |e_i| (\bar{x}_{e_i})_j,$$

which is precisely $G_E^T \mathbb{M}_E (\mathbb{N}_E)_j = G_E^T (\mathbb{R}_E)_j$ with $(\mathbb{R}_E)_j$ the j -th column of \mathbb{R}_E . We therefore only need to compare (2.13) with (7.2).

Let us first assume that \mathbb{M}_E satisfies (2.13). The generic formula (2.27) implies

$$\mathbb{R}_E^T (\mathbb{N}_E)_j = |E| (\Lambda_E)_j \quad (7.3)$$

and thus

$$\frac{1}{|E|} \mathbb{R}_E \Lambda_E^{-1} \mathbb{R}_E^T \mathbb{N}_E = \mathbb{R}_E. \quad (7.4)$$

Since $\mathbb{C}_E^T \mathbb{N}_E = 0$ by definition of \mathbb{C}_E , this shows that \mathbb{M}_E satisfies (7.2).

Let us now assume that \mathbb{M}_E satisfies (7.2) and let us consider the symmetric matrix

$$\tilde{\mathbb{M}}_E = \mathbb{M}_E - \frac{1}{|E|} \mathbb{R}_E \Lambda_E^{-1} \mathbb{R}_E^T. \quad (7.5)$$

By (7.2) and (7.4), we have $\tilde{\mathbb{M}}_E \mathbb{N}_E = 0$, and the columns of \mathbb{N}_E are therefore in the kernel of $\tilde{\mathbb{M}}_E$. The definition of \mathbb{C}_E shows that the columns of \mathbb{N}_E span the kernel of \mathbb{C}_E^T : we therefore have $\ker(\mathbb{C}_E^T) \subset \ker(\tilde{\mathbb{M}}_E)$ and we deduce the existence of a $k_E \times (k_E - d)$ matrix \mathbb{A} such that

$$\tilde{\mathbb{M}}_E = \mathbb{A} \mathbb{C}_E^T. \quad (7.6)$$

By symmetry of $\tilde{\mathbb{M}}_E$ we have $\mathbb{A} \mathbb{C}_E^T = \mathbb{C}_E \mathbb{A}^T$ and thus $\mathbb{A} \mathbb{C}_E^T \mathbb{C}_E = \mathbb{C}_E \mathbb{A}^T \mathbb{C}_E$; but $\mathbb{C}_E^T \mathbb{C}_E$ is an invertible $(k_E - d) \times (k_E - d)$ matrix (since \mathbb{C}_E is of rank $k_E - d$) and therefore $\mathbb{A} = \mathbb{C}_E \mathbb{A}^T \mathbb{C}_E (\mathbb{C}_E^T \mathbb{C}_E)^{-1} = \mathbb{C}_E \mathbb{U}_E$ for some $(k_E - d) \times (k_E - d)$ matrix \mathbb{U}_E . Gathering this result with (7.5) and (7.6), we have proved that \mathbb{M}_E satisfies (2.13) for some \mathbb{U}_E , and it remains to prove that this last matrix is symmetric definite positive to conclude. By (2.13) and the symmetry of \mathbb{M}_E and $\frac{1}{|E|} \mathbb{R}_E \Lambda_E^{-1} \mathbb{R}_E^T$ we have $\mathbb{C}_E \mathbb{U}_E \mathbb{C}_E^T = \mathbb{C}_E \mathbb{U}_E^T \mathbb{C}_E^T$ and, since \mathbb{C}_E^T is onto and \mathbb{C}_E is one-to-one, we deduce $\mathbb{U}_E = \mathbb{U}_E^T$. To prove that \mathbb{U}_E is definite positive, we use (2.13) and the fact that \mathbb{M}_E is definite positive to write

$$\forall \xi \in \ker(\mathbb{R}_E^T), \xi \neq 0 : (\mathbb{U}_E \mathbb{C}_E^T \xi) \cdot (\mathbb{C}_E^T \xi) > 0. \quad (7.7)$$

This shows in particular that $\ker(\mathbb{R}_E^T) \cap \ker(\mathbb{C}_E^T) = \{0\}$ and thus, since $\ker(\mathbb{R}_E^T)$ has dimension $k_E - d$, that the image by \mathbb{C}_E^T of $\ker(\mathbb{R}_E^T)$ is \mathbf{R}^{k_E-d} . Equation (7.7) then proves that \mathbb{U}_E is definite positive on the whole of \mathbf{R}^{k_E-d} and the proof of the lemma is complete. \square

7.2. An algebraic lemma

Lemma 7.3. *Let X, Y and Z be finite dimension vector spaces and $A : X \rightarrow Y$, $B : X \rightarrow Z$ be two linear mappings with identical kernel. Then, for all inner product $\{\cdot, \cdot\}_Y$ on Y , there exists an inner product $\{\cdot, \cdot\}_Z$ on Z such that, for all $(x, x') \in X^2$, $\{Bx, Bx'\}_Z = \{Ax, Ax'\}_Y$.*

Proof. Let $N = \ker(A) = \ker(B)$. The mappings A and B define one-to-one mappings $\bar{A} : X/N \rightarrow Y$ and $\bar{B} : X/N \rightarrow Z$ such that, if \bar{x} is the class of x , $Ax = \bar{A}\bar{x}$ and $Bx = \bar{B}\bar{x}$. We can therefore work with \bar{A} and \bar{B} on X/N rather than with A and B on X , and assume in fact that A and B are one-to-one.

Then $A : X \rightarrow \text{Im}(A)$ and $B : X \rightarrow \text{Im}(B)$ are isomorphisms and, if $\{\cdot, \cdot\}_Y$ is an inner product on Y , we can define the inner product $\{\cdot, \cdot\}_{\text{Im}(B)}$ on $\text{Im}(B)$ the following way: for all $z, z' \in \text{Im}(B)$, $\{z, z'\}_{\text{Im}(B)} = \{AB^{-1}z, AB^{-1}z'\}_Y$ (this means that $\{Bx, Bx'\}_{\text{Im}(B)} = \{Ax, Ax'\}_Y$ for all $x, x' \in X$). This inner product is only defined on $\text{Im}(B)$, but we extend it to Z by choosing W such that $\text{Im}(B) \oplus W = Z$, by taking any inner product $\{\cdot, \cdot\}_W$ on W and by letting $\{z, z'\}_Z = \{z_B, z'_B\}_{\text{Im}(B)} + \{z_W, z'_W\}_W$ for all $z = z_B + z'_W \in Z = \text{Im}(B) \oplus W$ and $z' = z'_B + z'_W \in Z$. This extension of $\{\cdot, \cdot\}_{\text{Im}(B)}$ preserves the property $\{Bx, Bx'\}_Z = \{Ax, Ax'\}_Y$. \square

Remark 7.1. The proof gives a way to explicitly compute $\{\cdot, \cdot\}_Z$ from $\{\cdot, \cdot\}_Y$, A and B : find a complementary subspace G of $\ker(A) = \ker(B)$ in X , compute an inverse of B between G and $\text{Im}(B)$, deduce $\{\cdot, \cdot\}_{\text{Im}(B)}$ and extend it to Z by finding a complementary subspace space of $\text{Im}(B)$.

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References

1. L. Agelas, D. Di Pietro, and J. Droniou. The G method for heterogeneous anisotropic diffusion on general meshes. submitted, november 2008.
2. L. Beirão da Veiga. A residual based error estimator for the mimetic finite difference method. *Numer. Math.*, 108(3):387–406, 2008.
3. L. Beirão da Veiga and G. Manzini. An a posteriori error estimator for the mimetic finite difference approximation of elliptic problems. *Internat. J. Numer. Methods Engrg.*, 76(11):1696–1723, 2008.
4. L. Beirão da Veiga and G. Manzini. A higher-order formulation of the mimetic finite difference method. *SIAM J. Sci. Comput.*, 31(1):732–760, 2008.

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5. F. Brezzi, K. Lipnikov, and M. Shashkov. Convergence of the mimetic finite difference method for diffusion problems on polyhedral meshes. *SIAM J. Numer. Anal.*, 43(5):1872–1896, 2005.
6. F. Brezzi, K. Lipnikov, and M. Shashkov. Convergence of mimetic finite difference method for diffusion problems on polyhedral meshes with curved faces. *Math. Models Methods Appl. Sci.*, 16(2):275–297, 2006.
7. F. Brezzi, K. Lipnikov, M. Shashkov, and V. Simoncini. A new discretization methodology for diffusion problems on generalized polyhedral meshes. *Comput. Methods Appl. Mech. Engrg.*, 196:3682–3692, 2007.
8. F. Brezzi, K. Lipnikov, and V. Simoncini. A family of mimetic finite difference methods on polygonal and polyhedral meshes. *Math. Models Methods Appl. Sci.*, 15(10):1533–1551, 2005.
9. A. Cangiani and G. Manzini. Flux reconstruction and solution post-processing in mimetic finite difference methods. *Comput. Methods Appl. Mech. Engrg.*, 197:933–945, 2008.
10. C. Chainais, J. Droniou, and R. Eymard. Benchmark on anisotropic problems, use of the mixed finite volume method. In R. Eymard and J.-M. Hérard, editors, *Finite Volumes for Complex Applications V*, pages 751–760. Wiley, 2008.
11. C. Chainais-Hillairet and J. Droniou. Convergence analysis of a mixed finite volume scheme for an elliptic-parabolic system modeling miscible fluid flows in porous media. *SIAM J. Numer. Anal.*, 45(5):2228–2258, 2007.
12. E. Chénier, R. Eymard, and Herbin R. A collocated finite volume scheme to solve free convection for general non-conforming grids. *J. Comput. Phys.*, accepted for publication.
13. D. A. Di Pietro and A. Ern. Discrete functional analysis tools for Discontinuous Galerkin methods with application to the incompressible Navier–Stokes equations. CERMIACS Research Report 381.
14. J. Droniou. Finite volume schemes for fully non-linear elliptic equations in divergence form. *M2AN Math. Model. Numer. Anal.*, 40(6):1069–1100, 2006.
15. J. Droniou and R. Eymard. A mixed finite volume scheme for anisotropic diffusion problems on any grid. *Numer. Math.*, 105(1):35–71, 2006.
16. J. Droniou and R. Eymard. Study of the mixed finite volume method for stokes and navier-stokes equations. *Numer. Meth. P. D. E.*, 25(1):137–171, 2009.
17. R. Eymard, T. Gallouët, and R. Herbin. Cell centred discretization of fully non linear elliptic problems on general multidimensional polyhedral grids. in preparation.
18. R. Eymard, T. Gallouët, and R. Herbin. Finite volume methods. In P. G. Ciarlet and J.-L. Lions, editors, *Techniques of Scientific Computing, Part III*, Handbook of Numerical Analysis, VII, pages 713–1020. North-Holland, Amsterdam, 2000.
19. R. Eymard, T. Gallouët, and R. Herbin. A new finite volume scheme for anisotropic diffusion problems on general grids: convergence analysis. *C. R., Math., Acad. Sci. Paris*, 344(6):403–406, 2007.
20. R. Eymard, T. Gallouët, and R. Herbin. Benchmark on anisotropic problems, SUSHI: a scheme using stabilization and hybrid interfaces for anisotropic heterogeneous diffusion problems. In R. Eymard and J.-M. Hérard, editors, *Finite Volumes for Complex Applications V*, pages 801–814. Wiley, 2008.
21. R. Eymard, T. Gallouët, and R. Herbin. Discretisation of heterogeneous and anisotropic diffusion problems on general non-conforming meshes, sushi: a scheme using stabilisation and hybrid interfaces. to appear in IMAJNA, see also <http://hal.archives-ouvertes.fr/docs/00/21/18/28/PDF/suchi.pdf>, 2008.
22. R. Eymard, T. Gallouët, and R. Herbin. Discretization schemes for linear diffusion

- operators on general non-conforming meshes. In R. Eymard and J.-M. Hérard, editors, *Finite Volumes for Complex Applications V*, pages 375–382. Wiley, 2008.
23. V. Gyrya and K. Lipnikov. High-order mimetic finite difference method for diffusion problems on polygonal meshes. *J. Comput. Phys.*, 227(20):8841–8854, 2008.
 24. R. Herbin. An error estimate for a finite volume scheme for a diffusion-convection problem on a triangular mesh. *Numer. Methods Partial Differential Equations*, 11(2):165–173, 1995.
 25. R. Herbin and F. Hubert. Benchmark on discretization schemes for anisotropic diffusion problems on general grids for anisotropic heterogeneous diffusion problems. In R. Eymard and J.-M. Hérard, editors, *Finite Volumes for Complex Applications V*, pages 659–692. Wiley, 2008.
 26. R. Herbin and O. Labergerie. Comparison of finite volume schemes for elliptic and hyperbolic on a triangular mesh. *Comput. Methods Appl. Mech. Engrg.*, 147(0):85–103, 1997.
 27. Y. Kuznetsov, K. Lipnikov, and M. Shashkov. The mimetic finite difference method on polygonal meshes for diffusion-type problems. *Comput. Geosci.*, 8(4):301–324 (2005), 2004.
 28. K. Lipnikov. Benchmark on anisotropic problems, mimetic finite difference method. In R. Eymard and J.-M. Hérard, editors, *Finite Volumes for Complex Applications V*, pages 843–850. Wiley, 2008.
 29. G. Manzini. Benchmark on anisotropic problems, the mimetic finite difference method. In R. Eymard and J.-M. Hérard, editors, *Finite Volumes for Complex Applications V*, pages 865–878. Wiley, 2008.
 30. G. Manzini. The mimetic finite difference method. In R. Eymard and J.-M. Hérard, editors, *Finite Volumes for Complex Applications V*, pages 119–134. Wiley, 2008.