Les Annales de l'Enit
Vol. 4, n° 2, Juillet 1990

ON SOME UPSTREAM WEIGHTING SCHEMES FOR OIL RECOVERY SIMULATION

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Abstract. We prove the convergence results of particular numerical approximation schemes for some nonlinear hyperbolic equation appearing in oil recovery simulation. These schemes use new technics of upstream weighting in accord with the physical meaning of the equations.

Sur des schémas décentrés amont pour la simulation de la récupération d'hydrocarbures.

Résumé. On démontre la convergence de certains schémas d'approximation numérique pour des équations hyperboliques non linéaires apparaissant dans la simulation de la récupération d'hydrocarbures. Ces schémas utilisent de nouvelles techniques de décentrement amont en accord avec le sens physique des équations.

حول الحلول التقريبية لبعض أنواع المعادلات التفاضلية أو ذوات الإشتقاقات الجزئية مع ضوارب متتابعة .

ملخص : نقدم بعض الصيغ التغيرية المتحصل عليها بادخال متغيرات ثانويّة تمكن من حلّ بعض المعادلات التفاضلية أو ذوات الإشتقاقات الجزئية مع ضوارب غير متتابعة أو منحلة بواسطة طرق الأجزاء المحدودة .

Reçu le 5 octobre 1989, révisé le 26 décembre 1989.

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1. Introduction.

1.1. A simple model for oil recovery simulation. We begin by giving a physical model in which the kind of equation studied in this paper will appear. It concerns the flow of a diphasic incompressible fluid in a porous medium. We assume that the phases are immiscible, and we neglect capillarity effects. For simplicity, we assume that the porosity of the medium is constant and that the permeability tensor is equal to the identity tensor. The two phases are (for instance) water and oil.

The principal unknowns are P and u, (pressure and water saturation), and the equations to solve are

(1) $\quad (u)_t + \operatorname{div} Q_w = F_w \quad$ (water conservation)

(2) $\quad -(u)_t + \operatorname{div} Q_o = F_o \quad$ (oil conservation)

(3) $\quad Q_w + M_w (\operatorname{grad} P - r_w g) = 0 \quad$ (Darcy's law)

(4) $\quad Q_o + M_o (\operatorname{grad} P - r_o g) = 0 \quad$ (Darcy's law)

$(\ )_t$ designes the partial derivative with respect to t.

F_w and F_o are given source terms r_o and r_w are supposed constant. g is a constant vector (gravity acceleration). M_w [resp. M_o] is a non-decreasing [resp. non increasing] function of u..

We set $M_t = M_w + M_o$, $F = F_w + F_o$, and $Q = Q_w + Q_o$. The system (1)-(4) can be replaced by the following system :

(5) $\quad \operatorname{div} Q = F$

(6) $\quad Q + M_t \operatorname{grad} P = (M_w r_w + M_o r_o)\, g$

(7) $$(u)_t + \operatorname{div} \frac{M_w}{M_t} Q + \operatorname{div} \frac{M_w M_o}{M_t} (r_w - r_o)\, g = F_w.$$

The spacial domain of the simulation is a given open set $\mathbb{R}^N$ ($N =$

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1.2. Numerical

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n open set $\mathbb{R}^N$ ($N =$

1, 2 or 3) ; and we want to solve (5)-(7) on a given time interval [0, T], with convenent conditions and initial conditions.

The system (5)-(7) appears to be a coupled system between an elliptic equation on P (equations (5)-(6) are the mixed from of that elliptic equation), and a nonlinear hyperbolic equation on u (equation (7)). The hyperbolic behaviour of (7) allows discontinuities to be created and to propagate.

The most commonly used procedures in Reservoir Simulation are IMPES procedures (IMplicit Pressure, Explicit Saturation). In that kind of procedure, one has to solve (5)-(6) with a given u and to solve (7) with a given Q, are a given P. In Eymard & al. [8], one solves (5)-(6) with a particular finite element method, and we introduce some new upstream weighting schemes in order to solve (7). In this paper we study these new upstream weighting schemes and we give some proofs of convergence. We don't describe the discretization of (5)-(6) (this is done in Eymard & al. [8].

There is many papers in the petroleum literature (SPE Papers) on some similar models. Some of these papers seem to be related to the methods used in this paper. One can see, for instance, Rozen [13], Bertiger [1], Caracotsios & al. [3]. Related schemes or more complicated models are also studied in Pfertzel [12] and Eymard [7]. Some numerical results can be seen in Pfertzel [12].

For simplicity we will assume that spacial domain of the simulation is $\mathbb{R}^N (N = 1, 2)$, $T = +\infty$ and $F_w = 0$. (the case $N = 3$ is very simular to the case $N = 2$).

1.2. Numerical method. The equation to solve is :

(8) $\quad (u)_t + \operatorname{div} f(u)\, Q + \operatorname{div} g(u)\, g = 0$, in $\mathbb{R}^N \times [0, \infty[$

with an initial datum

(6) $\quad u(.,0) = u_o$ in $\mathbb{R}^N$

Q and g are given and

$$f = \frac{f_1}{f_1 + f_2} \quad , \quad g = \frac{f_1 f_2}{f_1 + f_2},$$

f_1 is a nondecreasing function of u, and f_2 a nonincreasing function of u. In the following we will used finite volums technics with P_0 and P_1 discontinuous functions and with some upstream computation of some terms of (8).

For the second term (8) we use "vertex upstream weighting" scheme. The upstream direction is found with respect to **Q**. in our "first order scheme", this gives, in dimension $N = 1$, (section 2.2) the Godunov's scheme for the second term of (8) (if **g** = 0, we then obtain exactly the Godunov's scheme). Then in dimension $N = 2$, our first order scheme appears to be, on the second term of (8), a "two dimensional vertex Godunov's scheme" (section 3.2).

For the third term of (9) we use an "edge upstream weigthing" scheme. Here the upstream direction is found with respect to **g**. In dimension $N = 1$, our first order scheme (section 2.2) gives on the third term of (8) a scheme which looks a little like the Engquist-Osher scheme (see Engquist & al. [6]) the generalization to the dimension $N = 2$ is easy (section 3.3).

For both terms (second and third terms of (8)) we also describe technics in order to obtain "stable" second order schemes in space (see sections 2.3 and 3.4). This is done by using P_1 discontinuous functions and slope limiters (following in this way the ideas of Van leer [14]).

All the schemes described here have conservation-form and are consistent. For some of them we also prove sufficient stability properties to insure the convergence, in some appropriate space, to the weak, entropic consistant, solution of (8).

2. The one dimensional case.

2.1. Preliminaries. Let $N = 1$. The equation to solve becomes :

(10) $(u)_t + a\,(f(u))_x + b\,(g(u))_x = 0$ in $\mathbb{R} \times [0, \infty[$

(11) $u(.,0) = u_o$ in $\mathbb{R}$

a and b are some given constants. We assume that u_o, the initial datum, lies in the space $L^\infty(\mathbb{R}) \cap BV(\mathbb{R})$. and the hypotheses on f and g are

(12) $f = f_1 / (f_1 + f_2)$, $\quad g = f_1 f_2 / (f_1 + f_2)$

(13) $f_1, f_2 \quad C^1(\mathbb{R}, \mathbb{R}$

(14) f_1 [resp f_2] is a
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Let h be the space
jh and $x_j + 1/2 = (j + 1/2)$
desired approximative sol
$n \quad \mathbb{N}\} \subset \mathbb{R}$.

Indeed we have, by

$$u_j^o = \frac{1}{h} \int_{x_{j-1/2}}^{x_{j+1/2}} u_o(x)dx$$

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(13) $f_1, f_2 \quad C^1(\mathbb{R}, \mathbb{R})$ and $\exists\ \alpha > 0 f_1 + f_2 \geq \alpha$ in $\mathbb{R}$.

(14) f_1 [resp f_2] is a nonnegative nondecreasing [resp. nonincreasing] function.

Let h be the space step and k the time step. We set $\lambda = k/h$, $x_j = jh$ and $x_j + 1/2 = (j + 1/2)\,h$ for $j \in \mathbb{Z}$, and $t_n = nk$ for $n \in \mathbb{N}$. The desired approximative solution of (10)-(11) is defined by $\{n^{n_j}, j \in \mathbb{Z}, n \quad \mathbb{N}\} \subset \mathbb{R}$.

Indeed we have, by definition,

$$u_j^o = \frac{1}{h}\int_{x_{j-1/2}}^{x_{j+1/2}} u_o(x)dx \text{ , and we set for } x_{j-1/2} < x \leq x_{j+1/2}\,, j \in \mathbb{Z}$$

$$t^n \leq t < t^{n+1}\,, n \in \mathbb{N} \quad , \ u_{h,k}(x,t) = u_j^n\,,\ u^n(x) = u_j^n\,.$$

The way of computing u^{n+1} from u^n describes the scheme. Our principal aim is to find schemes for which $u_{h,k}$ goes to u, weak entropic solution of (10)-(11), in an appropriate sense, when $h,k \to 0$, with $k/h = \lambda$ constant, λ small enough.

The schemes we describe below have conservation form (see Harten & al. [11] or Crandall & al. [5]), they can be written under the following form

$$(15) \qquad \frac{1}{k}(u_j^{n+1} - u_j^n) + \frac{1}{h}\{a(F_{j+1/2}^n - F_{j-1/2}^n) + b(G_{n+1/2}^n - G_{j-1/2}^n)\} = 0$$

with

$$a\,F_{j+1/2}^n + b\,G_{j+1/2}^n = H_{j+1/2}^n = H(u_{j-p+1}^n\,,\ \ldots,\ u_{j+p}^n$$

where p is a positive integer and H a function of $2p$ real arguments.

In (15), $F^n_{j+1/2}$ can be viewed as an approximation of

$$f\,(u\,(x_{j+1/2}\,,\,t_n\,))$$

and $G^n_{j+1/2}$ as an approximation of

$$g\,(u\,(x_{j+1/2}\,,\,t_n\,))$$

In fact in order to obtain convergent schemes we need schemes like (15) with properties of consistancy and properties of L^∞ stability and BV stability. We recall that

$$||\,u^n\,||_\infty = \text{Sup}\,\{\,|\,u^n_j\,|\,,\,j\,\in\,\mathbb{Z}\}\ \text{and}\ TV\,(u^n) = \sum_{j\in\mathbb{Z}}|\,u^n_{j+1} - u^n_j\,|\,.$$

2.2. First order scheme. In this section we take following choice for F and G in (15)

$$(16)\quad \begin{cases} \text{if}\ a\,>\,0\ \ F^n_{j+1/2} = f\,(u^n_j\,) \\[2ex] \text{if}\ a\,>\,0\ \ F^n_{j+1/2} = f\,(u^n_{j+1}\,) \end{cases}$$

$$(17)\quad \begin{cases} \text{if}\ b\,>\,0\ \ G^n_{j+1/2} = \dfrac{f_1\,(u^n_j)\,f_2\,(u^n_{j+1}\,)}{f_1\,(u^n_j) + f_2\,(u^n_{j+1}\,)} \\[3ex] \text{if}\ b\,>\,0\ \ G^n_{j+1/2} = \dfrac{f_1\,(u^n_{j+1})\,f_2\,(u^n_j\,)}{f_1\,(u^n_{j+1}) + f_2\,(u^n_j\,)} \end{cases}$$

(In (17) we set if the denominator is equal to zero).

Remark 2. In the model described in section 1.1 the choices of (16) and (17) are quite natural. In fact if $b = 0$ and $a > 0$, the two phases (water and oil) are moving from left to right and (16) the classical upstream scheme (remark also that f is nondecreasing). If $a = 0$ and $b > 0$, the two phases are not moving in the same sense. Water is moving from left to right and oil from right to left, the choice of (17) is again physically natural because f_1 corresponds to the water mobility

and f_2 corresponds to

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THEOREM 1. *Le*

(12)-(14). *Let* α,β

(18) λ sup $\{|a\,|\,f'\,(u$

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Proof of Theor

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(17) can be written o

(19) $u^{n+1}_j = u^n_j - \lambda(H$

with

(20) $H^n_{j+1/2}\ =\ H\,($

(21) $H(u,v)\ =\ af\,(u)$

(In (21) we set $\dfrac{f_1(u}{f_1(u)}$

On this form w

and f_2 corresponds to oil mobility. From the mathematical point of view the choice of (17) correponds to an approximate Riemann solver.

THEOREM 1. *Let* $a,b \in$ u_o $L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$, *and* f,g *satisfy* (12)-(14). *Let* $\alpha,\beta \in \mathbb{R}$ *such that* $\alpha \le u_o \le \beta$ *a.e., and* λ *such that*

(18) $\lambda \sup \{|a|f'(u) + |b|f'_1(u) - |b|f'_2(u), u \in [\alpha,\beta]\} \le 1.$

Then the scheme (15)-(17) *is a consistent, conservation-from, monotone (on* $[\alpha,\beta]$*) scheme. Furthermore when* $h \to 0$, *and* $k/h = \lambda$ *(fixed) satisfy* (18), $u_{h,k}$ *converges toward the weak entropic solution of* (10)-(11), *in* $L^1(\mathbb{R})$ *uniformly for bounded* $t \ge 0$ *; that is*

$$\limsup_{h \to 0} \ _{0 \le t \le T} \int_{\mathbb{R}} |u_{h,k}(x,t) - u(x,t)| \, dx = 0 \quad \text{for each } T > 0.$$

Proof of Theorem 1. We prove the theorem 1 in the case, for instance, $a > 0$ and $b > 0$ (all the cases are similar). The scheme (15)-(17) can be written on the form

(19) $u_j^{n+1} = u_j^n - \lambda(H_{j+1/2}^n - H_{j-1/2}^n)$

with

(20) $H_{j+1/2}^n = H(u_j^n, u_{j+1}^n)$

(21) $H(u,v) = af(u) + b\,\dfrac{f_1(u)\,f_2(v)}{f_1(u) + f_2(v)}$

(In (21) we set $\dfrac{f_1(u)\,f_2(v)}{f_1(u) + f_2(v)} = 0$ if $f_1(u) = f_2(v) = 0$).

On this form we see that (15)-(17) is a 3 point conservation-form scheme. Its numerical flux is the function H. It is a consistent scheme since $H(u,u) = af(u) + bg(u)$.

We now prove that H is locally Lipschitz continous. This is a consequence of (12)-(14). We first remark that H is continous. Indeed, since f_1 and f_2 are nonnegative functions and continous, the only difficulty is to prove continuitty of H points (u,v) where $f_1(u) = 0 = f_2(v)$, but this continuity is then an easy consequence of

$$|H(u,v)| \leq |f_1(u)| \quad \text{and} \quad |H(u,v)| \leq |f_2(v)|.$$

For proving that H is locally Lipschitz continous, it suffices to notice that if $f_1(u) \neq 0$ or $f_2(v) \neq 0$, then H is derivable and

$$\left|\frac{\partial H}{\partial u}(u,v)\right| = \left|af'(u) + b\,\frac{f_2^2(v)\,f'_1(u)}{(f_1(u)+f_2(v))^2}\right| \leq |af'(u)| + bf_1'(u)|$$

$$\left|\frac{\partial H}{\partial u}(u,v)\right| = \left|b\,\frac{f_1^2(u)\,f_2(v)}{(f_1(u)+f_2(v))^2}\right| \leq |bf_2'(v)|$$

Then in order to prove Theorem 1 it only remains to prove that (15)-(17) is (under the hypothesis (18)) a monotone scheme on $[\alpha,\beta]$. (We recall in particular - see for instance Grandall & al. [12] - that a consistent, conservation-form, monotone on $[\alpha,\beta]$ scheme, with a locally Lipschitz continous numerical flux is convergent - if $\alpha \leq u_o \leq \beta$ a.e.- in the sense of Theorem 1). We now prove that (15)-(17) is (with (18)) a monotone scheme on $[\alpha,\beta]$.

Let $\mathcal{H}(u,v,w) = v - \lambda\,(H(v,w) - H(u,v))$.

We have to prove that $\mathcal{H}$ is nondecreasing with respect to its arguments when $u,v,w \in [\alpha,\beta]$ (we can notice that in this case we prove in particular $\alpha \leq u^n \leq \beta$, for all $n \in \mathbb{N}$). The monotonicity of $\mathcal{H}$ with respect to its first and third arguments is a very easy consequence of (12)-(14) (we do not make use of (18)). For proving the monotonicity of $\mathcal{H}$ with respect to its second argument, let $v_1 \geq v$, one has

$\mathcal{H}(u,v_1,w) - \mathcal{H}$

$= (v_1 - v) - \lambda a(f(v_1)$

Then, with (12)-(14),

$\mathcal{H}(u,v_1,w) - \mathcal{H}(u,v,w)$

and then, with (18),

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The proof Theorem 1

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$F^n_{j+1/2} = \frac{f}{f^n_{1,j+1}}$

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$$\mathcal{H}(u,v_1,w) - \mathcal{H}(u,v,w) = (v_1 - v) - \lambda\,(H(v_1,w) - (H(v,w)) + H(u,v_1) - H(u,v)).$$

$$= (v_1 - v) - \lambda a(f(v_1) - f(v)) - \lambda b\, \frac{f_2^2(w)(f_1(v_1) - f_1(v))}{(f_1(v_1) + f_2(w))\,(f_1(v) + f_2(w))} + \lambda b\, \frac{f_1^2(u)(f_2(v_1) - f_2(v))}{(f_1(u) + f_2(v_1))\,(f_1(u) + f_2(v))}$$

Then, with (12)-(14), we deduce

$$\mathcal{H}(u,v_1,w) - \mathcal{H}(u,v,w) \geq (v_1 - v) - \lambda\,\{ (af(v_1) + bf_1(v_1) - bf_2(v_1)) - (af(v) + bf_1(v) - bf_2(v)) \}$$

and then, with (18),

$$\mathcal{H}(u,v_1,w) - \mathcal{H}(u,v,w) \geq 0.$$

The proof Theorem 1 is complete.

Remark 3. The scheme (15)-(17) is a first order scheme and it introduces some numerical diffusion (which is necessary for the monotonicity of the scheme). Its disadvantage is, as usual, that it smeras the front. The numerical diffusion of that scheme is higher than that of the classical upstream weighting scheme used in reservoir simulation (which is also a monotone scheme - Brenier [2], Pfertzel [12]). For the sake of completness we recall that this classical upstream weighting scheme is (15) with, for $a \geq 0$ and $b \geq 0$ (for instance),

$$F^n_{j+1/2} = \frac{f^n_{1,j+1/2}}{f^n_{1,j+1/2} + f^n_{2,j+1/2}}\,,\quad G^n_{j+1/2} = \frac{f^n_{1,j+1/2} \cdot f^n_{2,j+1/2}}{f^n_{1,j+1/2} + f^n_{2,j+1/2}}\,,$$

$$f^n_{1,j+1/2} = f_1(u^n_j)\,,\ f^n_{2,j+1/2} = f_2(u^n_j) \text{ if } a - bf_1(u^n_j) \geq 0 \text{ and } f^n_{2,j+1/2} = f_2(u^n_{j+1}) \text{ if } a - bf_1(u^n_j) < 0.$$

2.3. Second order scheme. In order to reduce the numerical diffusion of the scheme described in 2.2, we now propose a "quasi" second order space discretization, associated with the usual Euler explicit time discretization. This corresponds to (15) where

$$\frac{1}{h}\left(a\left(F_{j+1/2}-F_{j-1/2}\right)+b\left(G_{j+1/2}-G_{j-1/2}\right)\right)$$

does not contain time discretization (it does not depend on the time step) and is a second order approximation of $a\,(f(u))_x + b\,(g(u))_x$ except eventually on the points (x,t) where u is not C^3 (we need u in C^3 for the computation of the truncation error) or where $(f(u))_x \cdot f_1(u))_x \cdot (f_2(u))_x = 0$. (This last condition is the sense of "quasi").

In fact the scheme we describe below has enough numerical diffusion in the neighbourghood of the front of the exact solution (this is due to the slope limiter) in order to avoid numerical oscillations, and is an antidiffusive scheme (due to the time discretization) elsewhere. Globally (space and time) it is a first order scheme, but it appears numerically that it gives results as good as that of quasi second order scheme (space-time), in particular the fronts are not smeared (this is not the case when we use an implicit scheme).

Remark 4. More precisely, when we use a second order space discretization, and the Euler explicit time discretization on the equation $u_t + (f(u))_x = 0$, we obtain a numerical scheme which can be viewed as a second order scheme (space-time) on the equation $u_t + k/2\, u_{tt} + (f(u))_x = 0$, or equivalently on the equation

$$u_t + (f(u))_x + \frac{k}{2}\left((f'(u))^2 u_x\right)_x = 0.$$

On this last form we see that we have a numerical antidiffusion.

In this scheme we take the following choice for F and G in (15), for $j \in \mathbb{Z}$.

$$(22)\quad \begin{cases} \text{if } a>0 \quad F^n_{j+1/2} = f^n_{0,j+1/2,-} \\ \text{if } a<0 \quad F^n_{j+1/2} = f^n_{0,j+1/2,+} \end{cases}$$

$$(23)\quad \begin{cases} \text{if } b>0 \quad G^n_{j+1} \\ \text{if } b<0 \quad G^n_{j+1} \end{cases}$$

(we set (23) $G^n_{j+1/2}$ =
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$$(24)\quad \begin{cases} f^n_{1,j+1/2,-} = \\ f^n_{1,j+1/2,+} = \end{cases}$$

$$(25)\quad \begin{cases} p^n_{1,j} = \text{sign } (q \\ \text{if } q^n_{1,j}, f_1(u \\ p^n_{1,j} = 0 \text{ if n} \end{cases}$$

$$(26)\quad q^n_{1,j} = \frac{f_1(u^n_{j+1}) -}{2}$$

Remark 5. The
condition gives that, f

Min $(f_1(u^n_j), f_1($

in particular, since f_1 a

$$(23)\quad \begin{cases} \text{if } b>0 \;\; G^n_{j+1/2} = \dfrac{f^n_{1,j+1/2,-}\, f^n_{2,j+1/2,+}}{f^n_{1,j+1/2,-} + f^n_{2,j+1/2,+}} \\[2ex] \text{if } b<0 \;\; G^n_{j+1/2} = \dfrac{f^n_{1,j+1/2,-}\, f^n_{2,j+1/2,-}}{f^n_{1,j+1/2,-} + f^n_{2,j+1/2,-}} \end{cases}$$

(we set (23) $G^n_{j+1/2} = 0$ if the denominator is zero).
With, for 1 = 0,1,2, setting $f_o = f$,

$$(24)\quad \begin{cases} f^n_{1,j+1/2,-} = f_1(u^n_j) + \dfrac{h}{2}\, p^n_{1,j} \\[2ex] f^n_{1,j+1/2,+} = f_1(u^n_j) + \dfrac{h}{2}\, p^n_{1,j} \end{cases}$$

$$(25)\quad \begin{cases} p^n_{1,j} = \text{sign}\,(q^n_{1,j} \;\text{Min}\left\{ |q_{1,j}| \,,\, \dfrac{2}{h}\, |f_1(u^n_{j+1}) - f_1(u^n_j)| \,,\; \dfrac{2}{h}\, |f_1(u^n_j) - f_1(u^n_{j-1})| \right\} \\[2ex] \quad \text{if } q^n_{1,j}, f_1(u^n_{j+1}) - f_1(u^n_j), f_1(u^n_j) - f_1(u^n_{j-1}) \text{ have same sign} \\ p^n_{1,j} = 0 \text{ if not} \end{cases}$$

$$(26)\quad q^n_{1,j} = \frac{f_1(u^n_{j+1}) - f_1(u^n_{j-1})}{2h}$$

Remark 5. The slope limiter is in the condition (25), in fact this condition gives that, for 1 = 0,1,2, one has

$$\text{Min}\,(f_1(u^n_j), f_1(^n_{j+1})) \le f^n_{1,j+1/2,\pm} \le \text{Max}\,(f_1(u^n_j), f_1(u^n_{j+1})),$$

in particular, since f_1 are nonnegative functions, we have also

$$f^n_{1,j+1/2,\pm} \geq 0$$

Remark 6: The scheme (15), with (22)-(26) is a finite volums scheme using P_1 discontinous functions and slope limiters. The originality of this scheme is that use P_1 discontinous functions and slope limiters on f, f_1 and f_2 instead of on the unknown u. This choice is suggested by reservoir simulation.

In fact in reservoir simulation we use P_1 discontinous functions (and slope limiters) on f_1 and f_2, associated with the upstream choice described in Remark 3. The case $b = 0$, P_1 discontinous functions and slope limiters on f is more completely studied in Chalabi & al. [4], were some implicit schemes are also described.

THEOREM 2. *Let* $a,b \in \mathbb{R}$ $u_o \in L^1(\mathbb{R}) \cap BV(\mathbb{R})$, *and* f,g *satisfy* (12)-(14). *Then the scheme* (15), *with* (22)-(26) *is a consistent, conservation-form scheme. It corresponds to quasi second order space discretization (in the sense defined before). Furthermore let* $\alpha,\beta \in \mathbb{R}$ *such that* $\alpha \leq u_0 \leq \beta$ *a.e, and* λ *such that*

$$(27)\quad \begin{cases} 2\lambda\,[\,|a|\,\mathrm{Sup}\,\{f'(s),\ \alpha \leq s \leq \beta\} + |b|\,(\mathrm{Sup}\,\{f_1'(s),\ \alpha \leq s \leq \beta\} \\ \qquad\qquad + \mathrm{Sup}\,\{-f_2'(s),\ \alpha \leq s \leq \beta\})] \leq 1 \end{cases}$$

The one has

$$(28)\quad \alpha \leq u^n \leq \beta \quad \forall\, n \in \mathbb{N}$$

$$(29)\quad TV(u_n) \leq TV(u_o)\ \ \forall\, n \in \mathbb{N}\ (TVD\ \text{scheme})$$

and for every sequence $(h_m, k_m)_{m\ \mathbb{N}}$, with $k_m = \lambda h_m$, λ (fixed) satisfying (27), and $h_m \to 0$ as $m \to +\infty$, there exists a subsquence, still denoted (h_m, k_m) such that $u_{h_m, k_m} \to u$ in $L^1(\mathbb{R})$, as $m \to +\infty$, uniformly for bounded $t \geq 0$, where u is a weak solution of (10)-(11).

Remark 7. Let

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Proof of Theore
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with

(31) $H^n_{j+1/2} = H(u^n_{j-1}$

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obviously have $H(u$
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P_1 discontinous functions
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d in Chalabi & al. [4], were

$\cap$ $BV(\mathbb{R})$, *and f,g satisfy*
22)-(26) *is a consistent,*
quasi second order space
. *Furthermore let* $\alpha,\beta \in \mathbb{R}$

Sup $\{f_1'(s), \alpha \leq s \leq \beta\}$
$\{-f_2'(s), \alpha \leq s \leq \beta\})] \leq 1$

e)

$= \lambda h_m$, λ (fixed) satis-
sts a subsequence, still
$\mathbb{R}$), as $m \to +\infty$, uni-
tion of (10)-(11).

Remark 7. Let $c > 0$ and $\alpha \in]0,1[$. If we add in the "Min" of the first frmula of (25) a fourth argument, $ch^{\alpha-1}$, then the conclusions of Theorem 2 are still true and we can proved that all the sequence is going to u (as $m \to \infty$) and that u is the weak entorpic solution of (10)-(11). This can be proved as it is done in Vila [15] for similar schemes. Numerically it is not necessary to use this modification of (25).

Proof of Theorem 2. As for Theorem 1, we prove Theorem 2 in particular case $a > 0$ and $b > 0$ (all the case similar). The scheme (15), (22)-(26) cans be written

(30) $u_j^{n+1} = u_j^n - \lambda(H_{j+1/2}^n - H_{j-1/2}^n)$

with

(31) $H_{j+1/2}^n = H(u_{j-1}^n, u_{j+1}^n, u_{j+2}^n) = aF_{j+1/2}^n + bG_{j+1/2}^n$,

and

$$F_{j+1/2}^n, \ G_{j+1/2}^n \text{ given by (22)-(26).}$$

Then the scheme (15),(22)-(26) is a five point conservation-form scheme. Its numerical flux is H. The scheme is consistent because we obviously have $H(u,u,u,u) = af(u) + bg(u)$.

The function H is locally Lipschitz continous. Indeed we see that the functions $p_{1,j}(1 = 0,1,2)$ are locally Lipschitz continous with respect to their arguments

$$u_{j-1}^n, \ u_j^n, \ u_{j+1}^n.$$

Then the function $F_{j+1/2}^n$ is also locally Lipschitz continous. The same argument works for $G_{j+1/2}^n$ except eventually if $f_{1,j+1/2,-}^n = f_{2,j+1/2,+}^n = 0$ (we recall that we always have $f_{1,j+1/2,-}^n \geq 0$ and $f_{2,j+1/2,+}^n \geq 0$, see remark 5). In this case we prove continuity and local Lipschitz continuity of $G_{j+1/2}^n$ with respect to its arguments

$$u^n_{j-1},\ u^n_j,\ u^n_{j+1},\ u^n_{j+2},$$

by using the fact that

$$\left(\frac{f^n_{1,j+1/2,-}}{f^n_{1,j+1/2,-}+f^n_{2,j+1/2,+}}\right)^2 \leq 1 \text{ and } \left(\frac{f^n_{2,j+1/2,+}}{f^n_{1,j+1/2,-}+f^n_{2,j+1/2,+}}\right)^2 \leq 1$$

(as in the proof of local Lipschitz continuity of H in Theorem 1). Thus we have proved that H is locally Lipschitz continious).

We now prove that the scheme (15), (22)-(26) corresponds to a quasi second order space discretization. Let u be C^3 function in the neighbourhood of a given point (x,t) and such that $(f_1(u))_x\,(x,t) \neq 0$ for $1 = 0,1,2$ (we recall that $f_0 = f$).

We denote by u_j the value $u(x,t)$ and by u_{j+i} the $u(x+ih,t)$, we have prove that, with

$$H_{j+1/2} = H(u_{j-1},\ u_j,\ u_{j+1},\ u_{j+2}), \text{ one has}$$

$$\frac{1}{h}\,(H_{j+1/2} - H_{j+1/2}) = a(f(u))_x\,(x,t) + b(g(u))_x\,(x,t) + h^2C$$

where C is bounded independently of h, as $h \to 0$.

In the following we denote by C various functions of h, bounded as $h \to 0$. These functions depend on the C^1, C^2, and C^3 derivatives of u.

By (25)-(26) one has, for 1= 0,1,2 (we remove the upper index n).

$$q_{1,j} = a(f(u))_x + h^2C$$

$$\frac{2}{h}\,(f_1(u_{j+1}) - f_1(u_j)) = 2(f_1(u))_x\,(x,t) + hC$$

$$\frac{2}{h}\,(f_1(u_j) - f_1(u_{j-1})) = 2(f_1(u))_x\,(x,t) + hC$$

Since $(f_1(u))_x \neq 0$, we the

$p_{1,j} =$

for h small enough. Simil

$p_{1,j\pm1}$

thus

$p_{1,j\pm1} =$ (

Then one has (see (24)),

$f_1 =\ (f_1(u))\,(x,t),$

(32) $f_{1,j+1/2,-} = f$; $f_{1,j-1/2,+} = f$

and

$f_{1,j+1/2,+} = f$

$f_{1,j-1/2,-} = f$

thus,

(33) $f_{1,j+1/2,+}$; $f_{1,j-1/2,-} =$

Since $(f_1(u))_x \neq 0$, we then deduce (from (25)), that

$$p_{1,j} = q_{1,j} = (f_1(u))_x\,(x,t) + h^2 C$$

for h small enough. Similary we have

$$p_{1,j\pm 1} = (f_1(u))_x\,(x \pm h,\, t) + h^2 C$$

thus

$$p_{1,j\pm 1} = (f_1(u))_x\,(x,t) \pm f_1(u))_{xx}\, h + h^2 C$$

Then one has (see (24)), setting (for simiplicity)

$$f_1 = (f_1(u))\,(x,t),\, f_1{}' = (f_1(u))_x\,(x,t),\, f_1{}'' = (f_1(u))_{xx}\,(x,t)$$

$$(32) \qquad \begin{cases} f_{1,j+1/2,-} = f_1 + \dfrac{h}{2} f_1{}' + h^3 C, \\ \\ f_{1,j-1/2,+} = f_1 - \dfrac{h}{2} f_1{}' + h^3 C, \end{cases}$$

and

$$f_{1,j+1/2,+} = f_1(u)\,(x+h,t) - \frac{h}{2} f_1{}' - \frac{h^2}{2} f_1{}'' + h^3 C,$$

$$f_{1,j-1/2,-} = f_1(u)\,(x-h,t) + \frac{h}{2} f_1{}' - \frac{h^2}{2} f_1{}'' + h^3 C,$$

thus,

$$(33) \qquad \begin{cases} f_{1,j+1/2,+} = f_1 + \dfrac{h}{2} f_1{}' + h^3 C, \\ \\ f_{1,j-1/2,-} = f_1 - \dfrac{h}{2} f_1{}' + h^3 C, \end{cases}$$

From (22) and (32)-(33) we deduce

$$F_{j\pm 1/2} = f_0 \pm (h/2)\, f_0' + h^3 C$$

and then,

$$(34)\quad (1/h)\,(F_{j+1/2} - F_{j-1/2}) = (f(u))_x\,(x,t) + h^2 C$$

From (23) and (32)-(33) we deduce

$$G_{j\pm 1/2} = \frac{(f_1 \pm (h/2) f_1' + h^3 C)(f_2 \pm (h/2) f_2' + h^3 C)}{(f_1 + f_2) \pm (h/2)(f_1' + f_2') + h^3 C}$$

Then easy computation (and a continuity argument for the cases where the denominator of $G_{j\pm 1/2}$ is zero) show that

$$(35)\quad (1/h)\,(G_{j+1/2} - G_{j-1/2}) = (g(u))_x\,(x,t) + h^2 C.$$

By (34) and (35) we have

$$(1/h)\,(H_{j+1/2} - H_{j-1/2}) = a(f(u))_x(x,t) + b(g(u))_x\,(x,t) + h^2 C\,,$$

and we have proved that the scheme (15), (22)-(26) corresponds to a quasi second order space discretization.

To prove Theorem 2 it only remains to prove (28)-(29), since we know that a consistent conservation-form scheme, wich satisfies (28)-(29) and with a locally Lipschitz continous numerical flux is convergent in the sense of Theorem 2. (This is essentially due to Harten, see [10]). We now prove (28)-(29).

From $\alpha \le u_0 \le \beta$ a.e., we deduce $\alpha \le u_0 \le \beta$.

Thus we only have to show that $\alpha \le u_n \le \beta$ implies $\alpha \le u_{n+1} \le \beta$ and $TV(u_{n+1}) \le TV(u_n)$ for all $n \in \mathbb{N}$. Let, $\alpha \le u_n \le \beta$.

From (15) and (22)-(26) we have (for $j \in \mathbb{Z}$).

$$u_j^{n+1} = u_j^n - \lambda\, a(f^n_{0,j+1/2,-} - f^n_{0,j-1/2,-})$$

$- \lambda \left(\frac{f^n_{1,j+1/2}}{f^n_{1,j+1/2}} \right.$

This can be written, wi

$(36)\quad u_j^{n+1} = u_j^n + \lambda\, A^n_{j+}$

with (setting $f^n_{1,j\pm 1/2,\pm}$

$(37)\quad A^n_{j+1/2} = - \frac{b\, f}{\Delta_+ u_j^n\, (f}$

$(38)\quad B^n_{j-1/2} = \frac{a}{\Delta_+ u^n_{j-1}}\, (f$

We first remark that A

$1 = 1{,}2$, and

$\frac{f_{2++} - f_{2-+}}{\Delta_+ u_j^n}$

Indeed one has, for in

$f_{1+-} - f_{1--}$

By (25), $p^n_{1,j}$ and $p^n_{1,}$

$$- \lambda \left(\frac{f^n_{1,j+1/2,-}\, f^n_{2,j+1/2,+}}{f^n_{1,j+1/2,-} + f^n_{2,j+1/2,+}} - \frac{f^n_{1,j-1/2,-}\, f^n_{2,j-1/2,+}}{f^n_{1,j-1/2,-} + f^n_{2,j-1/2,+}} \right)$$

This can be written, with $\Delta_+ u^n_j = u^n_{j+1} - u^n_j$,

(36) $u^{n+1}_j = u^n_j + \lambda A^n_{j+1/2} \Delta_+ u^n_j - \lambda B^n_{j-1/2} \Delta_+ u^n_{j-1}$

with (setting $f^n_{1,j\pm1/2,\pm} = f_{1\pm\pm}$)

(37) $$A^n_{j+1/2} = - \frac{b\, f_{1--}\, f_{1+-}\, (f_{2++} - f_{2-+})}{\Delta_+ u^n_j\, (f_{1+-} + f_{2++})\, (f_{1--} + f_{2-+})} \text{, if } \Delta_+ u^n_j \neq 0$$

(38) $$B^n_{j-1/2} = \frac{a}{\Delta_+ u^n_{j-1}} (f_{0+-} - f_{o--}) + \frac{b}{\Delta_+ u^n_{j-1}} \frac{f_{2-+}\, f_{2++}\, (f_{1+-} - f_{1--})}{(f_{1+-} + f_{2++})\, (f_{1--} + f_{2-+})} \text{, } if\ \Delta_+ u^n_{j-1} \neq 0.$$

We first remark that $A^n_{j+1/2} \geq 0$ and $B^n_{j-1/2} \geq 0$, since $f^n_{1\,..} \geq 0$, for, 1 = 1,2, and

$$\frac{f_{2++} - f_{2-+}}{\Delta_+ u^n_j} \geq 0, \quad \frac{f_{0+-} - f_{0--}}{\Delta_+ u^n_{j-1}} \geq 0, \quad \frac{f_{1+-} - f_{1--}}{\Delta_+ u^n_{j-1}} \geq 0,$$

Indeed one has, for instance,

$$f_{1+-} - f_{1--} = f_1(u^n_j) + \frac{h}{2} p^n_{1,j} - (u^n_{j-1}) - \frac{h}{2} p^n_{1,j-1} \cdot$$

By (25), $p^n_{1,j}$ and $p^n_{1,j-1}$ have the same sign and

$$|p^n_{1,j-1}|, |p^n_{1,j}| \le -\frac{2}{h} |\Delta_+ f_1(u^n_{j-1})| .$$

We then conlude that $\dfrac{f_{1+-} - f_{1--}}{\Delta_+ u^n_{j-1}} \ge 0$ (since $f_1' \ge 0$).

A similar argument works for the two other terms.

We now make use of (27) in order to prove that

$$(39) \qquad \begin{cases} 1 - \lambda A^n_{j+1/2} - \lambda B^n_{j+1/2} \ge 0 \\ 1 - \lambda A^n_{j+1/2} - \lambda B^n_{j-1/2} \ge 0 \end{cases}$$

one has $A^n_{j+1/2} \le b \left| \dfrac{f_{2++} - f_{2-+}}{\Delta_+ u^n_j} \right|$

$$\le 2b \text{ Sup } \{-f_2'(s) , \alpha \le s \le \beta \}$$

$$B^n_{j+1/2} \le 2a \text{ Sup } \{f_2'(s) , \alpha \le s \le \beta \}$$

$$+ 2b \text{ Sup } \{f_2'(s) , \alpha \le s \le \beta \}$$

This prove (39) if (27) is satisfied.

The first part of (39) show (29). Indeed on has

$$TV(u^{n+1}) = \sum_{j \in \mathbb{Z}} |u^{n+1}_{j+1} - u^{n+1}_j|$$

$$= \sum_{j \in \mathbb{Z}} |(1 - \lambda A^n_{j+1/2} - \lambda B^n_{j+1/2}) \Delta_+ u^n_j + \lambda A^n_{j+3/2} \Delta_+ u^n_{j+1} + \lambda B^n_{j-1/2} \Delta_+ u^n_{j-1}|$$

$\le \sum_{j \in \mathbb{Z}} |(1 - \lambda A^n_{j+}$

(we have used the first pa
is (29).

The second part of (
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Then one has $\alpha \le u^{n+1} \le$

The proof of Theorem 2 i

Remark 8. We rec
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time) which satisfy (28)-
to (27). (Then the schem
let $f^n_{j+1/2}$ and $G^n_{j+1/2}$ def
slope limiter (for instante

$H^n_{j+1/2}$ b

The new scheme is then
formula (30). The valu
$b(g(u))$ at the time $t_n + k$
convenient correction fro
scheme is more complic
obtained by the scheme
done in Chalabi & al. [4

$$\leq \sum_{j\in \mathbb{Z}} |(1 - \lambda A^n_{j+1/2} - \lambda B^n_{j+1/2})|\Delta_+ u^n_j| + \sum_{j\in \mathbb{Z}} \lambda A^n_{j+1/2}| \Delta_+ u^n_j| + |$$

$$+ \sum_{j\in \mathbb{Z}} \lambda A^n_{j+1/2}| \Delta_+ u^n_j |$$

(we have used the first part of (39)), and then $TV(u^{n+1}) \leq TV(u^n)$. This is (29).

The second part of (39) show (28). Indeed (36) show that u_j^{n+1} is the convex hull of

$$u^n_{j-1},\ u^n_j,\ u^n_{j+1}.$$

Then one has $\alpha \leq u^{n+1} \leq \beta$ since $\alpha \leq u^n \leq \beta$.

The proof of Theorem 2 is complete.

Remark 8. We recall that the scheme we have described here (scheme (15), (22)-(26) is only a first order scheme (in space-time). It is also possible to construct a "quasi" second order scheme (in space-time) which satisfy (28)-(19) under a convenient *CFL* condition similar to (27). (Then the scheme is convergent in the since of Theorem 2.) We let $f^n_{j+1/2}$ and $G^n_{j+1/2}$ defined by (22)-(26) but with a more restrictive slope limiter (for instante 1 instead of 2 in (25)). This define

$$H^n_{j+1/2} \text{ by } H^n_{j+1/2} = a\, F^n_{j+1/2} + b\, F^n_{j+1/2}$$

The new scheme is then obtained by setting $H^{n+1/2}_{j\pm 1/2}$ instead $H^n_{j\pm 1/2}$ in formula (30). The value $H^{n+1/2}_{j+1/2}$ is an approximation of $a(f(u)) + b(g(u))$ at the time $t_n + k/2$ (and point $(j + 1/2)$h) and is obtained with a convenient correction from $H^n_{j+1/2}$ (using equation (10)). The obtained scheme is more complicate and, numerically, give results similar to that obtained by the scheme (15), (22)-(26). A more complete description is done in Chalabi & al. [4] in the case $b = 0$.

Our method to obtain "quasi" second order scheme (in space or in space-time) works also with the classical upstream weighting scheme used in reservoir simulation briefly described in Remark 3 (see Pfertzel [12]).

3. The two dimensional case.

3.1. Preliminaries. Let N = 2. The equation to solve is

(40) $(u)_t + \operatorname{div} f(u)\,\mathbf{Q} + \operatorname{div} g(u)\,\mathbf{g} = 0$, in $\mathbb{R} \times [0,\infty[$

(41) $u\,(.0) = u_0$ in $\mathbb{R}$

Q and g are some given constant vectors. We set $\mathbf{Q} = (a,b)^t$ and $\mathbf{g} = (c,d)^t$. We assume that u_0, the initial datum, lies in the space

$$L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2).$$

The hypotheses on f and g are (12)-(14) (as in Section 2).
For simplicity we take the same space step, say h, in the two directions (denoted by x and y), and k is the time step. We set

$\lambda = k/h$, $M_{i,j} = (ih,jh)$, $M_{\pm 1/2,j \pm 1/2} = ((i \pm 1/2)h, (j \pm 1/2h))$, $j \in \mathbb{Z}$, and $t_n = nk$, $n \in \mathbb{N}$.

The desired approximate solution of (40)-(41) is defined by

$$\{u^n_{i,j},\ i,j \in \mathbb{Z},\ n \quad \mathbb{N}\} \in \mathbb{R}$$

As in Section 2 we have, by definition,

$$u^o_{i,j} = \frac{1}{h^2} \int_{k_{i,j}} u_o(x,y)\ dxdy$$

where $K_{i,j}$ is the square of vertice $M_{i\pm 1/2,j \pm 1/2}$. We set for $M \in K_{i,j}$

i,j

$u_{h,}$

The schemes we descr
& al. [5], Harten & al.
form

(42) $\frac{1}{k}\,(u^{n+1}_{i,j} - u^n_{i,j}) + \cdot$

with

$H^n_{i+1/2}$

$H^n_{i,j+1/}$

where p is a positive i
variables.

To obtain the sc
Volums discretization
tization in time. This

(43) $h^2 \frac{u^{n+1}_{i,j} - u^n_{i,j}}{k}$

where $\partial K_{i,j}$ denotes
$K_{i,j}$, and $\mathbf{n}_{i,j}$ is the

The schemes in
$f(u)$ and $g(u)$ on ∂K_i

3.2. Case g =
g = 0, and describe o
$\mathbb{Z}$. We have to descr

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upstream weighting scheme
ed in Remark 3 (see Pfertzel

equation to solve is

$\mathbb{R} \times [0,\infty[$

s. We set $\mathbf{Q} = (a,b)^t$ and
m, lies in the space

).

in Section 2).
say h, in the two directions
We set

$/2)h, (j\pm 1/2h)), \; j \in \mathbb{Z}$;

)-(41) is defined by

$\in \mathbb{R}$

dy

We set for $M \in K_{i,j}$

$$i,j \in \mathbb{Z}, t^n \le t \le t^{n+1}, \; n \in \mathbb{N}$$

$$u_{h,k}(M,t) = u^n_{i,j}, \; u^n(M) = u^n_{i,j}$$

The schemes we describe below have conservation form (see Crandall & al. [5], Harten & al. [11]), they can be written under the following form

$$(42) \quad \frac{1}{k}(u^{n+1}_{i,j} - u^n_{i,j}) + \frac{1}{h}(H^n_{i+1/2,j}) + \frac{1}{h}(H^n_{i,j+1/2} - H^n_{i,j-1/2}) = 0$$

with

$$H^n_{i+1/2,j} = H_1(u^n_{i-p+1,j-p}, \ldots, u^n_{i+p,j+p}),$$
$$H^n_{i,j+1/2,j} = H_2(u^n_{i-p,j-p+1}, \ldots, u^n_{i+p,j+p}),$$

where p is a positive integer and H_1, H_2 are functions of $2p(2p+1)$ real variables.

To obtain the schemes of the following sections, we use a Finite Volums discretization in space combined with an Explicite Euler discretization in time. This gives

$$(43) \qquad h^2\,\frac{u^{n+1}_{i,j} - u^n_{i,j}}{k} + \int_{\partial K_{i,j}} f(u)\,\mathbf{Q}\,\mathbf{n}_{i,j}\,d\sigma + \int_{\partial K_{i,j}} g(u)\,\mathbf{g}\,\mathbf{n}_{i,j}\,d\sigma = 0$$

where $\partial K_{i,j}$ denotes the boundary of $K_{i,j}$, $d\sigma$ the length element on $K_{i,j}$, and $\boldsymbol{n}_{i,j}$ is the normal to $\sigma K_{i,j}$, exterior to $K_{i,j}$

The schemes in the following section are given by the choice of $f(u)$ and $g(u)$ on $\partial K_{i,j}$.

3.2. Case $\mathbf{g} = 0$, first order scheme. In this section we let $\mathbf{g} = 0$, and describe our "vertex upstream weighting scheme". Let $i,j \in \mathbb{Z}$. We have to describe in (43) the choice $f(u)$ on $\partial K_{i,j}$. The four ver-

tice of $K_{i,j}$ are M_k, $k \quad I_{i,j}$, with $I_{i,j} = \{(i+1/2, j+1/2), (i+1/2, j-1/2), (i-1/2, j+1/2), (i-1/2, j-1/2)\}$.
For $k \quad I_{i,j}$, let $\partial K_{i,j}(M_k)$ be the part of $\partial K_{i,j}$ composed of the two half-edges of $\partial K_{i,j}$ containing M_k.
The one has

$$\delta K_{i,j} = \bigcup_{k \in I_{i,j}} \delta K_{i,j}(M_k)$$

We take in (43)

$$(44) \quad \int_{\delta K_{i,j}} f(u)\, \mathbf{Q}\, \mathbf{n}_{i,j}\, d\sigma = \sum_{k \in I_{i,j}} \int_{\delta K_{i,j(Mk)}} f(u)\, \mathbf{Q}\, \mathbf{n}_{i,j}\, d\sigma$$

with

$$(45) \quad \begin{cases} f(u) = f(u^n_{i,j}) \text{ on } \delta K_{i,j}(M_k) \text{ if } \displaystyle\int_{\delta K_{i,j(Mk)}} \mathbf{Q}\, \mathbf{n}_{i,j}\, d\sigma \geq 0 \\ f(u) = f_{M_k} \text{ on } \delta K_{i,j}(M_k) \text{ if } \displaystyle\int_{\delta K_{i,j(Mk)}} \mathbf{Q}\, \mathbf{n}_{i,j}\, d\sigma < 0 \end{cases}$$

We now describe the choice of f_{M_k} in (45). This choice is made in order to obtain, with (43)-(45), a scheme which has conservation form. Indeed let $M_{i+1/2,j+1/2}$, $(i,j \in \mathbb{Z})$, be a vertex of the a mesh. $M_{i+1/2,j+1/2}$ is a vertex of the four squares $K_{i+\alpha,j+\beta}$, $(\alpha,\beta) \in J$ with $J = \{(0,0), (0,1), (1,0), (1,1)\}$. We set

$$(46) \quad \begin{cases} J^+_{i,j} = \{(\alpha,\beta) \in j, \ldots \\ J^+_{i,j} = \mathcal{N}^+_{i,j} \ldots \end{cases}$$

and f_{M_k} is defined by $(M_k$ …

$$(47) \quad \begin{cases} f_{M_{i+1/2,j+1/2}} \big(- \sum_{(\alpha,\beta) \in} \ldots \\ \sum_{(\alpha,\beta) \in j^+_{i,j}} f(u^n_{i+\alpha,j+\beta}) \ldots \end{cases}$$

It is easy to verify that the …
cause one has, for all i,j

$$\sum_{(\alpha,\beta) \in j} \int_{\partial K_{i+\alpha,j\ldots}} \ldots$$

(This is the case when w…
ization of (5)-(6) made in …
We first prove that …
and is consistent with (4…

PROPOSITION 1. *The* …

+1/2, j+1/2), (i+1/2,j-1/2), (i-

$_{i,j}$ composed of the two half-

M_k)

u) Q $\mathrm{n}_{i,j}\, d\sigma$

Q $\mathrm{n}_{i,j}\, d\sigma \geq 0$

$j(Mk)$

Q $\mathrm{n}_{i,j}\, d\sigma < 0$

k)

(45). This choice is made

which has conservation

a vertex of the a mesh.

$+\alpha,j+\beta$, $(\alpha,\beta)\in J$ with J =

$$(46)\quad \begin{cases} \mathcal{J}^+_{i,j} = \{(\alpha,\beta)\in j, \displaystyle\int_{\delta K_{i+\alpha,j+\beta}(M_{i+1/2,j+1/2})} \mathrm{Q}.\mathrm{n}_{i+\alpha,j+\beta}\, d\sigma \geq 0\} \\ \mathcal{J}^+_{i,j} = \mathcal{N}^+_{i,j} \end{cases}$$

and f_{M_k} is defined by ($M_k = M_{i+1/2,j+1/2}$)

$$(47)\quad \begin{cases} f_{M_{i+1/2,j+1/2}}\Big(-\displaystyle\sum_{(\alpha,\beta)\in J_{i,j}} \int_{\partial K_{i+\alpha,j+\beta}(M_{i+1/2,j+1/2})} \mathrm{Q}.\mathrm{n}_{i+\alpha,j+\beta}\, d\sigma\Big) = \\ \displaystyle\sum_{(\alpha,\beta)\in j^+_{i,j}} f(u^n_{i+\alpha,j+\beta}) \int_{\partial K_{i+\alpha,j+\beta}(M_{i+1/2,j+1/2})} \mathrm{Q}.\mathrm{n}_{i+\alpha,j+\beta}\, d\sigma \end{cases}$$

It is easy to verify that the scheme (43)-(47) has conservation form, because one has, for all i,j

$$\sum_{(\alpha,\beta)\in j} \int_{\partial K_{i+\alpha,j+\beta}(M_{i+1/2,j+1/2})} \mathrm{Q}.\mathrm{n}_{i+\alpha,j+\beta}\, d\sigma = 0$$

(This is the case when we use the velocities Q obtained by the discretization of (5)-(6) made in Eymard & al. [8]).

We first prove that the scheme (43)-(47) has conservation form and is consistent with (40) (with g = 0).

PROPOSITION 1. *The scheme* (43)-(47) *is a consistent, conservation-*

form approximation of (40) (*when* $g = 0$). *Furthermore the numerical fluxes of this scheme are* C^1 *functions.*

Proof of Proposition 1. We prove the proposition 1 in the particular case $a \geq b > 0$ (the other cases are similar). In this case (47) gives for all $i,j \in \mathbb{Z}$.

$$(48)\quad (2a)f_{M_{i+1/2,j+1/2}} = (a+b)f(u^n_{i,j}) + (a-b)f(u^n_{i,j+1}).$$

Then (44) is

$$\int_{\partial K_{i,j}} f(u)\,\mathbf{Q}.\mathbf{n}_{i,j}\,d\sigma = 2a\,\frac{h}{2}f(u^n_{i,j}) - (a-b)\frac{h}{2}f_{M_{i-1/2,j+1/2}} - (a+b)\frac{h}{2}f_{M_{i-1/2,j+1/2}}$$

this gives, with (48) ,

$$(49)\quad \frac{2}{h}\int_{\partial K_{i,j}} f(u)\,\mathbf{Q}.\mathbf{n}_{i,j}\,d\sigma = 2a\,f(u^n_{i,j}) - \frac{a^2-b^2}{a}\,f(u^n_{i,1,j}) - \frac{(a^2-b^2)}{2a}f(u^n_{i,1,j}) - \frac{(a-b)^2}{2a}f(u^n_{i,1,j})$$

which can also be written

$$(50)\quad \left\{ \frac{2}{h}\int_{\partial K_{i,j}} f(u)\,\mathbf{Q}.\mathbf{n} \ldots + [(-\frac{(a-b}{2a} \ldots \right.$$

Then we see that sch
scheme (43)-(47) is t

$H^n_{i,j+1/2} = H_2(u^n_{i-1,\ldots}$

The numerical fluxe
are C^1 functions by

The proof of propo
gence result.

THEOREM 3.

($\mathbb{R}^2$), *and f,g satisf*
and λ *such that*

(51) λ Max ($|a|$, $|b$

Then the scheme

). *Furthermore the numerical*

e proposition 1 in the particu-
nilar). In this case (47) gives

$)f(u^n_{i,j+1})$.

$f_{M_{i-1/2,j+1/2}}$ -

$- (a+b)\frac{h}{2} f_{M_{i-1/2,j+1/2}}$

$\frac{b^2}{} f(u^n_{i,1,j})$ -

$f(u^n_{i,1,j}) - \frac{(a-b)^2}{2a} f(u^n_{i,1,j})$

$$(50)\quad \frac{2}{h}\int_{\partial K_{i,j}} f(u)\, \mathbf{Q}.\mathbf{n}_{i,j}\, d\sigma = [2a\, f(u^n_{i,j}) - 2a\, f(u^n_{i-1,j})] + \\ + [(-\frac{(a-b)^2}{2a} f(u^n_{i-1,j+1}) + \frac{(a+b)^2}{2a} f(u^n_{i-1,j+1}) + \frac{(a+b)^2}{2a} f(u^n_{i-1,j}) - \\ -(-\frac{(a-b)^2}{2a} f(u^n_{i-1,j}) + \frac{(a+b)^2}{2a} f(u^n_{i-1,j-1}))]$$

Then we see that scheme (43)-(47) has conservation form. Indeed the scheme (43)-(47) is the scheme (42) with

$$H^n_{i+1/2,j} = H_1(u^n_{i,j}) = af(u^n_{i,j})$$

$$H^n_{i,j+1/2} = H_2(u^n_{i-1,j+1}, u^n_{i-1,j}) = -\frac{(a-b)^2}{4a} f(u^n_{i-1,j+1}) + \frac{(a+b)^2}{4a} f(u^n_{i-1,j}).$$

The numerical fluxes of the scheme, which are the functions H_1 and H_2, are C^1 functions by hypothesis (13). The scheme is consistent since

$$H_1(u) = af(u),$$

$$H_2(u,u) = bf(u).$$

The proof of proposition 1 is complete. We can now prove a convergence result.

THEOREM 3. *Let* $Q = (a,b)^t \in \mathbb{R}^2$, $g = 0$, $u_0 \; L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$, *and f,g satisfy* (12)-(14). *Let* $\alpha,\beta \in \mathbb{R}$ *such that* $\alpha \le u_0 \le \beta$ *a.e, and* λ *such that*

(51) $\lambda \,\mathrm{Max}\,(|a|, |b|)\ \mathrm{Sup}\ \{f'(s), \alpha \le s \le \beta\} \le 1.$

Then the scheme (43)-(47) is a consistent, conservation-form, monotone (on [α,β] scheme. Furthermore when $h \to 0$, and $k/h = \lambda$ (fixed) satisfy (51), $u_{h,k}$ converges toward the weak entropic solution of (40)-(41), in $L^1(\mathbb{R}^2)$ uniformly for bounded $t \geq 0$.

Proof of Theorem 3. By proposition 1 and classical results on monotone schemes (see Crandall & al. [5]) we have only to prove that the scheme is monotone on [α, β]. As in proposition 1 we let $a \geq b > 0$. (The other cases are similar.) Then we have to prove that $\mathcal{H}$ is a nondecreasing function of each arguement. $u,v,w,z \in [\alpha,\beta]$, with

$$\mathcal{H}\ (u,v,w,z) = u\ - \lambda(H_1(u) - H_1(u)) - \lambda(h_2(z,v) - H_2(v,w))$$

$$H_1(u) = af(u),\ H_2(u,v) = -\ \frac{(a-b)^2}{4a}f(u) + \frac{(a+b)^2}{4a}f(v)$$

[see the proof of proposition 1].

An easy computation gives

$$\mathcal{H}\ (u,v,w,z) = u - \lambda\ af(u) + \lambda\ \frac{a^2-b^2}{2a}f(v) + \lambda\ \frac{(a-b)^2}{4a}f(z) + 1\ \frac{(a+b)^2}{4a}f(w)$$

Then we see that $\mathcal{H}$ is nondecreasing with respect to v,w and z, since f is nondecreasing (we don't make use of (51)). The condition (51) gives that $\mathcal{H}$ is nondecreasing with respect to u, for $u \in [\alpha,\beta]$.

The proof of Theorem 3 is complete.

Remark 9. The scheme (43)-(47) is a first order scheme and it introduces some numerical diffusion. In fact, let $a \geq b \geq 0$, the space-discretization is second-order discretization of

$$\operatorname{div} Q f(u) - \frac{h}{2}\ (a\ (f(u))_{xx} + 2b\ (f(u))_{xy} + \frac{a^2-b^2}{2a}\ (f(u))_{yy}).$$

(Recall that $f' \geq 0$.) In the case of the classical upstream weighting scheme, one has a second-order discretization of

$$\operatorname{div} Q f(u) - \frac{h}{2}\ (a\ (f(u))_{xx} + b\ (f(u))_{yy}\ ).$$

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Eymard & al. [8]. I
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3.3. Gravity
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(52) $\int_{\partial K_{i,j}} g(u)\ \mathbf{g}\ \mathbf{n}$

To do this, we will
edges of $K_{i,j}$, and w
edge of the mesh (
and Remark 2). No
not satisfy $g' \geq$

Remark 10.
(5)-(6), the vector
(5)-(6) is discretize
Finite Element M
associated to the f
Let $E_{i,j,k}$, k
1,2,3,4, let $i_k, j_k \in$

ı $h \to 0$, and $k/h = \lambda$ (fixed)
ak entropic solution of (40)-
0.

ı 1 and classical results on
we have only to prove that
position 1 we let $a \geq b > 0$.
to prove that $\mathcal{H}$ is a nonde-
$z \in [\alpha,\beta]$, with

$\lambda(h_2(z,v) - H_2(v,w))$

$(u) + \frac{(a+b)^2}{4a} f(v)$

$\frac{(a-b)^2}{4a} f(z) + 1 \ \frac{(a+b)^2}{4a} f(w)$

spect to v,w and z, since f
The condition (51) gives
$u \in [\alpha,\beta]$.

t order scheme and it in-
let $a \geq b \geq 0$, the space-

$+ \frac{a^2-b^2}{2a} (f(u))_{yy})$.

cal upstream weighting
f

$u))_{yy}$).

We can notice that this (spacial) numerical diffusion is exactly $-(h/2a)(f(u)_{QQ}$ when $a = b$ for the scheme (43)-(47), and when $b = 0$ for the classical scheme, with $(.)_Q = a(.)_x + b(.)_y)$.

We can also remark that the stability condition (CFL-condition) is more restrictive for the classical upstream weigthing scheme. Indeed this condition is $\lambda\,(a+b)$ Sup $\{f'(s),\ \alpha \leq s \leq \beta\} \leq 1$ for the classical scheme, and is (51) for the scheme (43)-(47).

The scheme (43)-(47) is well adapted to the discretization of the "pressure equation" (equation (5)-(6)) made in Eymard & al. [8], this is not the case of the classical upstream weighting scheme (for details, see Eymard & al. [8]. It is also very easy to generalize the scheme (43)-(47) to other mesh, that is to mesh composed of quadrilaterals or even to mesh composed of triangles.

3.3. Gravity terms, first order scheme. We de not assume here that $\mathbf{g} = 0$. Then one has to discretize in (43) the term

$$(52) \quad \int_{\partial K_{i,j}} g(u)\ \mathbf{g}\ \mathbf{n}_{i,j}\ d\sigma$$

To do this, we will decompose (52) in four parts, associated to the four edges of $K_{i,j}$, and we will use on approximated Riemann solver on each edge of the mesh (this is natural generalization of case $N = 1$, see (17) and Remark 2). Note that, as for $N = 1$, the function g, generally, does not satisfy $g' \geq 0$.

Remark 10. Even when equation (7) is coupled with equations (5)-(6), the vector $\mathbf{g}$ is constant (this is not true for $\mathbf{Q}$). Then, even if (5)-(6) is discretized as it is done in Eymard & al. [8] (using an Hybrid Finite Element Method), it is possible to decompose (52) in four parts associated to the four edges of $K_{i,j}$.

Let $E_{i,j,k}$, k = 1,2,3,4, be the four edges of $K_{i,j}$, and, for k = 1,2,3,4, let $\mathrm{i}_k, \mathrm{j}_k \in \mathbb{Z}$ such that

$$K_{i,j} \cap K_{i_k j_k} = E_{i,j,k}$$

Then, for the discretization of (52), we take

$$(53)\qquad \int_{\delta K_{i,j}} g(u)\ \mathbf{g}.\mathbf{n}_{i,j}\ d\sigma = \sum_{k=1}^{4} \int_{E_{i,j,k}} g(u)\ \mathbf{g}.\mathbf{n}_{i,j}\ d\sigma$$

with

$$(54)\qquad \begin{cases} g(u) = \dfrac{f_1(u^n_{i,j})\, f_2(u^n_{i_k j_k})}{f_1(u^n_{i,j}) + f_2(u^n_{i_k j_k})} \text{ on } E_{i,j,k} \text{ if } \displaystyle\int_{E_{i,j,k}} g(u)\ \mathbf{g}.\mathbf{n}_{i,j}\ d\sigma \geq 0 \\[2ex] g(u) = \dfrac{f_1(u^n_{i_k j_k})\, f_2(u^n_{i,j})}{f_1(u^n_{i_k j_k}) + f_2(u^n_{i,j})} \text{ on } E_{i,j,k} \text{ if } \displaystyle\int_{E_{i,j,k}} g(u)\ \mathbf{g}.\mathbf{n}_{i,j}\ d\sigma < 0 \end{cases}$$

(In (54) we set $g(u) = 0$ if the denominator is equal to zero. We recall that $f_1 \geq 0$ and $f_2 \geq 0$.) We can now proceed as in the previous section.

PROPOSITION 2. The scheme (43)-(47), (53)-(54) is a consistent, conservation-form approximation of (40). Furthermore the numerical fluxes of this scheme are locally Lipschitz continuous.

The proof of proposition 2 is very easy, we do not give it. We just remark that the locally Lipschitz continuity of the numerical fluxes is done as in the proof of Theorem 1.

THEOREM 4. *Let* $\mathbf{Q} = (a,b)^t \in \mathbb{R}^2$, $\mathbf{g} = (c,d)^t \in \mathbb{R}^2$, $u_0 \quad L^1(\mathbb{R}^2) \cap$

$L^\infty(\mathbb{R}^2)$, *and f,g sati*

a.e., and λ *such that*

(55) λ Sup {Max

Then the scheme (4
form, monotone (on
= λ (fixed) satisfy (5
tion, u, of (40)-(41), i

lim sup $h \to 0$ $0 \leq t \leq T$

Proof of Theor
to prove that the sch
$d \geq 0$ (the other case

(56) $u^{n+1}_{i,j}$ = $(u^n_{i,j},$

with

(57) $\mathcal{H}\ (u,v,w$

Then we are that $\mathcal{H}$

$L^\infty(\mathbb{R}^2)$, and f,g satisfy (12)-(14). *Let α,β such that $\alpha \le u_0 \le \beta$ a.e., and λ such that*

$$(55) \qquad \lambda \,\mathrm{Sup}\,\{\mathrm{Max}\,(|a|, |b|)\, f'(s) + (|c| + |d|)(f_1'(s)),\ s \quad [\alpha,\beta]\} \le 1$$

Then the scheme (43)-(47), (53)-(54) is a consistent, conservation-form, monotone (on $[\alpha, \beta]$) scheme. Furthermore, when $h \to 0$, and $k/h = \lambda$ (fixed) satisfy (55), $u_{h,k}$ converges toward the weak entropic solution, u, of (40)-(41), in $L^1(\mathbb{R}^2)$ uniformly for bounded $t \ge 0$. That is

$$\limsup_{h \to 0\ 0 \le t \le T} \int_{\mathbb{R}^2} |u_{h,k}(x,t) - u(x,t)|\, dx = 0 \text{ for each } T > 0$$

Proof of Theorem 4. As in the proof of Theorem 3, we have only to prove that the scheme is monotone on $[\alpha,\beta]$. We let $a \ge b > 0$, $c \ge 0$, $d \ge 0$ (the other case are similar). Then the scheme can be written

$$(56) \quad u_{i,j}^{n+1} = \quad (u_{i,j}^n,\ u_{i-1,j}^n,\ u_{i-1,j-1}^n,\ u_{i-1,j+1}^n,\ u_{i,j-1}^n,\ u_{i,j+1}^n,\ u_{i+1,j}^n)$$

with

$$(57) \quad \begin{cases} \mathcal{H}\,(u,v,w,z,r,s,t) = u - \lambda\, a f(u) + \lambda\, \dfrac{a^2-b^2}{2a} f(v) \\ \qquad + \lambda\, \dfrac{(a-b)^2}{4a} f(z) + \lambda\, \dfrac{(a+b)^2}{4a} f(w) \\ \qquad - \lambda c \left(\dfrac{f_1(u)\, f_2(t)}{f_1(u) + f_2(t)} - \dfrac{f_1(v)\, f_2(u)}{f_1(v) + f_2(u)} \right) \\ \qquad - \lambda d \left(\dfrac{f_1(u)\, f_2(s)}{f_1(u) + f_2(s)} - \dfrac{f_1(r)\, f_2(u)}{f_1(r) + f_2(u)} \right) \end{cases}$$

Then we are that $\mathcal{H}$ is nondecreasing with respect to $v,wz,r,s,$ and t

(here we only use that f, f_1 and $(-f_2)$ are nondecreasing). For proving the monotonicity of   with respect to u, let $u_1 \geq u$ (and u, u_1 $[\alpha,\beta]$), one has

$$\mathcal{H}(u_1,v,w,z,r,s,t) - \mathcal{H}(u,v,w,z,r,s,t) = (u_1 - u) - \lambda a\,(f(u_1) - f(u))$$

$$- \lambda c \left(\frac{f_2^2(u)\,(f_1(u_1) - f_1(u))}{f_1(u) + f_2(t))(f_1(u_1) + f_2(s))} - \frac{f_1^2(v)(\,f_2(u_1) - f_2(u))}{f_1(v) + f_2(u))\,(f_1(v) + f_2(u_1))} \right)$$

$$- \lambda c \left(\frac{f_2^2(s)\,(f_1(u_1) - f_1(u))}{f_1(u) + f_2(ts)(f_1(u_1) + f_2(s))} - \frac{f_1^2(r)(\,f_2(u_1) - f_2(u))}{f_1(r) + f_2(u))\,(f_1(r) + f_2(u_1))} \right)$$

$$\geq (u_1) - u) - \lambda a(f(u_1) - f(u)) - \lambda(c+d)((f_1(u_1) - f_1(u)) - (f_2(u_1) - f(u)))$$

Then, with (55), we conclude

$$\mathcal{H}(u_1,v,w,z,r,s,t) - \mathcal{H}(u,v,w,z,r,s,t) \geq 0$$

The proof of Theorem 4 is complete.

3.4. Second order scheme. The scheme described in sections 3.2-3.3 has a wide numerical diffusion. In order to reduce it we can proceed as in the case $N = 1$ (see section 2.3) for construct less diffusive schemes. We described this procedure in the case $g = 0$ (The discretization of the gravity terms can be made with a similar method.)

With u^n, we first compute an affine approximation $f(u)$ in each element K, at instant t_n. Indeed we set, (for instance), for $i,j \in \mathbb{Z}$,

(58)
$$P^n_{i,j} = \frac{f(u^n_{i+1,j}) - f(u^n_{i-1,j})}{2h}$$

$$Q^n_{i,j} = \frac{f(u^n_{i+1,j}) - f(u^n_{i-1,j})}{2h}$$

Then, with the notatio

(59) $f_{i,j,k} = f(u^n_{i,j}) \pm$

In fact $f_{i,j,k}$ is an appr

In (59) the choice of ε

(60) $\varepsilon^n_{i,j} \in [0,1]$ is
Min $\{f(u^n_{i_k j_k}$
where $\{K_{i_k j_k}$
M_k is a verte

Then the scheme is (

(61) $f(u) = fu_i$
$f(u) = f_M$

and for choice of f_k

: nondecreasing). For proving
u, let $u_1 \geq u$ (and u, u_1 [α,β]),

$- u) - \lambda a\ (f(u_1) - f(u))$

$\frac{{}^2_1(v)(f_2(u_1) - f_2(u))}{+ f_2(u))\ (f_1(v) + f_2(u_1))}\Bigg)$

$\frac{{}^2_1(r)(f_2(u_1) - f_2(u))}{+ f_2(u))\ (f_1(r) + f_2(u_1))}\Bigg)$

$) - f_1(u)) - (f_2(u_1) - f(u)))$

ɜme described in sections
order to reduce it we can
3) for construct less diffu-
in the case g = 0 (The dis-
ith a similar method.)

oximation $f(u)$ in each ele-
nce), for $i,j \in Z$,

Then, with the notations of 3.2, we set

$$(59) \qquad f_{i,j,k} = f(u^n_{i,j}) \pm \frac{h}{2}\, \varepsilon^n_{i,j}\, P^n_{i,j} \pm \frac{h}{2}\, \varepsilon^n_{i,j}\, Q^n_{i,j},$$
$$\text{when } k = (i \pm \tfrac{1}{2}, j \pm \tfrac{1}{2}) \quad I_{i,j}$$

In fact $f_{i,j,k}$ is an approximation of $f(u)$ in M_k. (M_k is a vertex of $K_{i,j}$).

In (59) the choice of $\varepsilon^n_{i,j}$ (with does not depend on k) is made such that

$$(60) \quad \begin{cases} \varepsilon^n_{i,j} \in [0,1] \text{ is as great as possible in order to have for all } k \quad I_{i,j} \\ \text{Min } \{f(u^n_{i_k j_k}),\ (i_k j_k) \in J_k\} \leq f_{i,j,k} \leq \text{Max } \{f(u^n_{i_k j_k}),\ (i_k j_k) \quad j_k \\ \text{where } \{K_{i_k j_k},\ (i_k j_k) \in J_k\} \text{ are the four elements which} \\ M_k \text{ is a vertex} \end{cases}$$

Then the scheme is (43)-(44) with, instead of (45),

$$(61) \quad \begin{cases} f(u) = fu_{i,j}) \text{ on } \partial K_{i,j}(M_k) \text{ if } \displaystyle\int_{\partial K_{i,j(Mk)}} Q\ \mathrm{n}_{i,j}\, d\sigma \geq 0 \\ f(u) = f_{M_k} \text{ on } \partial K_{i,j}(M_k) \text{ if } \displaystyle\int_{\partial K_{i,j(Mk)}} Q\ \mathrm{n}_{i,j}\, d\sigma < 0 \end{cases}$$

and for choice of f_{km} we replace (47) by

$$
(62)\begin{cases} f_{M_{i+1/2,j+1/2}}\Big(- \displaystyle\sum_{(\alpha,\beta)\;J_{i,j}} \int_{\partial K_{i+\alpha,j+\beta}(M_{i+1/2,j+1/2})} \mathbf{Q}.\mathbf{n}_{i+\alpha,j+\beta}\, d\sigma\Big) = \\[2ex] \displaystyle\sum_{(\alpha,\beta)\;j^{+}_{i,j}} f(u^{n}_{i+\alpha,j+\beta}) \int_{\partial K_{i+\alpha,j+\beta}(M_{i+1/2,j+1/2})} \mathbf{Q}.\mathbf{n}_{i+\alpha,j+\beta}\, d\sigma \end{cases}
$$

The scheme (43)-(44), (58)-(61), (46), (62) is a consistent, conservation-form scheme. We omit the proof of these results. It is not a monotone scheme. We can prove (under a stability condition, similar to (27) in Theorem 2) a result of L^{∞} stability but not of BV stability. Therefore we do not have a convergence result. (In order to obtain a convergence result we need L^{∞} stability and BV stability, in particular a monotone scheme is L^{∞} stable and BV- stable.) In fact we suspect that (in two dimensions) schemes of this kind are never BV-stables (see Goodman & al. [9] in this direction). However the scheme (43)-(44), (58)-(61), (46), (62) gives, numerically, good results (as the scheme described in section 2.3, for the case $N = 1$). Results can be seen in Eymard & al. [8].

The schemes described can also be easily generalized to other mesh, that is to mesh composed of quadrilaterals or to mesh composed of triangles.

Acknowledgement. We thank for many discusions on this subject.

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