Convergence of the marker-and-cell scheme for the semi-stationary compressible Stokes problem

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Abstract

We prove in this paper the convergence of the marker-and-cell (MAC) scheme for the discretization of the semi-stationary compressible Navier-Stokes equations on two or three dimensional Cartesian grids. Existence of a solution to the scheme is stated, followed by estimates on approximate solutions, which yields the convergence of the approximate solutions, up to a subsequence, and in an appropriate sense. We then prove that the limit of the approximate solutions satisfy the mass balance and mass momentum equations, as well as the equation of state, which is the main difficulty of this study.

Keywords: Compressible fluids, Navier-Stokes equations, Cartesian grids, Marker-and-cell scheme, Convergence.

2000 MSC: 35Q30, 65N12, 76N10, 76N15, 76M12, 76M20

1. Introduction

The aim of this paper is to prove the convergence of the marker-and-cell (MAC) scheme for the discretization of the semi-stationary (barotropic) compressible Stokes system, introduced in [41]. These equations are posed on the time-space domain $Q_T = (0, T) \times \Omega$, where Ω is a bounded domain of $\mathbb{R}^d$ adapted to the MAC scheme (see section 3),

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$d = 2, 3$ is the space dimension and $(0, T) > 0$ is the time interval, and read:

$$\partial_t \varrho + \operatorname{div}(\varrho \boldsymbol{u}) = 0, \tag{1.1a}$$

$$-\mu \Delta \boldsymbol{u} - (\mu + \lambda) \nabla \operatorname{div} \boldsymbol{u} + \nabla p = \boldsymbol{0}, \tag{1.1b}$$

$$p = \varrho^{\gamma}, \ \varrho \geq 0, \tag{1.1c}$$

supplemented with the initial condition

$$\varrho(0, \boldsymbol{x}) = \varrho_0(\boldsymbol{x}), \tag{1.2}$$

where ϱ_0 is a given function from Ω to $\mathbb{R}_+^*$, and the homogeneous Dirichlet boundary condition

$$\boldsymbol{u}_{|(0,T)\times\partial\Omega} = 0. \tag{1.3}$$

In the above equations, the unknown functions are the pressure $p(t, \boldsymbol{x})$, the scalar density field $\varrho(t, \boldsymbol{x}) \geq 0$ and vector velocity field $\boldsymbol{u} = (u_1, \ldots, u_d)(t, \boldsymbol{x})$, where $t \in (0, T)$ denotes the time variable and $\boldsymbol{x} \in \Omega$ the space variable. The viscosity coefficients μ and λ are such that

$$\mu > 0, \qquad \lambda + \mu \geq 0. \tag{1.4}$$

In the compressible barotropic Navier-Stokes equations, the pressure and the density are linked through a constitutive law. Here we assume that the fluid is a perfect gas obeying Boyle's law, given by (1.1c). Typical values of γ range from a maximum value of $\frac{5}{3}$ for *monoatomic gases* or $\frac{7}{5}$ for *diatomic gases* including air, to a minimum value close to 1 for *polyatomic gases* at high temperature. For the sake of simplicity the constant a will be taken equal to 1. The convergence result that we prove in this paper is valid for $\gamma \geq \frac{3}{2}$ (if $d = 3$).

There are several motivations for the study of the model (1.1). First of all, the solutions of this system of equations is used to build solutions of the compressible Navier-Stokes equations which exhibit persistent oscillations (see [41]). Next, this model is derived for the dynamics of vortices in the Ginzburg-Landau theories in superconductivity. Finally, it is the simplest system derived from a model existing for biological flows in a compressible tissue (see [4] and [9]) or in compressible porous media in petroleum engineering (see [19]). We refer to [45, 3] and [41, 13, 42] for further development and tools for the analysis of the incompressible and compressible Navier-Stokes equations.

The mathematical analysis of numerical schemes for the discretization of the steady and/or non steady compressible Navier-Stokes and/or compressible Stokes equations has been the object of some recent work. The convergence of the discrete solutions to the weak solutions of the compressible stationary Stokes was shown for a finite volume– non conforming P1 finite element [23, 12, 18], for the wellknown MAC scheme [11] and for a more complicated finite volume - Nedelec finite element scheme [39] (for a Navier slip boundary condition in this latter work); The finite volume– non conforming P1 finite element was adapted in [40] to deal with the unsteady barotropic Navier-Stokes equations albeit only in the case $\gamma > 3$ (there is a real difficulty in the realistic case $\gamma \leq 3$ arising from the treatment of the non linear convection term). Let us also mention the works [14] and [15] where the study of the mixed scheme is extended to regular domains and to the Navier-Stokes-Fourier equations.

Problem (1.1) with a Navier boundary condition was discretized in [37] with a Nedelec finite element scheme, discretizing the weak form of the momentum equation with the bilinear form $\int \mathrm{rot}\mathbf{u} \cdot \mathrm{rot}\mathbf{v} + \int \mathrm{div}\mathbf{u}\, \mathrm{div}\mathbf{v}$ instead of $\int \nabla\mathbf{u} : \nabla\mathbf{v}$, the authors of Ref. 30 discretize the term $\int \mathrm{rot}\mathbf{u} \cdot \mathrm{rot}\mathbf{v} + \int \mathrm{div}\mathbf{u}\, \mathrm{div}\mathbf{v}$. Because of this, the resulting scheme is not coercive, and a stabilizing term must be introduced. In the present work, we deal with the same set of equations, but discretized with the MAC scheme and with Dirichlet boundary conditions. The resulting scheme is naturally coercive and is easier to generalize to other boundary conditions.

Since the very beginning of the introduction of the marker-and-cell (MAC) scheme [30], it is claimed that this discretization is suitable for both the incompressible and compressible flow problems (see [28, 29] for the seminal papers, [5, 34, 35, 36, 1, 8, 31, 32, 46, 47, 48] for subsequent developments and [49] for a review). The proof of convergence for the MAC scheme in primitive variables has been recently been completed [25]. Here we give a convergence proof of the MAC scheme for the system (1.1). To our knowledge the proof of convergence of the MAC scheme for this system is new; it is also a preliminary work to a proof of convergence of the MAC scheme for the compressible barotropic Navier-Stokes equations. It relies on some recents time compactness results obtained in [7] and [27], in which the famous Aubin-Simon theorem is generalized to piewewise constant functions.

The paper is organized as follows. The fundamental setting of the problem in the continuous case is recalled in Section 2, and its discretization in Section 3: the discrete functional spaces and the numerical scheme are defined; the convergence of the scheme, which is the main result of the paper, is stated in Theorem 1. The remaining sections are devoted to the proof of Theorem 1. In Section 4 we derive some a priori estimates from the scheme along with the existence of a discrete solution. In Section 5, we prove the convergence of the numerical scheme in the sense of Theorem 1 toward a weak solution of Problem 1.1. Finally, In Section 8, we list some functional analysis results used to prove Theorem 1.

2. The continuous problem

A weak solution of Problem (1.1)–(1.4) is defined as follows.

Definition 1 (Weak solutions)**.** *Let $\varrho_0 : \Omega \to \mathbb{R}_+^* \in \mathrm{L}^\gamma(\Omega)$. Let $\gamma > 1$. We shall say that the triplet $(\boldsymbol{u}, p, \varrho)$ is a weak solution to the problem (1.1)–(1.4) emanating from the initial data ϱ_0 if:*

1. $\varrho \in \mathrm{L}^\infty(0, T; \mathrm{L}^\gamma(\Omega)) \cap \mathrm{L}^{2\gamma}((0, T) \times \Omega)$, $\varrho \geq 0$ *a.e in* $(0, T) \times \Omega$, $p \in \mathrm{L}^2((0, T) \times \Omega)$ *and* $\boldsymbol{u} \in \mathrm{L}^2(0, T; \mathrm{H}_0^1(\Omega)^d)$.
2. *The continuity equation (1.1a) is satisfied in the following weak sense,*
$$\int_0^T \int_\Omega \varrho\partial_t\varphi + \varrho\boldsymbol{u} \cdot \nabla\varphi \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t = -\int_\Omega \varrho_0\varphi(0, \cdot)\,\mathrm{d}\boldsymbol{x},\ \forall\varphi \in C^\infty([0, T] \times \overline{\Omega}),\ \varphi(T, \cdot) = 0. \tag{2.1}$$
3. *The momentum equation (1.1b) is satisfied in the weak sense, that is for any $\boldsymbol{\psi} \in C_c^\infty((0, T) \times \Omega)^d$,*
$$\int_0^T \int_\Omega \mu\nabla\boldsymbol{u} : \nabla\boldsymbol{\psi} + (\mu + \lambda)\mathrm{div}\boldsymbol{u}\ \mathrm{div}\boldsymbol{\psi} - p\mathrm{div}\boldsymbol{\psi}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t = 0, \tag{2.2}$$

4. $p = \varrho^\gamma$ *a.e in* $(0, T) \times \Omega$,

5. The following energy inequality is satisfied a.e in $(0, T)$,

$$\int_\Omega \frac{1}{\gamma - 1} \varrho^\gamma(\tau)\,\mathrm{d}\boldsymbol{x} + \int_0^\tau \int_\Omega \Big(\mu |\nabla \boldsymbol{u}|^2 + (\mu + \lambda) |\operatorname{div} \boldsymbol{u}|^2\Big)\,\mathrm{d}\boldsymbol{x}\,\mathrm{dt} \le \mathcal{E}_0, \tag{2.3}$$

where $\mathcal{E}_0 = \int_\Omega \frac{1}{\gamma-1} \varrho_0^\gamma\,\mathrm{d}\boldsymbol{x}$ *stands for the initial energy.*

Remark 1. 1. Note that the existence of weak solutions emanating from the finite energy initial data is wellknown on bounded Lipschitz domains provided $\gamma > 1$, see [41, Theorem 8.6].

2. The natural functional spaces for the unknows which are introduced in the above definition are obtained when looking for some a priori estimates (see for instance [41]).

3. The numerical scheme

3.1. Space discretization

Let Ω be a connected subset of $\mathbb{R}^d$ consisting in a union of rectangles ($d = 2$) or orthogonal parallelepipeds ($d = 3$); without loss of generality, the edges (or faces) of these rectangles (or parallelepipeds) are assumed to be orthogonal to the canonical basis vectors, denoted by $(\boldsymbol{e}_1, \ldots, \boldsymbol{e}_d)$.

Definition 2 (MAC grid)**.** *A discretization of* Ω *with a MAC grid, denoted by* $\mathcal{D}$*, is defined by* $\mathcal{D} = (\mathcal{M}, \mathcal{E})$*, where:*

- $\mathcal{M}$ *stands for the primal grid, and consists in a conforming structured partition of* Ω *in possibly non uniform rectangles* ($d = 2$) *or rectangular parallelepipeds* ($d = 3$)*. A generic cell of this grid is denoted by* K*, and its mass center by* $\boldsymbol{x}_K$*. The pressure is associated to this mesh, and* $\mathcal{M}$ *is also sometimes referred to as "the pressure mesh".*
- *The set of all faces of the mesh is denoted by* $\mathcal{E}$*; we have* $\mathcal{E} = \mathcal{E}_{\rm int} \cup \mathcal{E}_{\rm ext}$*, where* $\mathcal{E}_{\rm int}$ *(resp.* $\mathcal{E}_{\rm ext}$*) are the edges of* $\mathcal{E}$ *that lie in the interior (resp. on the boundary) of the domain. The set of faces that are orthogonal to* $\boldsymbol{e}_i$ *is denoted by* $\mathcal{E}^{(i)}$*, for* $i = 1, \ldots, d$*. We then have* $\mathcal{E}^{(i)} = \mathcal{E}^{(i)}_{\rm int} \cup \mathcal{E}^{(i)}_{\rm ext}$*, where* $\mathcal{E}^{(i)}_{\rm int}$ *(resp.* $\mathcal{E}^{(i)}_{\rm ext}$*) are the edges of* $\mathcal{E}^{(i)}$ *that lie in the interior (resp. on the boundary) of the domain.*

  For $\sigma \in \mathcal{E}_{\rm int}$*, we write* $\sigma = K|L$ *if* $\sigma = \partial K \cap \partial L$*. A dual cell* D_σ *associated to a face* $\sigma \in \mathcal{E}$ *is defined as follows:*

  - *if* $\sigma = K|L \in \mathcal{E}_{\rm int}$ *then* $D_\sigma = D_{K,\sigma} \cup D_{L,\sigma}$*, where* $D_{K,\sigma}$ *(resp.* $D_{L,\sigma}$*) is the half-part of* K *(resp.* L*) adjacent to* σ *(see Fig. 1 for the two-dimensional case);*
  - *if* $\sigma \in \mathcal{E}_{\rm ext}$ *is adjacent to the cell* K*, then* $D_\sigma = D_{K,\sigma}$*.*

  We obtain d *partitions of the computational domain* Ω *as follows:*

$$\Omega = \cup_{\sigma \in \mathcal{E}^{(i)}} D_\sigma, \quad i \in [\![1,\ d]\!],$$

and the i^{th} of these partitions is called i^{th} dual mesh, and is associated to the i^{th} velocity component, in a sense which is precised below. The set of the faces of the i^{th} dual mesh is denoted by $\widetilde{\mathcal{E}}^{(i)}$ (note that these faces may be orthogonal to any vector of the basis of $\mathbb{R}^d$ and not only $\boldsymbol{e}_i$) and is decomposed into the internal and boundary edges: $\widetilde{\mathcal{E}}^{(i)} = \widetilde{\mathcal{E}}^{(i)}_{\mathrm{int}} \cup \widetilde{\mathcal{E}}^{(i)}_{\mathrm{ext}}$. The dual face separating two duals cells D_σ and $D_{\sigma'}$ is denoted by $\epsilon = \sigma|\sigma'$.

The set of faces of a primal cell K and a dual cell D_σ are denoted by $\mathcal{E}(K)$ and $\widetilde{\mathcal{E}}(D_\sigma)$ respectively. For $\sigma \in \mathcal{E}$, we denote by $\boldsymbol{x}_\sigma$ the mass center of σ. The vector $\boldsymbol{n}_{K,\sigma}$ stands for the unit normal vector to σ outward K. In some cases, we need to specify the orientation of a geometrical quantity with respect to the axis:

- a primal cell K is denoted $K = [\overrightarrow{\sigma\sigma'}]$ if $\sigma, \sigma' \in \mathcal{E}^{(i)} \cap \mathcal{E}(K)$ for some $i \in [\![1,\ d]\!]$ are such that $(\boldsymbol{x}_{\sigma'} - \boldsymbol{x}_\sigma) \cdot \boldsymbol{e}_i > 0$;
- we write $\sigma = \overrightarrow{K|L}$ if $\sigma \in \mathcal{E}^{(i)}$ and $\overrightarrow{\boldsymbol{x}_K \boldsymbol{x}_L} \cdot \boldsymbol{e}_i > 0$ for some $i \in [\![1,\ d]\!]$;
- the dual face ϵ separating D_σ and $D_{\sigma'}$ is written $\epsilon = \overrightarrow{\sigma|\sigma'}$ if $\overrightarrow{\boldsymbol{x}_\sigma \boldsymbol{x}_{\sigma'}} \cdot \boldsymbol{e}_i > 0$ for some $i \in [\![1,\ d]\!]$.

The definition of the discrete momentum diffusion operator involves a distance d_ϵ associated to a face ϵ as sketched on Figure 1.

$$d_\epsilon = \begin{cases} d(\boldsymbol{x}_\sigma, \boldsymbol{x}_{\sigma'}) & \text{if } \epsilon = \sigma|\sigma' \in \widetilde{\mathcal{E}}^{(i)}_{\mathrm{int}}, \\ d(\boldsymbol{x}_\sigma, \epsilon) & \text{if } \epsilon \in \widetilde{\mathcal{E}}^{(i)}_{\mathrm{ext}} \cap \tilde{\mathcal{E}}(D_\sigma), \end{cases} \quad \text{, for} \qquad \epsilon \in \widetilde{\mathcal{E}}(D_\sigma), \sigma \in \mathcal{E}^{(i)}, i \in [\![1,\ d]\!]. \tag{3.1}$$

where $d(\cdot,\cdot)$ denotes the Euclidean distance in $\mathbb{R}^d$.

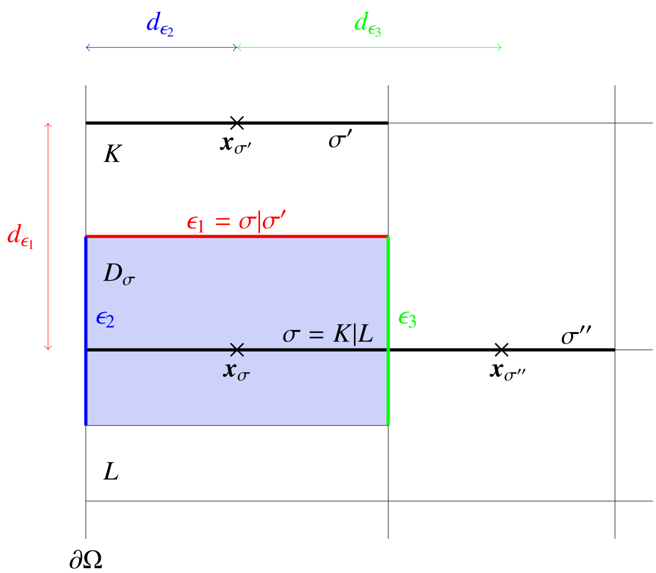

Figure 1: Notations for control volumes and dual cells (for the second component of the velocity).

The size $h_{\mathcal{M}}$ and the regularity $\eta_{\mathcal{M}}$ of the mesh $\mathcal{M}$ are defined by:

$$h_{\mathcal{M}} = \max\{\mathrm{diam}(K), K \in \mathcal{M}\}, \tag{3.2}$$

$$\eta_{\mathcal{M}} = \min\Big\{\frac{|\sigma|}{|\sigma'|},\ \sigma \in \mathcal{E}^{(i)},\ \sigma' \in \mathcal{E}^{(j)},\ i, j \in [\![1, d]\!],\ i \neq j\Big\}, \tag{3.3}$$

where $|\cdot|$ stands for the $(d-1)$-dimensional measure of a subset of $\mathbb{R}^{d-1}$ (in the sequel, it is also used to denote the d-dimensional measure of a subset of $\mathbb{R}^d$).

The spatially discrete velocity unknowns are associated to the velocity cells and are denoted by $(u_\sigma)_{\sigma\in\mathcal{E}^{(i)}}$, $i \in [\![1,\ d]\!]$, while the discrete pressure unknowns are associated to the primal cells and are denoted by $(p_K)_{K\in\mathcal{M}}$. The spatially discrete pressure space $L_{\mathcal{M}}$ is defined as the set of piecewise constant functions over each of the grid cells K of $\mathcal{M}$, and the discrete i^{th} velocity space $H_{\mathcal{E}}^{(i)}$ as the set of piecewise constant functions over each of the grid cells D_σ, $\sigma \in \mathcal{E}^{(i)}$. As in the continuous case, the Dirichlet boundary conditions are (partly) incorporated into the definition of the velocity spaces, by means of the spaces $H_{\mathcal{E},0}^{(i)} \subset H_{\mathcal{E}}^{(i)}$, $i \in [\![1,\ d]\!]$, defined as follows:

$$H_{\mathcal{E},0}^{(i)} = \Big\{u \in H_{\mathcal{E}}^{(i)},\ u(\boldsymbol{x}) = 0\ \forall \boldsymbol{x} \in D_\sigma,\ \sigma \in \widetilde{\mathcal{E}}_{\mathrm{ext}}^{(i)}\Big\}.$$

We then set $\mathbf{H}_{\mathcal{E},0} = \prod_{i=1}^d H_{\mathcal{E},0}^{(i)}$. Note that if $\boldsymbol{u} \in \mathbf{H}_{\mathcal{E},0}$ then $u_\sigma = 0$ for any $\sigma \in \widetilde{\mathcal{E}}_{\mathrm{ext}}^{(i)}$ for $i \in [\![1,\ d]\!]$. Defining the characteristic function $1\!\!1_A$ of any subset $A \subset \Omega$ by $1\!\!1_A(\boldsymbol{x}) = 1$ if $\boldsymbol{x} \in A$ and $1\!\!1_A(\boldsymbol{x}) = 0$ otherwise, the d components of a function $\boldsymbol{u} \in \mathbf{H}_{\mathcal{E},0}$ and a function $p \in L_{\mathcal{M}}$ may then be written:

$$u_i = \sum_{\sigma\in\mathcal{E}^{(i)}} u_\sigma 1\!\!1_{D_\sigma},\ i \in [\![1,\ d]\!] \quad \text{and} \quad p = \sum_{K\in\mathcal{M}} p_K 1\!\!1_K.$$

3.2. *Time discretization*

Consider a partition $0 = t^0 < t^1 < \cdots < t^N = T$ of the time interval $(0, T)$, with constant time step $\delta t = t^m - t^{m-1}$; hence $t^m = m\delta t$ for $m \in \{0, \cdots, N\}$. Let $\mathcal{D} = (\mathcal{M}, \mathcal{E})$ of Ω be a MAC grid in the sense of Definition 2, then the sets $\{u_\sigma^m, \sigma \in \mathcal{E}_{\mathrm{int}}^{(i)}, i \in \{1, \cdots, d\}, m \in \{0, \cdots, N\}\}$, $\{p_K^m, K \in \mathcal{M}, m \in \{1, \cdots, N\}\}$ and $\{\varrho_K^m, K \in \mathcal{M}, m \in \{1, \cdots, N\}\}$ are respectively the sets of discrete velocity, pressure and density unknowns; we define the corresponding piecewise constant functions $\boldsymbol{u} = (u_1, \ldots, u_d)$, p and ϱ. The approximate velocity $\boldsymbol{u}$ is thus of the form:

$$u_i(t, \boldsymbol{x}) = \sum_{m=1}^{N} \sum_{\sigma\in\mathcal{E}_{\mathrm{int}}^{(i)}} u_\sigma^m\, 1\!\!1_{D_\sigma}(\boldsymbol{x})\, 1\!\!1_{(t^{m-1},t^m)}(t),$$

where $1\!\!1_{(t^{m-1},t^m)}$ is the characteristic function of the interval (t^{m-1}, t^m). We denote by $X_{i,\mathcal{E},\delta t}$ the set of such piecewise constant functions on time intervals and dual cells, and we set $\boldsymbol{X}_{\mathcal{E},\delta t} = \prod_{i=1}^d X_{i,\mathcal{E},\delta t}$. The approximate pressure and the density functions are thus respectively of the form:

$$p(t, \boldsymbol{x}) = \sum_{m=1}^{N} \sum_{K\in\mathcal{M}} p_K^m 1\!\!1_K(\boldsymbol{x})\, 1\!\!1_{(t^{m-1},t^m)}(t), \text{ and } \varrho(t, \boldsymbol{x}) = \sum_{m=1}^{N} \sum_{K\in\mathcal{M}} \varrho_K^m 1\!\!1_K(\boldsymbol{x})\, 1\!\!1_{(t^{m-1},t^m)}(t).$$

and we denote by $Y_{\mathcal{M},\delta t}$ the space of such piecewise constant functions.

The initial approximation for ϱ is the average of the initial condition $\varrho_0 \in L^\gamma(\Omega, \mathbb{R}_+^*)$ on the primal cells:

$$\varrho^0(\boldsymbol{x}) = \mathcal{P}_{\mathcal{M}}\varrho_0 = \sum_{K\in\mathcal{M}} \varrho_K^0 \, 1\!\!1_K(\boldsymbol{x}), \text{ with } \forall K \in \mathcal{M}, \quad \varrho_K^0 = \frac{1}{|K|}\int_K \varrho_0(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}. \tag{3.4}$$

In particular, by virtue of the convexity of $t \to t^\gamma$, the discrete initial energy $\mathcal{E}_{0,\mathcal{M}} = \frac{1}{\gamma-1}\int_\Omega (\varrho^0)^\gamma \,\mathrm{d}\boldsymbol{x}$ is bounded from above by $\mathcal{E}_0$.

We then define the discrete time derivative $\eth_t\varrho \in Y_{\mathcal{M},\delta t}$ by $\eth_t\varrho = \displaystyle\sum_{m=1}^{N} \frac{1}{\delta t}(\varrho^m - \varrho^{m-1})\, 1\!\!1_{(t^{m-1},t^m)}(t)$.

3.3. *The fully discrete scheme*

The implicit-in-time scheme reads in its fully discrete form, for $1 \le m \le N$:

$$\frac{1}{\delta t}(\varrho^m - \varrho^{m-1}) + \mathrm{div}_{\mathcal{M}}^{\mathrm{up}}(\varrho^m \boldsymbol{u}^m) = 0, \tag{3.5a}$$

$$-\mu\Delta_{\mathcal{E}}\boldsymbol{u}^m - (\mu+\lambda)\nabla_{\mathcal{E}}\,\mathrm{div}_{\mathcal{M}}\,\boldsymbol{u}^m + \nabla_{\mathcal{E}}p^m = 0, \tag{3.5b}$$

$$p^m = p(\varrho^m) = (\varrho^m)^\gamma. \tag{3.5c}$$

where the discrete operators introduced for each discrete equation are defined hereafter.

3.3.1. *Mass balance equation*

Equation (3.5a) is a finite volume discretization of the mass balance (1.1*a*) over the primal mesh. Given $(\varrho, \boldsymbol{u}) \in L_{\mathcal{M}} \times \mathbf{H}_{\mathcal{E},0}$, the discrete function $\mathrm{div}_{\mathcal{M}}^{\mathrm{up}}(\varrho\boldsymbol{u}) \in L_{\mathcal{M}}$ is defined by

$$\mathrm{div}_{\mathcal{M}}^{\mathrm{up}}(\varrho\boldsymbol{u})(\boldsymbol{x}) = \frac{1}{|K|}\sum_{\sigma\in\mathcal{E}(K)} F_{K,\sigma}, \ \forall \boldsymbol{x} \in K, \ \forall K \in \mathcal{M},$$

where $F_{K,\sigma} = F_{K\sigma}(\varrho, \boldsymbol{u})$ stands for the mass flux across σ outward K, which, because of the Dirichlet boundary conditions, vanishes on external faces and is given on the internal faces by:

$$\forall \sigma = K|L \in \mathcal{E}_{\mathrm{int}}, \qquad F_{K,\sigma} = |\sigma|\,\varrho_\sigma^{\mathrm{up}}\, u_{K,\sigma}, \tag{3.6}$$

where $u_{K,\sigma}$ is an approximation of the normal velocity to the face σ outward K, defined by:

$$u_{K,\sigma} = u_\sigma\, \boldsymbol{e}_i \cdot \boldsymbol{n}_{K,\sigma} \text{ for } \sigma \in \mathcal{E}^{(i)} \cap \mathcal{E}(K). \tag{3.7}$$

Thanks to the boundary conditions, $u_{K,\sigma}$ vanishes for any external face σ. The density at the internal face $\sigma = K|L$ is obtained by an upwind technique:

$$\varrho_\sigma^{\mathrm{up}} = \left| \begin{array}{ll} \varrho_K & \text{if } u_{K,\sigma} \ge 0, \\ \varrho_L & \text{otherwise.} \end{array} \right. \tag{3.8}$$

Note that thanks to the upwind choice, any solution $(\varrho^m, \boldsymbol{u}^m) \in L_{\mathcal{M}} \times \mathbf{H}_{\mathcal{E},0}$ to (3.5a) satisfies $\varrho_K^m > 0$, $\forall K \in \mathcal{M}$ provided $\varrho_K^{m-1} > 0$, $\forall K \in \mathcal{M}$ and in particular $p(\varrho^m)$ makes sense. For fixed $\boldsymbol{u}$, the convergence of the upwind scheme (3.6) is shown in [6] in the case $u \in C^1$, and in [17, chapter 2] and [2] under the minimal regularity assumption $u \in L^1(0, T, W^{1,1}(\Omega)), \mathrm{div} u \in L^1(0, T; L^\infty(\Omega))$. This latter assumption, which is needed to obtain an estimate on ρ, is unfortunately not attainable in the framework of the compressible Stokes or Navier-Stokes equations: another technique, relying on the coupling between mass and momentum equation is used to obtained this estimate, see Lemma 2 below. Note that the upwind choice in the scheme ensures the positivity of the density ϱ^m in (3.5a) is not enforced in the scheme but results from the above upwind choice. Indeed, for a given velocity field, the discrete mass balance (3.5a) is a linear system for ϱ^m whose matrix is an invertible matrix with a non negative inverse [18, Lemma C.3].

Note also that, with this definition, we have the usual finite volume property of local conservativity of the flux through a primal face $\sigma = K|L$ (*i.e.* $F_{K,\sigma} = -F_{L,\sigma}$), and that the flux through a dual face included in the boundary still vanishes. Consequently, summing (3.5a) over $K \in \mathcal{M}$ immediately yields the total conservation of mass, which reads:

$$\forall m = 1, ...N, \quad \int_\Omega \varrho^m \,\mathrm{d}\boldsymbol{x} = \int_\Omega \varrho^0 \,\mathrm{d}\boldsymbol{x}. \tag{3.9}$$

3.3.2. The momentum equation

Discrete divergence and gradient - The discrete divergence operator $\mathrm{div}_{\mathcal{M}}$ is defined by:

$$\mathrm{div}_{\mathcal{M}} : \left| \begin{array}{l} \mathbf{H}_{\mathcal{E}} \longrightarrow L_{\mathcal{M}} \\ \boldsymbol{u} \longmapsto \mathrm{div}_{\mathcal{M}} \boldsymbol{u} = \displaystyle\sum_{K \in \mathcal{M}} \frac{1}{|K|} \sum_{\sigma \in \mathcal{E}(K)} |\sigma| u_{K,\sigma} \, \mathbb{1}_K, \end{array} \right. \tag{3.10}$$

where $u_{K,\sigma}$ is defined in (3.7). The discrete divergence of $\boldsymbol{u} = (u_1, \ldots, u_d) \in \mathbf{H}_{\mathcal{E},0}$ may also be written as $\mathrm{div}_{\mathcal{M}}(\boldsymbol{u}) = \displaystyle\sum_{i=1}^{d} \sum_{K \in \mathcal{M}} (\eth_i u_i)_K \mathbb{1}_K$, where the discrete derivative $(\eth_i u_i)_K$ of u_i on K is defined by

$$(\eth_i u_i)_K = \frac{|\sigma|}{|K|}(u_{\sigma'} - u_\sigma) \text{ with } K = [\overrightarrow{\sigma\sigma'}], \sigma, \sigma' \in \mathcal{E}^{(i)}. \tag{3.11}$$

The gradient in the discrete momentum balance equation is defined as follows:

$$\boldsymbol{\nabla}_{\mathcal{E}} : \left| \begin{array}{l} L_{\mathcal{M}} \longrightarrow \mathbf{H}_{\mathcal{E},0} \\ p \longmapsto \boldsymbol{\nabla}_{\mathcal{E}} p, \end{array} \right. \tag{3.12}$$

where for $\boldsymbol{x} \in \Omega$, $\nabla_{\mathcal{E}} p(\boldsymbol{x}) = (\eth_1 p(\boldsymbol{x}), \ldots, \eth_d p(\boldsymbol{x}))^t$, and $\eth_i p \in H_{\mathcal{E},0}^{(i)}$ is the discrete derivative of p in the i-th direction, defined by:

$$\eth_i p(\boldsymbol{x}) = \frac{|\sigma|}{|D_\sigma|} (p_L - p_K) \quad \forall \boldsymbol{x} \in D_\sigma, \text{ for } \sigma = \overrightarrow{K|L} \in \mathcal{E}_{\mathrm{int}}^{(i)}, \ i = 1, \ldots, d. \tag{3.13}$$

Note that in fact, the discrete gradient of a function of $L_{\mathcal{M}}$ should only defined on the internal faces, and does not need to be defined on the external faces; we set it here in $\mathbf{H}_{\mathcal{E},0}$ (that is zero on the external faces) for the sake of simplicity.

The gradient in the discrete momentum balance equation is built as the dual operator of the discrete divergence which means:

Lemma 1 (Discrete div $-\boldsymbol{\nabla}$ duality)**.** *Let* $q \in L_{\mathcal{M}}$ *and* $\boldsymbol{v} \in \mathbf{H}_{\mathcal{E},0}$ *then we have:*

$$\int_\Omega q \operatorname{div}_{\mathcal{M}} \boldsymbol{v} \, \mathrm{d}\boldsymbol{x} + \int_\Omega \nabla_{\mathcal{E}} q \cdot \boldsymbol{v} \, \mathrm{d}\boldsymbol{x} = 0. \tag{3.14}$$

Discrete Laplace operator - For $i = 1 \ldots, d$, we classically define the discrete Laplace operator on the i-th velocity grid by:

$$-\Delta_{\mathcal{E}}^{(i)} : \left| \begin{array}{l} H_{\mathcal{E},0}^{(i)} \longrightarrow H_{\mathcal{E},0}^{(i)} \\ u_i \longmapsto -\Delta_{\mathcal{E}}^{(i)} u_i \end{array} \right.$$

$$-\Delta_{\mathcal{E}}^{(i)} u_i(\boldsymbol{x}) = \frac{1}{|D_\sigma|} \sum_{\epsilon \in \tilde{\mathcal{E}}(D_\sigma)} \phi_{\sigma,\epsilon}(u_i), \qquad \forall \boldsymbol{x} \in D_\sigma, \ \text{ for } \sigma \in \mathcal{E}_{\mathrm{int}}^{(i)}, \tag{3.15}$$

where $\tilde{\mathcal{E}}(D_\sigma)$ denotes the set of faces of D_σ, and

$$\phi_{\sigma,\epsilon}(u_i) = \begin{cases} \dfrac{|\epsilon|}{d_\epsilon}(u_\sigma - u_{\sigma'}) & \text{if } \epsilon = \sigma|\sigma' \in \widetilde{\mathcal{E}}_{\mathrm{int}}^{(i)}, \\ \\ \dfrac{|\epsilon|}{d_\epsilon} u_\sigma & \text{if } \epsilon \in \widetilde{\mathcal{E}}_{\mathrm{ext}}^{(i)} \cap \tilde{\mathcal{E}}(D_\sigma) \end{cases} \tag{3.16}$$

where d_ϵ is defined by (3.1). Note that we have the usual finite volume property of local conservativity of the flux through an interface $\epsilon = \sigma\,|\sigma'$:

$$\phi_{\sigma,\epsilon}(u_i) = -\phi_{\sigma',\epsilon}(u_i), \quad \forall \epsilon = \sigma|\sigma' \in \widetilde{\mathcal{E}}_{\mathrm{int}}^{(i)}. \tag{3.17}$$

Then the discrete Laplace operator of the full velocity vector is defined by

$$\begin{aligned} -\Delta_{\mathcal{E}} : \quad & \mathbf{H}_{\mathcal{E},0} \longrightarrow \mathbf{H}_{\mathcal{E},0} \\ & \boldsymbol{u} \mapsto -\Delta_{\mathcal{E}} \boldsymbol{u} = (-\Delta_{\mathcal{E}}^{(1)} u_1, \ldots, -\Delta_{\mathcal{E}}^{(d)} u_d)^t. \end{aligned} \tag{3.18}$$

3.4. The main result

In the following we denote

$$q(d) = \begin{cases} \text{any real number greater than 1 if } d = 2, \\ 6 \text{ if } d = 3. \end{cases}$$

Theorem 1 (Convergence of the MAC scheme)**.** *Let* Ω *be an open bounded subset of* $\mathbb{R}^d$*,* $d = 2$ *or* $d = 3$*, adapted to the MAC-scheme (that is any finite union of rectangles in 2D or rectangular in 3D). Let* $\gamma \geq \frac{3}{2}$ *if* $d = 3$ *and* $\gamma > 1$ *if* $d = 2$ *and* $\varrho_0 \in L^\gamma(\Omega)$ *such that* $\varrho_0 > 0$ *a.e in* Ω*.*

Consider a partition I_n *of the time interval* $[0, T]$*, which, for the sake of simplicity, we suppose uniform. Let* δt_n *be the constant time step going to* 0 *as* $n \to \infty$*.*

Consider a sequence of MAC grids $(\mathcal{D}_n = (\mathcal{M}_n, \mathcal{E}_n))_{n\in\mathbb{N}}$, with step size $h_{\mathcal{M}_n}$ (as defined by (3.2)) going to zero as $n \to \infty$ and satisfying $\eta \leq \eta_{\mathcal{M}_n}$ for $\eta > 0$ uniformly with respect to n and where the regularity of the mesh $\eta_{\mathcal{M}_n}$ is defined by (3.3).

Let, for any $n \in \mathbb{N}$, $(\boldsymbol{u}_n, p_n, \varrho_n) \in \boldsymbol{X}_{\mathcal{E}_n,\delta t_n} \times Y_{\mathcal{M}_n,\delta t_n} \times Y_{\mathcal{M}_n,\delta t_n}$ be a discrete solution of (3.5) (with respect to the mesh $\mathcal{M}_n$ and the partition I_n) emanating form the initial data $\mathcal{P}_{\mathcal{M}_n}\varrho_0$ where $\mathcal{P}_{\mathcal{M}_n}$ is defined in (3.4). Then there exists $(\boldsymbol{u}, p, \varrho) \in \mathrm{L}^2(0,T;\mathrm{H}_0^1(\Omega)^d) \times \mathrm{L}^2((0,T)\times\Omega) \times \Big(\mathrm{L}^\infty(0,T;\mathrm{L}^\gamma(\Omega)) \cap \mathrm{L}^{2\gamma}((0,T)\times\Omega)\Big)$, $\varrho \geq 0$ a.e in $(0,T)\times\Omega$, such that up to a subsequence, the following limits hold

$$\begin{aligned}
&\varrho_n \to \varrho \text{ strongly in } \mathrm{L}^q((0,T)\times\Omega), \text{ for any } q \in [1,2\gamma), \quad \text{and weakly in } \mathrm{L}^{2\gamma}((0,T)\times\Omega).\\
&p_n \to p \text{ strongly in } \mathrm{L}^q((0,T)\times\Omega), \text{ for any } q \in [1,2), \quad \text{and weakly in } \mathrm{L}^2((0,T)\times\Omega).\\
&\boldsymbol{u}_n \to \boldsymbol{u} \text{ weakly in } \mathrm{L}^2(0,T;\mathrm{L}^{q(d)}(\Omega)^d),\\
&\nabla_{\mathcal{E}_n}\boldsymbol{u}_n \to \nabla\boldsymbol{u} \text{ weakly in } \mathrm{L}^2(0,T;\mathrm{L}^2(\Omega)^{d\times d}).
\end{aligned}$$

and such that $(\boldsymbol{u}, p, \varrho)$ is a weak solution of Problem 1.1 emanating from ϱ_0 in the sense of Definition 1.

Remark 2. In the case of the compressible Stokes system, one can prove the convergence of the MAC scheme up to $\gamma > 1$ (see [11]). For the steady compressible Navier-Stokes system, we need to assume $\gamma > 3$ to prove the convergence of the MAC scheme (this is ongoing work). Here we are restricted to $\gamma \geq \frac{3}{2}$ if $d = 3$ and $\gamma > 1$ if $d = 2$.

Remark 3. In both two and three space dimensions, we can also include a non-zero external force on the right-hand side of the momentum equation, i.e. we have additionally the term $\varrho\boldsymbol{g} + \boldsymbol{f}$ on the right-hand side of (1.1*b*). For $(\boldsymbol{g}, \boldsymbol{f}) \in \mathrm{L}^\infty((0,T)\times\Omega)^3 \times \mathrm{L}^2((0,T)\times\Omega)^3$ we would get the same result as in Theorem 1.

3.5. Weak form of the momentum equation

For the analysis of the scheme, it is convenient to work with a weak form of the momentum equation. To this purpose, we first recall the definition of the discrete H_0^1 inner product [10, Chapter III], which is obtained by multiplying the discrete Laplace operator scalarly by a test function $\boldsymbol{v} \in \mathbf{H}_{\mathcal{E},0}$ and integrating over the computational domain. A simple reordering of the sums (which may be seen as a discrete integration by parts) yields, thanks to the conservativity of the diffusion flux (3.17):

$$\begin{aligned}
&\forall (\boldsymbol{u}, \boldsymbol{v}) \in {\mathbf{H}_{\mathcal{E},0}}^2, \qquad \int_\Omega -\Delta_{\mathcal{E}}\boldsymbol{u}\cdot\boldsymbol{v}\,\mathrm{d}\boldsymbol{x} = [\boldsymbol{u}, \boldsymbol{v}]_{1,\mathcal{E},0} = \sum_{i=1}^{d} [u_i, v_i]_{1,\mathcal{E}^{(i)},0},\\
&\text{with } [u_i, v_i]_{1,\mathcal{E}^{(i)},0} = \sum_{\substack{\epsilon\in\widetilde{\mathcal{E}}^{(i)}_{\mathrm{int}}\\ \epsilon=\sigma|\sigma'}} \frac{|\epsilon|}{d_\epsilon}\,(u_\sigma - u_{\sigma'})\,(v_\sigma - v_{\sigma'}) + \sum_{\substack{\epsilon\in\widetilde{\mathcal{E}}^{(i)}_{\mathrm{ext}}\\ \epsilon\subset\partial(D_\sigma)}} \frac{|\epsilon|}{d_\epsilon}\,u_\sigma\,v_\sigma.
\end{aligned} \tag{3.20}$$

The bilinear forms $\left| \begin{array}{l} H^{(i)}_{\mathcal{E},0} \times H^{(i)}_{\mathcal{E},0} \to \mathbb{R} \\ \\ (u, v) \mapsto [u_i, v_i]_{1,\mathcal{E}^{(i)},0} \end{array} \right.$ and $\left| \begin{array}{l} \mathbf{H}_{\mathcal{E},0} \times \mathbf{H}_{\mathcal{E},0} \to \mathbb{R} \\ \\ (\boldsymbol{u}, \boldsymbol{v}) \mapsto [\boldsymbol{u}, \boldsymbol{v}]_{1,\mathcal{E},0} \end{array} \right.$ are inner products on $H^{(i)}_{\mathcal{E},0}$ and $\mathbf{H}_{\mathcal{E},0}$ respectively, which induce the following discrete H^1_0 norms:

$$\|u_i\|^2_{1,\mathcal{E}^{(i)},0} = [u_i, u_i]_{1,\mathcal{E}^{(i)},0} = \sum_{\substack{\epsilon \in \widetilde{\mathcal{E}}^{(i)}_{\text{int}} \\ \epsilon = \overrightarrow{\sigma|\sigma'}}} \frac{|\epsilon|}{d_\epsilon} (u_\sigma - u_{\sigma'})^2 + \sum_{\substack{\epsilon \in \widetilde{\mathcal{E}}^{(i)}_{\text{ext}} \\ \epsilon \subset \partial(D_\sigma)}} \frac{|\epsilon|}{d_\epsilon} u_\sigma^2 \tag{3.21a}$$

$$\|\boldsymbol{u}\|^2_{1,\mathcal{E},0} = [\boldsymbol{u}, \boldsymbol{u}]_{1,\mathcal{E},0} = \sum_{i=1}^{d} \|u_i\|^2_{1,\mathcal{E}^{(i)},0}. \tag{3.21b}$$

Since we are working on Cartesian grids, this inner product may be formulated as the L^2 inner product of discrete

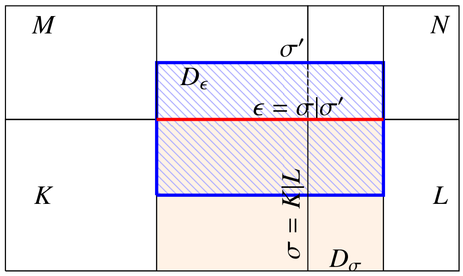

Figure 2: Full grid for definition of the derivative of the velocity.

gradients. Indeed, consider the following discrete gradient of each velocity component u_i.

$$\nabla_{\mathcal{E}^{(i)}} u_i = (\eth_1 u_i, \dots, \eth_d u_i) \text{ with } \eth_j u_i = \sum_{\substack{\epsilon \in \widetilde{\mathcal{E}}^{(i)} \\ \epsilon \perp \boldsymbol{e}_j}} (\eth_j u_i)_{D_\epsilon} \, \mathbb{1}_{D_\epsilon}, \tag{3.22}$$

$$(\eth_j u_i)_{D_\epsilon} = \begin{cases} \dfrac{u_{\sigma'} - u_\sigma}{d_\epsilon} \text{ with } \epsilon = \overrightarrow{\sigma|\sigma'}, \text{ and } D_\epsilon = \epsilon \times \boldsymbol{x}_\sigma \boldsymbol{x}_{\sigma'} \\ \\ -\dfrac{u_\sigma}{d_\epsilon} \boldsymbol{e}_j \cdot \boldsymbol{n}_{D_\sigma,\epsilon} \text{ with } \epsilon \in \widetilde{\mathcal{E}}^{(i)}_{\text{ext}} \cap \tilde{\mathcal{E}}(D_\sigma), \text{ and } D_\epsilon = \epsilon \times \boldsymbol{x}_\sigma \boldsymbol{x}_\epsilon \end{cases} \tag{3.23}$$

where $\boldsymbol{n}_{D_\sigma,\epsilon}$ stands for the unit normal vector to ϵ outward D_σ and $\boldsymbol{x}_\epsilon$ stands for the mas center of ϵ (see Figure 2).

This definition is compatible with the definition of the discrete derivative $(\eth_i u_i)_K$ given by (3.11), since, if $\epsilon \subset K$ then $D_\epsilon = K$. With this definition, it is easily seen that

$$\int_\Omega \nabla_{\mathcal{E}^{(i)}} u \cdot \nabla_{\mathcal{E}^{(i)}} v \, \mathrm{d}\boldsymbol{x} = [u, v]_{1,\mathcal{E}^{(i)},0}, \forall u, v \in H^{(i)}_{\mathcal{E},0}, \forall i = 1, \dots, d. \tag{3.24}$$

where $[u, v]_{1,\mathcal{E}^{(i)},0}$ is the discrete H^1_0 inner product defined by (3.20). We may then define

$$\nabla_{\mathcal{E}} \boldsymbol{u} = (\nabla_{\mathcal{E}^{(1)}} u_1, \dots, \nabla_{\mathcal{E}^{(d)}} u_d), \text{ so that } \int_\Omega \nabla_{\mathcal{E}} \boldsymbol{u} : \nabla_{\mathcal{E}} \boldsymbol{v} \, \mathrm{d}\boldsymbol{x} = [\boldsymbol{u}, \boldsymbol{v}]_{1,\mathcal{E},0}.$$

Thanks to the previous definition, we then introduce the following discrete dual norms on $\mathbf{H}_{\mathcal{E},0}$.

$$\boldsymbol{v} \in \mathbf{H}_{\mathcal{E},0} \mapsto \|\boldsymbol{v}\|_{-1,\mathcal{E},0} = \max\{\left| \int_\Omega \boldsymbol{v} \cdot \boldsymbol{\varphi} \, \mathrm{d}\boldsymbol{x} \right| \; ; \; \boldsymbol{\varphi} \in \mathbf{H}_{\mathcal{E},0} \text{ and } \|\boldsymbol{\varphi}\|_{1,\mathcal{E},0} \leq 1\}, \tag{3.25}$$

With these notations, a weak formulation of the momentum equation reads for any $t \in (0,T)$ and $\boldsymbol{v} \in \mathbf{H}_{\mathcal{E},0}$:

$$\mu[\boldsymbol{u}(t),\boldsymbol{v}]_{1,\mathcal{E},0} + (\mu+\lambda)\int_\Omega \operatorname{div}_{\mathcal{M}}[\boldsymbol{u}(t)]\operatorname{div}_{\mathcal{M}}\boldsymbol{v}\,\mathrm{d}\boldsymbol{x} - \int_\Omega p(t)\operatorname{div}_{\mathcal{M}}\boldsymbol{v}\,\mathrm{d}\boldsymbol{x} = 0. \tag{3.26}$$

4. Mesh independent estimates

For the sake of clarity, we shall perform the proofs only in the most interesting three dimensional case. The two dimensional case is simpler and mostly requires modifications due to the different Sobolev embeddings.. Assume that there exists $\eta > 0$ such that all the considered meshes considered satisfy $\eta \le \eta_{\mathcal{M}}$, where $\eta_{\mathcal{M}}$ is defined by (3.3). From now on, the letter c denotes a positive number which may depend on $|\Omega|$, $\operatorname{diam}(\Omega)$, γ, λ, μ, η, ρ_0 and on other parameters; the dependency on these other parameters (if any) is always explicitly indicated.

4.1. Existence and estimates

Lemma 2 (Existence of approximate solutions and estimates)**.** *There exists at least a solution* $(\boldsymbol{u},p,\varrho) \in \boldsymbol{X}_{\mathcal{M},\delta t} \times Y_{\mathcal{M},\delta t} \times Y_{\mathcal{M},\delta t}$ *satisfying* (3.5)*. Any solution is such that* $\varrho > 0$ *a.e in* Ω*. Furthermore, there exists* $c(\gamma,\mu,\rho_0)$ *and* $c(\gamma,\mu,\rho_0,\Omega)$ *such that:*

$$\|\varrho\|_{L^\infty(0,T;L^\gamma(\Omega))} + \|\boldsymbol{u}\|_{L^2(0,T;\mathbf{H}_{\mathcal{E},0})} \le c(\gamma,\mu,\rho_0), \tag{4.1a}$$

$$\|\boldsymbol{u}\|_{L^2(0,T;L^6(\Omega)^3)} \le c(\gamma,\mu,\rho_0,\Omega). \tag{4.1b}$$

Proof. The existence follows by the Brower fixed point theorem: for fixed m, assume that the existence of a solution to the scheme (3.5) with $\rho^k \ge 0$ is known for $k = 0,\dots m-1$. Let $M = \int_\Omega \rho^{m-1}\,\mathrm{d}\boldsymbol{x}$, and let $C = \{\rho \in L_{\mathcal{M}} : \rho \ge 0, \int_\Omega \rho^{m-1}\,\mathrm{d}\boldsymbol{x} \le M$. Let $\rho \in C$, $p = \rho^\gamma$, and let $\boldsymbol{u} \in \mathbf{H}_{\mathcal{E}}$ be the unique solution to $-\mu\Delta_{\mathcal{E}}\boldsymbol{u} - (\mu+\lambda)\nabla_{\mathcal{E}}\operatorname{div}_{\mathcal{M}}\boldsymbol{u} + \nabla_{\mathcal{E}}p = 0$ (which is an invertible linear system); finally, let $\bar\varrho$ be the solution to $\frac{1}{\delta t}(\bar\varrho - \varrho) + \operatorname{div}^{\mathrm{up}}_{\mathcal{M}}(\bar\varrho\boldsymbol{u}) = 0$, which is also an invertible linear system; by conservativity we have $\int_\Omega \bar\rho\,\mathrm{d}\boldsymbol{x} = \int_\Omega \rho\,\mathrm{d}\boldsymbol{x} = M$ and by the upwind choice $\bar\varrho \ge 0$ so that $\bar\varrho \in C$. It is clear that p depends continuously on ϱ, that $\boldsymbol{u}$ depends continuously on p and thus on ϱ, and that $\bar\varrho$ depends also continously on $\boldsymbol{u}$ and ϱ, and therefore on ϱ. Hence we have constructed a continuous mapping $T : \varrho \in C \mapsto \bar\varrho \in C$, which admits a fixed point by Brower's theorem. This concludes to the existence of a solution to (3.5).

Let us then prove (4.1a). Testing (3.26) by $\boldsymbol{u}(t)$ we infer that for a.e. $s \in (0,T)$:

$$\mu\|\boldsymbol{u}\|^2_{L^2(0,s;\mathbf{H}_{\mathcal{E},0})} + (\mu+\lambda)\|\operatorname{div}_{\mathcal{M}}\boldsymbol{u}\|^2_{L^2(0,s;L^2(\Omega))} - \int_0^s\int_\Omega p\operatorname{div}_{\mathcal{M}}\boldsymbol{u}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \le 0. \tag{4.2}$$

Multiplying (3.5a) by $\frac{\gamma}{\gamma-1}(\varrho^m)^{\gamma-1}$ and following [20] one has for a.a $s \in (0,T)$:

$$\frac{1}{\gamma-1}\int_\Omega \varrho(s)^\gamma\,\mathrm{d}\boldsymbol{x} + \int_0^s\int_\Omega p\operatorname{div}_{\mathcal{M}}\boldsymbol{u}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \le \mathcal{E}_0 \tag{4.3}$$

Summing (4.2) and (4.3) we infer that

$$\frac{1}{\gamma-1}\int_\Omega \varrho(s)^\gamma\,\mathrm{d}\boldsymbol{x} + \mu\|\boldsymbol{u}\|^2_{L^2(0,s;\mathbf{H}_{\mathcal{E},0})} + (\mu+\lambda)\|\operatorname{div}_{\mathcal{M}}\boldsymbol{u}\|^2_{L^2(0,s;L^2(\Omega))} \le \mathcal{E}_0\,. \tag{4.4}$$

which gives (4.1a).

The estimate (4.1b) follows by the discrete Sobolev inequality (see Theorem 2 in the appendix). $\square$

The following Lemma can be seen as a discrete version of [16, Lemma NN] OU AUTRE ?

Lemma 3 (Discrete renormalized continuity equation)**.** *Let $(\boldsymbol{u}, p, \varrho) \in X_{\mathcal{M},\delta t} \times Y_{\mathcal{M},\delta t} \times Y_{\mathcal{M},\delta t}$ satisfying (3.5). Then for any $B \in C^2(\mathbb{R}_+^*) \cap C(\mathbb{R}_+)$ such that B is convex on $\mathbb{R}_+^*$ we have, for a.a $t \in (0, T)$,*

$$\int_\Omega B(\varrho(t))\,\mathrm{d}\boldsymbol{x} - \int_\Omega B(\varrho_0)\,\mathrm{d}\boldsymbol{x} + \int_0^t \int_\Omega (\varrho B'(\varrho) - B(\varrho))\,\mathrm{div}_{\mathcal{M}}\,\boldsymbol{u}\,\mathrm{d}\boldsymbol{x}\,\mathrm{dt} \le 0. \tag{4.5}$$

4.2. Higher integrability of the pressure

Up to now, the only avaible estimates on the pressure p have been deduced from Lemma 2 and it gives

$$||p||_{L^1((0,T)\times\Omega)} \le C(\gamma, \mu, \mathcal{E}_0).$$

The non-reflexive Banach space L^1 is not very convenient as bounded sequences are not necessarily weakly precompact. We need more integrability of the pressure to pass to the limit in the pressure term. We shall prove that the discrete pressure is in fact bounded (with respect to $h_{\mathcal{M}}$ and δt) in $L^2((0, T) \times \Omega)$, using the two following lemmas.

Lemma 4 (Nečas, [16, see e.g. Lemma 10.10])**.** *Let Ω be a bounded Lipschitz domain of $\mathbb{R}^d$ where $d \ge 2$. Let $q \in L^2(\Omega)$ such that $\int_\Omega q\,\mathrm{d}\boldsymbol{x} = 0$. Then, there exists $\boldsymbol{v} \in (H_0^1(\Omega))^d$ such that $\mathrm{div}(\boldsymbol{v}) = q$ a.e. in Ω and $\|\boldsymbol{v}\|_{(H_0^1(\Omega))^d} \le C\,\|q\|_{L^r(\Omega)}$ where C only depends on Ω.*

To adapt the proof of the continuous case, we introduce a so-called Fortin interpolation operator, which preserves the divergence.

Lemma 5 (Fortin interpolation operator, [24, Theorem 1])**.** *Let $\mathcal{D} = (\mathcal{M}, \mathcal{E})$ be a MAC grid of Ω. For $\boldsymbol{v} \in C_c^\infty(\Omega)^d$, we define $\widetilde{\mathcal{P}}_{\mathcal{E}}\boldsymbol{v}$ by*

$$\begin{aligned}
&\widetilde{\mathcal{P}}_{\mathcal{E}}\boldsymbol{v} = \left(\widetilde{\mathcal{P}}_{\mathcal{E}}^{(1)} v_1, \cdots, \widetilde{\mathcal{P}}_{\mathcal{E}}^{(1)} v_d\right) \in \mathbf{H}_{\mathcal{E},0}, \text{ where for } i = 1, \ldots d,\\
&\widetilde{\mathcal{P}}_{\mathcal{E}}^{(i)} : \quad C_c^\infty(\Omega) \longrightarrow H_{\mathcal{E},0}^{(i)}\\
&\qquad\quad v_i \;\longmapsto \widetilde{\mathcal{P}}_{\mathcal{E}} v_i \;;\; i = 1, \cdots, d,\\
&\qquad\quad \widetilde{\mathcal{P}}_{\mathcal{E}}^{(i)} v_i(\boldsymbol{x}) = \frac{1}{|\sigma|}\int_\sigma v_i(\boldsymbol{x})\,\mathrm{d}\gamma(\boldsymbol{x}),\ \forall \boldsymbol{x} \in D_\sigma,\ \sigma \in \mathcal{E}^{(i)}.
\end{aligned} \tag{4.6}$$

Let $\eta_{\mathcal{M}} > 0$ be defined by (3.3). Let $\boldsymbol{\varphi} \in (C_c^\infty(\Omega))^d$, then

$$||\widetilde{\mathcal{P}}_{\mathcal{E}}\boldsymbol{\varphi} - \boldsymbol{\varphi}||_{L^\infty(\Omega)} \le C_{\boldsymbol{\varphi}} h_{\mathcal{M}}, \tag{4.7a}$$

$$\mathrm{div}_{\mathcal{M}}(\widetilde{\mathcal{P}}_{\mathcal{E}}\boldsymbol{\varphi}) = \mathcal{P}_{\mathcal{M}}(\mathrm{div}\boldsymbol{\varphi}), \tag{4.7b}$$

$$||\nabla_{\mathcal{E}}\widetilde{\mathcal{P}}_{\mathcal{E}}\boldsymbol{\varphi}||_{(L^2(\Omega))^d} \le C_{\eta_{\mathcal{M}}}||\nabla\boldsymbol{\varphi}||_{(L^2(\Omega))^d}, \tag{4.7c}$$

where $C_{\eta_{\mathcal{M}}}$ depends only on $\eta_{\mathcal{M}}$ in a decreasing way, on Ω and where $\mathcal{P}_{\mathcal{M}}$ is defined in (3.4).

Let us now prove that the pressure is bounded in $L^2((0,T)\times\Omega)$ with respect to the size of the discretizations.

Proposition 1. *Any solution* $(\boldsymbol{u}, p, \varrho) \in \boldsymbol{X}_{\mathcal{M},\delta t} \times Y_{\mathcal{M},\delta} \times Y_{\mathcal{M},\delta t}$ *of* (3.5) *satisfy*

$$\|p\|_{L^2((0,T)\times\Omega)} \le C, \tag{4.8}$$

and in particular

$$\|\varrho\|_{L^{2\gamma}((0,T)\times\Omega)} \le C, \tag{4.9}$$

where the constant C depends on $T, \Omega, \mu, \lambda, \gamma, \eta, \mathcal{E}_0$.

Proof. Let us introduce the quantity

$$P(\varrho^m, \boldsymbol{u}^m) = p^m - (\mu+\lambda)\mathrm{div}_{\mathcal{M}}\boldsymbol{u}^m.$$

Let $\boldsymbol{v}^m \in H_0^1(\Omega)^d$ (see Lemma 4) such that

$$\mathrm{div}\boldsymbol{v}^m = P(\varrho^m, \boldsymbol{u}^m) - \frac{1}{|\Omega|}\int_\Omega P(\varrho^m, \boldsymbol{u}^m)\,\mathrm{d}\boldsymbol{x}. \tag{4.10}$$

Testing (3.26) by $\widetilde{\mathcal{P}}_{\mathcal{E}}(\boldsymbol{v}^m)$ and using Lemma 5 we get

$$\|P(\varrho^m, \boldsymbol{u}^m)\|^2_{L^2(\Omega)} = \mu[\boldsymbol{u}^m, \widetilde{\mathcal{P}}_{\mathcal{E}}(\boldsymbol{v}^m)]_{1,\mathcal{E},0} + \frac{1}{|\Omega|}\Big(\int_\Omega P(\varrho^m, \boldsymbol{u}^)\,\mathrm{d}\boldsymbol{x}\Big)^2.$$

Consequently,

$$\begin{aligned}\|P(\varrho^m, \boldsymbol{u}^m) - \frac{1}{|\Omega|}\int_\Omega P(\varrho^m, \boldsymbol{u}^m)\,\mathrm{d}\boldsymbol{x}\|^2_{L^2(\Omega)} =&\mu[\boldsymbol{u}^m, \widetilde{\mathcal{P}}_{\mathcal{E}}(\boldsymbol{v}^m)]_{1,\mathcal{E},0}\\ &\le c(\Omega,\mu,\epsilon,\eta)\|\boldsymbol{u}^m\|^2_{1,\mathcal{E},0} + \epsilon\|P(\varrho^m, \boldsymbol{u}^m) - \frac{1}{|\Omega|}\int_\Omega P(\varrho^m, \boldsymbol{u}^m)\,\mathrm{d}\boldsymbol{x}\|^2_{L^2(\Omega)}\end{aligned}$$

for any $\epsilon > 0$. Since $\int_\Omega P(\varrho^m, \boldsymbol{u}^m)\,\mathrm{d}\boldsymbol{x} = \int_\Omega p^m\,\mathrm{d}\boldsymbol{x} \le c(\gamma, \mathcal{E}_0)$ we obtain

$$\|P(\varrho^m, \boldsymbol{u}^m)\|^2_{L^2(\Omega)} \le c(\mathcal{E}_0, \mu, \gamma, \Omega, \eta_{\mathcal{M}})(1 + \|\boldsymbol{u}^m\|^2_{1,\mathcal{E},0})$$

Consequently,

$$\|p^m\|^2_{L^2(\Omega)} \le c(\mathcal{E}_0, \mu, \lambda, \Omega, \eta_{\mathcal{M}})(1 + \|\boldsymbol{u}^m\|^2_{1,\mathcal{E},0} + \|\mathrm{div}_{\mathcal{M}}\boldsymbol{u}^m\|^2_{L^2(\Omega)})$$

which gives

$$\|p\|^2_{L^2((0,T)\times\Omega)} \le c(\mathcal{E}_0, \mu, \lambda, \Omega, \eta_{\mathcal{M}})(T + \|\boldsymbol{u}\|^2_{L^2(0,T;\mathbf{H}_{\mathcal{E},0})}).$$

We conclude by using Lemma 2. □

Remark 4. From estimates (4.1*b*) and (4.9) we infer that

$$\|\varrho\boldsymbol{u}\|_{L^{\frac{2\gamma}{\gamma+1}}(0,T;L^{\frac{6\gamma}{\gamma+3}}(\Omega)^3)} \le C. \tag{4.11}$$

In particular, since $\gamma \ge \frac{3}{2}$,

$$\|\varrho\boldsymbol{u}\|_{L^{\frac{2\gamma}{\gamma+1}}(0,T;L^2(\Omega)^3)} \le C. \tag{4.12}$$

The following estimate (4.13) is obtained thanks to the numerical diffusion due to the upwinding, as is classical in the framework of hyperbolic conservation laws, see e.g. [10, Chapter V]. We refer the reader to [26, Lemma 4.2] for a proof. This estimate is useful to pass to the limit in the continuity equation.

Lemma 6. *Let $(\boldsymbol{u}, p, \varrho) \in \boldsymbol{X}_{\mathcal{M},\delta t} \times Y_{\mathcal{M},\delta t} \times Y_{\mathcal{M},\delta t}$ satisfying (3.5). Then there exists C only depending on the data such that*

$$\delta t \sum_{m=1}^{N} \sum_{\sigma\in\mathcal{E}_{\rm int},\sigma=K|L} |\sigma| \frac{(\varrho_K^m - \varrho_L^m)^2}{\max(\varrho_K^m, \varrho_L^m)} |u_{K,\sigma}^m| \leq C. \tag{4.13}$$

The following Lemma deals with an estimate discrete time derivative of the density, which is crucial to pass to the limit in the equation of state (3.5*c*).

Lemma 7 (Estimates on the dual norm of the discrete time derivative of the density)**.** *Let $(\boldsymbol{u}, p, \varrho) \in \boldsymbol{X}_{\mathcal{E},\delta t} \times Y_{\mathcal{M},\delta t} \times Y_{\mathcal{M},\delta t}$ be a solution to (3.5). Then there exists $C > 0$ depending only on the data such that:*

$$\|\eth_t \varrho\|_{L^1(0,T;W^{-1,1}(\Omega))} \leq C, \tag{4.14}$$

$$\|\eth_t \varrho\|_{L^1(0,T;(L_{\mathcal{M}})')} \leq C. \tag{4.15}$$

In particular the solution of $-\Delta_{\mathcal{M}}[w(t)] = \varrho(t)$ in the sense of Proposition 3, where $t \in (0, T)$, satisfies

$$\|\eth_t w\|_{L^1(0,T;L^2(\Omega))} \leq C. \tag{4.16}$$

Proof. In the sake of brievity we only prove (4.15). The inequality (4.14) comes from a trivial adaption of the following proof. The inequality (4.16) is a consequence of (5.15).

Let $\phi \in L_{\mathcal{M}}$ such that $\|\phi\|_{1,\mathcal{M}} \leq 1$. Multiplying (3.5*a*) by ϕ we obtain for any $n \in \{1, ..., N\}$ and $t \in (t^{m-1}, t^m)$:

$$\int_\Omega \eth_t \varrho(t) \phi \, \mathrm{d}\boldsymbol{x} = - \sum_{\sigma\in\mathcal{E}_{\rm int},\sigma=K|L} F_{K,\sigma}^m (\phi_K - \phi_L) = - \sum_{\sigma\in\mathcal{E}_{\rm int},\sigma=K|L} |\sigma| \varrho_\sigma^{m,\rm up} u_{K,\sigma}^m (\phi_K - \phi_L).$$

By virtue of Holder's inequality we infer that

$$\begin{aligned} \Big| \sum_{\sigma\in\mathcal{E}_{\rm int},\sigma=K|L} |\sigma| \varrho_\sigma^{m,\rm up} u_{K,\sigma}^m (\phi_K - \phi_L) \Big| &\leq \Big[\sum_{\sigma\in\mathcal{E}_{\rm int},\sigma=K|L} |\sigma| d_{KL} (\varrho_K^m + \varrho_L^m)^2 |u_{K,\sigma}^m|^2 \Big]^{1/2} \times \Big[\sum_{\sigma\in\mathcal{E}_{\rm int},\sigma=K|L} \frac{|\sigma|}{d_{KL}} (\phi_K - \phi_L)^2 \Big]^{1/2} \\ &\leq C \|\varrho(t) \boldsymbol{u}(t)\|_{L^2(\Omega)^3} \|\phi\|_{1,\mathcal{M}} \end{aligned}$$

where C depends only on the data. Using (4.12) gives (4.15). □

5. Convergence analysis

The aim of this section is to prove Theorem 1. We begin with the introduction of an interpolate operator used in convergence analysis of the discrete continuity equation (3.5*a*).

Lemma 8 (Full grid velocity interpolate)**.** *For a given MAC mesh $(\mathcal{M},\mathcal{E})$, we define, for $i = 1,2,3$:*

$$\mathcal{R}^{(i)}_{\mathcal{M}} : \left| \begin{array}{l} H^{(i)}_{\mathcal{E},0} \longrightarrow L_{\mathcal{M}} \\ u \longmapsto \displaystyle\sum_{K\in\mathcal{M}} (\mathcal{R}^{(i)}_{\mathcal{M}} u)_K \, \chi_K, \end{array} \right.$$

where

$$(\mathcal{R}^{(i)}_{\mathcal{M}} u)_K = \frac{1}{2} \sum_{\sigma\in\mathcal{E}^{(i)}(K)} u_\sigma. \tag{5.1}$$

We also define

$$\mathcal{R}_{\mathcal{M}} : \left| \begin{array}{l} \mathbf{H}_{\mathcal{E},0} \longrightarrow L^3_{\mathcal{M}} \\ \boldsymbol{u} = (u_1, ..., u_d) \longmapsto (\mathcal{R}^{(1)}_{\mathcal{M}}(u_1), ..., \mathcal{R}^{(d)}_{\mathcal{M}}(u_d)). \end{array} \right.$$

Then we have for $i = 1,2,3$ and $1 \le p < +\infty$:

$$\| \mathcal{R}^{(i)}_{\mathcal{M}}(u_i)\|_{L^p(\Omega)} \le \|u_i\|_{L^p(\Omega)}, \ \forall u_i \in H^{(i)}_{\mathcal{E},0}.$$

and

$$\|u_i - \mathcal{R}^{(i)}_{\mathcal{M}}(u_i)\|_{L^p(\Omega)} \le h_{\mathcal{M}} \|\eth_i u_i\|_{L^p(\Omega)}, \ \forall u_i \in H^{(i)}_{\mathcal{E},0}. \tag{5.2}$$

Proposition 2. *Let $\eta > 0$ and $(\mathcal{D}_n = (\mathcal{M}_n, \mathcal{E}_n))_{n\in\mathbb{N}}$ be a sequence of MAC grids with step size $h_{\mathcal{M}_n}$ tending to zero as $n \to \infty$. Assume that $\eta \le \eta_{\mathcal{M}_n}$, $\forall n \in \mathbb{N}$ where $\eta_{\mathcal{M}_n}$ is defined by (3.3). Consider for any $n \in \mathbb{N}$ a partition I_n of the time interval $[0,T]$, which, for the sake of simplicity, we suppose uniform. Let δt_n be the constant time step going to 0 as $n \to \infty$. Consider for any $n \in \mathbb{N}$ a solution $(\boldsymbol{u}_n, p_n, \varrho_n) \in \boldsymbol{X}_{\mathcal{E}_n,\delta t_n} \times Y_{\mathcal{M}_n,\delta t_n} \times Y_{\mathcal{M}_n,\delta t_n}$ to problem (3.5) (with respect to the mesh $\mathcal{D}_n$ and the partition I_n). Then, up to the extraction of a subsequence:*

1. *the sequence $(\boldsymbol{u}_n)_{n\in\mathbb{N}}$ converges weakly in $\mathrm{L}^2(0,T;(\mathrm{L}^6(\Omega))^3)$ to a function $\boldsymbol{u} \in \mathrm{L}^2(0,T;(\mathrm{H}^1_0(\Omega))^3)$ and the sequence $(\nabla_{\mathcal{E}_n}\boldsymbol{u}_n)_{n\in\mathbb{N}}$ converges weakly in $(\mathrm{L}^2((0,T)\times\Omega))^{3\times3}$ to $\nabla\boldsymbol{u}$.*

2. *the sequence $(\varrho_n)_{n\in\mathbb{N}}$ weakly converges to a function ϱ in $\mathrm{L}^{2\gamma}((0,T)\times\Omega)$,*

3. *the sequence $(p_n)_{n\in\mathbb{N}}$ weakly converges to a function p in $\mathrm{L}^2((0,T)\times\Omega)$,*

4. *the sequence $(\varrho_n\boldsymbol{u}_n)_{n\in\mathbb{N}}$ weakly converges to the function $\varrho\boldsymbol{u}$ in $\mathrm{L}^{\frac{2\gamma}{\gamma+1}}(0,T;\mathrm{L}^{\frac{6\gamma}{\gamma+3}}(\Omega)^3)$,*

5. *$\boldsymbol{u}$ and ϱ satisfy the continuous mass equation (1.1a) in the weak sense that is (2.1).*

6. *$\boldsymbol{u}$ and p satisfy the momentum balance equation (1.1b) in the weak sense that is (2.2).*

7. *$\varrho \ge 0$ a.e in $(0,T)\times\Omega$.*

Proof. The stated convergences (*i.e.* points *1.* to *3.*) are straightforward consequences of the uniform bounds for the sequence of solutions, together, for the velocity, with the compactness theorem 3. Point *4.* comes from Lemma 13 combined with Proposition 5, estimates ((4.11) and (4.14). Point *7.* is an easy consequence of point *2.*. Let us then prove point *5.* *i.e.* that $\boldsymbol{u}, \varrho$ satisfy (2.1).

Let $\varphi \in C^\infty([0,T]\times\overline{\Omega})$ such that $\varphi(T,\cdot)=0$. Multiplying (3.5*a*) by φ and integrating over $(0,T)\times\Omega$ gives

$$\int_0^T \int_\Omega \eth_t \varrho_n \varphi + \mathrm{div}^{\mathrm{up}}_{\mathcal{M}_n}(\varrho_n \boldsymbol{u}_n)\varphi \,\mathrm{d}\boldsymbol{x}\,\mathrm{dt} = T_{\mathrm{time}} + T_{\mathrm{space}} = 0. \tag{5.3}$$

A discrete summation by parts gives

$$T_{\mathrm{time}} = \int_0^T \int_\Omega \eth_t \varrho_n \varphi \,\mathrm{d}\boldsymbol{x}\,\mathrm{dt} = -\int_0^T \int_\Omega \varrho_n(t,\boldsymbol{x})\frac{1}{\delta t_n}(\varphi(t+\delta t_n,\boldsymbol{x})-\varphi(t,\boldsymbol{x}))\,\mathrm{d}\boldsymbol{x}\,\mathrm{dt}$$
$$+\frac{1}{\delta t_n}\int_{T-\delta t_n}^T \int_\Omega \varrho_n(t,\boldsymbol{x})\varphi(t,\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\,\mathrm{dt} - \frac{1}{\delta t_n}\int_0^{\delta t_n}\int_\Omega \mathcal{P}_{\mathcal{M}_n}(\varrho_0)\varphi(t,\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\,\mathrm{dt}\,.$$

By virtue of $\max_{[T-\delta t_n,T]\times\overline{\Omega}}|\varphi| \le C_\varphi \delta t_n$ we obtain

$$\frac{1}{\delta t_n}\int_{T-\delta t_n}^T \int_\Omega \varrho_n(t,\boldsymbol{x})\varphi(t,\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\,\mathrm{dt} \to 0 \text{ as } n\to\infty.$$

Since $\mathcal{P}_{\mathcal{M}_n}(\varrho_0) \to_{n\to\infty} \varrho_0$ in $L^1(\Omega)$ we infer that

$$\frac{1}{\delta t_n}\int_0^{\delta t_n}\int_\Omega \mathcal{P}_{\mathcal{M}_n}(\varrho_0)\varphi(t,\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \to \int_\Omega \varrho_0\varphi(0,\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \text{ as } n\to\infty.$$

Clearly, since φ is smooth, we observe that $\frac{1}{\delta t_n}(\varphi(t+\delta t_n,\boldsymbol{x})-\varphi(t,\boldsymbol{x})) \to \partial_t\varphi$ in $L^\infty((0,T)\times\Omega)$ as $n\to\infty$. Using the weak convergence of the sequence $(\varrho_n)_{n\in\mathbb{N}}$ we get

$$\lim_{n\to\infty}\int_0^T \int_\Omega \varrho_n(t,\boldsymbol{x})\frac{1}{\delta t_n}(\varphi(t+\delta t_n,\boldsymbol{x})-\varphi(t,\boldsymbol{x}))\,\mathrm{d}\boldsymbol{x}\,\mathrm{dt} \to \int_0^T\int_\Omega \varrho\partial_t\varphi\,\mathrm{d}\boldsymbol{x}\,\mathrm{dt}\,.$$

Consequently

$$T_{\mathrm{time}} \to -\int_0^T\int_\Omega \varrho\partial_t\varphi\,\mathrm{d}\boldsymbol{x}\,\mathrm{dt} - \int_\Omega \varrho_0\varphi(0,\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \text{ as } n\to\infty. \tag{5.4}$$

For the study the term T_{space}, for the sake of readability, the dependency of the discrete function on n is omitted. The partition of $[0,T]$ is denoted $0=t^0<t^1<\ldots<t^N=T$. In the following we denote $\varphi_K^m = \frac{1}{\delta t}\int_{t^{m-1}}^{t^m}\frac{1}{|K|}\int_K \varphi(t,\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\,\mathrm{dt}$, $\varphi_\sigma^m = \frac{1}{\delta t}\int_{t^{m-1}}^{t^m}\frac{1}{|\sigma|}\int_\sigma \varphi(t,\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\,\mathrm{dt}$.

$$T_{\mathrm{space}} = \delta t\sum_{m=1}^N\sum_{K\in\mathcal{M}}\sum_{\sigma\in\mathcal{E}(K)}|\sigma|(\varrho_\sigma^{m,\mathrm{up}}-\varrho_K^n)(\boldsymbol{u}_\sigma^m\cdot\boldsymbol{n}_{K,\sigma})(\varphi_K^m-\varphi_\sigma^m) + \delta t\sum_{m=1}^N\sum_{K\in\mathcal{M}}\sum_{\sigma\in\mathcal{E}(K)}|\sigma|\varrho_K^m(\boldsymbol{u}_\sigma^m\cdot\boldsymbol{n}_{K,\sigma})(\varphi_K^m-\varphi_\sigma^m)$$
$$= R_1 - \delta t\sum_{m=1}^N\sum_{K\in\mathcal{M}}\sum_{\sigma\in\mathcal{E}(K)}|\sigma|\varrho_K^m(\boldsymbol{u}_\sigma^m\cdot\boldsymbol{n}_{K,\sigma})(\varphi_\sigma^m-\varphi_K^m).$$

Using Holder's inequality we infer that

$$|R_1| \le C_\varphi\sqrt{h_{\mathcal{M}}}\Big(\delta t\sum_{m=1}^N\sum_{\sigma\in\mathcal{E}_{\mathrm{int}},\sigma=K|L}|\sigma|\frac{(\varrho_K^m-\varrho_L^m)^2}{\max(\varrho_K^m,\varrho_L^m)}|u_{K,\sigma}^m|\Big)^{1/2}\times\Big(\|\varrho\boldsymbol{u}\|_{L^1((0,T)\times\Omega)}\Big)^{1/2}.$$

Consequently, by virtue of (4.13) and (4.12), one has $R_1 \to 0$ as $n \to \infty$. Now we can write

$$
\begin{aligned}
&\delta t \sum_{m=1}^{N} \sum_{K\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}(K)} |\sigma| \varrho_K^m (\boldsymbol{u}_\sigma^m \cdot \boldsymbol{n}_{K,\sigma})(\varphi_\sigma^m - \varphi_K^m) \\
&\qquad = \delta t \sum_{m=1}^{N} \sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}(K)} |\sigma| \varrho_K^m (\boldsymbol{u}_\sigma^m \cdot \boldsymbol{n}_{K,\sigma}) \varphi_\sigma^m - \int_0^T \int_\Omega \varrho\varphi \operatorname{div}_{\mathcal{M}}[\boldsymbol{u}(t)] \,\mathrm{d}\boldsymbol{x} \,\mathrm{dt} \\
&\qquad = \delta t \sum_{m=1}^{N} \sum_{K\in\mathcal{M}} \varrho_K^m \sum_{\substack{j=1,\\ \sigma,\sigma'\in\mathcal{E}^{(j)}(K),\\ K=[\overrightarrow{\sigma\sigma'}]}}^{3} |\sigma| (u_{\sigma'}^m \varphi_{\sigma'}^m - u_\sigma^m \varphi_\sigma^m) - \int_0^T \int_\Omega \varrho\varphi \operatorname{div}_{\mathcal{M}}[\boldsymbol{u}(t)] \,\mathrm{d}\boldsymbol{x} \,\mathrm{dt} \\
&= \delta t \sum_{m=1}^{N} \sum_{K\in\mathcal{M}} |K| \varrho_K^m \sum_{\substack{j=1,\\ \sigma,\sigma'\in\mathcal{E}^{(j)}(K),\\ K=[\overrightarrow{\sigma\sigma'}]}}^{3} \Big(\frac{|\sigma|}{|K|} (u_{\sigma'}^m - u_\sigma^m) \varphi_{\sigma'}^m + \frac{|\sigma|}{|K|} u_\sigma^m (\varphi_{\sigma'}^m - \varphi_\sigma^m) \Big) - \int_0^T \int_\Omega \varrho\varphi \operatorname{div}_{\mathcal{M}}[\boldsymbol{u}(t)] \,\mathrm{d}\boldsymbol{x} \,\mathrm{dt} \\
&\quad = \delta t \sum_{m=1}^{N} \sum_{K\in\mathcal{M}} |K| \varrho_K^m \sum_{j=1}^{3} \Big((\eth_j u_j^m)_K \varphi_{\sigma'}^m + u_\sigma^m (\eth_j \widetilde{\mathcal{P}}_{\mathcal{E}}^{(j)}(\varphi^m))_K \Big) - \int_0^T \int_\Omega \varrho\varphi \operatorname{div}_{\mathcal{M}}[\boldsymbol{u}(t)] \,\mathrm{d}\boldsymbol{x} \,\mathrm{dt} \\
&= \delta t \sum_{m=1}^{N} \sum_{K\in\mathcal{M}} |K| \varrho_K^m \sum_{j=1}^{3} \Big(\mathcal{R}_{\mathcal{M}}^{(j)}(u_j^m)_K (\eth_j \widetilde{\mathcal{P}}_{\mathcal{E}}^{(j)}(\varphi^m))_K + (\eth_j u_j^m)_K \widetilde{\mathcal{P}}_{\mathcal{E}}^{(j)}(\varphi^m)_K \Big) - \int_0^T \int_\Omega \varrho\varphi \operatorname{div}_{\mathcal{M}}[\boldsymbol{u}(t)] \,\mathrm{d}\boldsymbol{x} \,\mathrm{dt} \\
&\qquad\qquad = \int_0^T \int_\Omega \varrho_n \mathcal{R}_{\mathcal{M}_n}(\boldsymbol{u}_n) \cdot \nabla\varphi \,\mathrm{d}\boldsymbol{x} \,\mathrm{dt} + R_1 + R_2
\end{aligned}
$$

where the remainder R_1 and R_2 are given by

$$
< R_1, \varphi >= \delta t \sum_{m=1}^{N} \sum_{j=1}^{3} \int_\Omega \varrho^m \eth_j (u_j^m) (\mathcal{R}_{\mathcal{M}}^{(j)}(\widetilde{\mathcal{P}}_{\mathcal{E}}^{(j)}(\varphi^m)) - \varphi^m) \,\mathrm{d}\boldsymbol{x},
$$

$$
< R_2, \varphi >= \delta t \sum_{m=1}^{N} \sum_{j=1}^{3} \int_\Omega \varrho^m \mathcal{R}_{\mathcal{M}}^{(j)}(u_j^m) (\eth_j \widetilde{\mathcal{P}}_{\mathcal{E}}^{(j)}(\varphi^m) - \partial_j \varphi^m) \,\mathrm{d}\boldsymbol{x}.
$$

A straightforward computation gives

$$
|R_1| \le C_\varphi h_{\mathcal{M}} \|\varrho\|_{L^2((0,T)\times\Omega)} \|\boldsymbol{u}\|_{L^2((0,T)\times\Omega)^3}
$$

and

$$
|R_2| \le C_\varphi h_{\mathcal{M}} \|\varrho\|_{L^2((0,T)\times\Omega)} \| \mathcal{R}_{\mathcal{M}} \boldsymbol{u}\|_{L^2((0,T)\times\Omega)^3}.
$$

Using (4.1b) and (4.9) we get $R_1, R_2 \to 0$ as $n \to \infty$.

By virtue (4.1b), (4.8) and (5.2) we have

$$
\varrho_n(\boldsymbol{u}_n - \mathcal{R}_{\mathcal{M}_n}(\boldsymbol{u}_n)) \to 0 \text{ in } L^1((0,T)\times\Omega) \text{ as } n \to \infty.
$$

Consequently, by virtue of the weak convergence of $\varrho_n \boldsymbol{u}_n$ towards $\varrho \boldsymbol{u}$ in , we obtain

$$\int_0^T \int_\Omega \varrho_n \mathcal{R}_{\mathcal{M}_n}(\boldsymbol{u}_n) \cdot \nabla \varphi \, \mathrm{d}\boldsymbol{x} \, \mathrm{dt} \to \int_0^T \int_\Omega \varrho \boldsymbol{u} \cdot \nabla \varphi \, \mathrm{d}\boldsymbol{x} \, \mathrm{dt} \ \text{ as } n \to \infty.$$

Summing up the previous limits and passing to the limit in (5.3) we obtain that $\boldsymbol{u}, \varrho$ satisfy (2.1) for $\varphi \in C_c^\infty([0,T) \times \overline{\Omega})$.

Let us then prove point *6. i.e.* that $\boldsymbol{u}$, p satisfy (1.1b) in the weak sense.

Let $\boldsymbol{\psi}$ be a function of $\mathrm{C}_c^\infty((0,T) \times \Omega)^3$.

Taking $\widetilde{\mathcal{P}}_{\mathcal{E}_n}(\boldsymbol{\psi}(t))$ as a test function in (3.26) and integrating over $(0,T)$ we infer:

$$\mu \int_0^T \int_\Omega \nabla_{\mathcal{E}_n} \boldsymbol{u}_n : \nabla_{\mathcal{E}_n} [\widetilde{\mathcal{P}}_{\mathcal{E}_n}(\boldsymbol{\psi}(t))] \, \mathrm{d}\boldsymbol{x} \, \mathrm{dt} + (\mu + \lambda) \int_0^T \int_\Omega \mathrm{div}_{\mathcal{M}_n} \boldsymbol{u}_n \, \mathrm{div}_{\mathcal{M}_n} [\widetilde{\mathcal{P}}_{\mathcal{E}_n}(\boldsymbol{\psi}(t))] \, \mathrm{d}\boldsymbol{x} \, \mathrm{dt} \\ - \int_0^T \int_\Omega p_n \, \mathrm{div}_{\mathcal{M}_n} [\widetilde{\mathcal{P}}_{\mathcal{E}_n}(\boldsymbol{\psi}(t))] \, \mathrm{d}\boldsymbol{x} \, \mathrm{dt} = 0.$$

The convergence of the first term may be proven by slight modifications of a classical result [10, Chapter III]:

$$\lim_{n \to \infty} \int_0^T \int_\Omega \nabla_{\mathcal{E}_n} \boldsymbol{u}_n : \nabla_{\mathcal{E}_n} [\widetilde{\mathcal{P}}_{\mathcal{E}_n}(\boldsymbol{\psi}(t))] \, \mathrm{d}\boldsymbol{x} \, \mathrm{dt} = \int_0^T \int_\Omega \nabla \boldsymbol{u} : \nabla \boldsymbol{\psi} \, \mathrm{d}\boldsymbol{x} \, \mathrm{dt} \, .$$

From the definition of $\boldsymbol{\varphi}_n$ and thanks to the L^2 weak convergence of the pressure, we have:

$$\int_0^T \int_\Omega p_n \, \mathrm{div}_{\mathcal{M}_n} [\widetilde{\mathcal{P}}_{\mathcal{E}_n}(\boldsymbol{\psi}(t))] \, \mathrm{d}\boldsymbol{x} \, \mathrm{dt} = \int_0^T \int_\Omega p_n \, \mathrm{div} \boldsymbol{\psi} \, \mathrm{d}\boldsymbol{x}$$

and therefore

$$\lim_{n \to \infty} \int_0^T \int_\Omega p_n \, \mathrm{div}_{\mathcal{M}_n} [\widetilde{\mathcal{P}}_{\mathcal{E}_n}(\boldsymbol{\psi})(t))] \, \mathrm{d}\boldsymbol{x} \, \mathrm{dt} = \int_0^T \int_\Omega p \, \mathrm{div} \boldsymbol{\psi} \, \mathrm{d}\boldsymbol{x} \, \mathrm{dt} \, .$$

By virtue of the L^2 weak convergence of $\mathrm{div}_{\mathcal{M}_n} \boldsymbol{u}_n$, we have also:

$$\lim_{n \to \infty} \int_0^T \int_\Omega \mathrm{div}_{\mathcal{M}_n} [\boldsymbol{u}_n(t)] \, \mathrm{div}_{\mathcal{M}_n} [\widetilde{\mathcal{P}}_{\mathcal{E}_n}(\boldsymbol{\psi}(t))] \, \mathrm{d}\boldsymbol{x} \, \mathrm{dt} = \int_0^T \int_\Omega \mathrm{div} \, \boldsymbol{u} \, \mathrm{div} \boldsymbol{\psi} \, \mathrm{d}\boldsymbol{x} \, \mathrm{dt} \, .$$

Finally $\boldsymbol{u}, p$ satisfy point *6.* and the proof of Proposition 2 is complete. □

5.1. *Passing to the limit in the equation of the state*

The goal of this part is to pass to the limit in the nonlinearity (3.5*c*). The aim is to prove that sequence the $(\varrho_n)_{n \in \mathbb{N}}$ converges to ϱ in $\mathrm{L}^1((0,T) \times \Omega)$.

5.1.1. *The effective viscous flux*

To overtake this difficulty in the continuous case we introduce the quantity $p(\varrho) - (\lambda + 2\mu) \, \mathrm{div} \, \boldsymbol{u}$ usually called the effective viscous flux. The effective viscous flux enjoys many remarkable properties for which we refer to Hoff [33], Lions [41], or Serre [43]. Note that this quantity is nothing other than the amplitude of the normal viscous stress augmented by the hydrostatic pressure p, that is, the "real" pressure acting on a volume element of the fluid. The passage to the limit on the effective flux is stated in Proposition 4 below; it is a discrete version of [13, Proposition 6.1], which necessitates some preliminary lemmas which we now state.

5.1.2. On the discrete Laplace equation

First of all, we introduce a modification of the discrete gradient which, contrary to $\nabla_{\mathcal{E}}$, takes account the value of a function $w \in L_{\mathcal{M}}$ at the external faces. It reads:

$$\overline{\nabla}_{\mathcal{E}} : \left| \begin{array}{l} L_{\mathcal{M}} \longrightarrow \mathbf{H}_{\mathcal{E}} \\ w \longmapsto \overline{\nabla}_{\mathcal{E}} w \\ \overline{\nabla}_{\mathcal{E}} w(\boldsymbol{x}) = (\overline{\eth}_1 w(\boldsymbol{x}), \ldots, \overline{\eth}_d w(\boldsymbol{x}))^t, \end{array} \right. \tag{5.5}$$

where $\overline{\eth}_i w \in H_{\mathcal{E}}^{(i)}$ is the discrete derivative of w in the i-th direction, defined by:

$$\overline{\eth}_i w(\boldsymbol{x}) = \begin{cases} \frac{|\sigma|}{|D_\sigma|} (w_L - w_K) & \forall \boldsymbol{x} \in D_\sigma, \text{ for } \sigma = \overrightarrow{K|L} \in \mathcal{E}_{\text{int}}^{(i)}, \\ \\ -\frac{|\sigma|}{|D_\sigma|} w_K \boldsymbol{n}_{K,\sigma} \cdot \boldsymbol{e}_i & \forall \boldsymbol{x} \in D_\sigma, \text{ for } \sigma \in \mathcal{E}(K) \cap \mathcal{E}_{\text{ext}}^{(i)}. \end{cases} \tag{5.6}$$

The discrete curl operator of a function $\boldsymbol{v} = (v_1, ..., v_d) \in \mathbf{H}_{\mathcal{E}}$ is defined as follows :

$$\operatorname{curl}_{\mathcal{M}} \boldsymbol{v} = \begin{cases} \eth_1 v_2 - \eth_2 v_1 & \text{if } d = 2, \\ \\ \left(\eth_2 v_3 - \eth_3 v_2, \eth_3 v_1 - \eth_1 v_3, \eth_1 v_2 - \eth_2 v_1\right) & \text{if } d = 3, \end{cases} \tag{5.7}$$

where the functions $(\eth_j v_i)_{1 \le i,j \le d}$ are introduced in (3.22). More precisely, in (3.22), the quantities $(\eth_j v_i)_{1 \le i,j \le d}$ are defined for $\boldsymbol{v} \in \mathbf{H}_{\mathcal{E},0}$; they are naturally extended here to the case $\boldsymbol{v} \in \mathbf{H}_{\mathcal{E}}$.

Lemma 9. *Let $\mathcal{D} = (\mathcal{M}, \mathcal{E})$ be a MAC grid, $(\boldsymbol{v}, \boldsymbol{w}) \in (\mathbf{H}_{\mathcal{E},0})^2$. Then the following discrete identity holds:*

$$\int_\Omega \nabla_{\mathcal{E}} \boldsymbol{v} : \nabla_{\mathcal{E}} \boldsymbol{w} \, \mathrm{d}\boldsymbol{x} = \int_\Omega \operatorname{div}_{\mathcal{M}} \boldsymbol{v} \operatorname{div}_{\mathcal{M}} \boldsymbol{w} \, \mathrm{d}\boldsymbol{x} + \int_\Omega \operatorname{curl}_{\mathcal{M}} \boldsymbol{v} \cdot \operatorname{curl}_{\mathcal{M}} \boldsymbol{w} \, \mathrm{d}\boldsymbol{x}. \tag{5.8}$$

We define the discrete Laplace operator on the primal mesh by:

$$-\Delta_{\mathcal{M}} : \left| \begin{array}{l} L_{\mathcal{M}} \longrightarrow L_{\mathcal{M}} \\ w \longmapsto -\Delta_{\mathcal{M}} w \end{array} \right.$$

$$-\Delta_{\mathcal{M}} w(\boldsymbol{x}) = \frac{1}{|K|} \sum_{\sigma \in \mathcal{E}(K)} \phi_{K,\sigma}(w), \quad \forall \boldsymbol{x} \in K, \ \forall K \in \mathcal{M}, \tag{5.9}$$

where

$$\phi_{K,\sigma}(w) = \begin{cases} \dfrac{|\sigma|}{d_{KL}} (w_K - w_L) & \text{if } \sigma = K|L \in \mathcal{E}_{\text{int}}, \\ \\ \dfrac{|\sigma|}{d_{K,\sigma}} w_K & \text{if } \sigma \in \mathcal{E}_{\text{ext}} \cap \mathcal{E}(K) \end{cases} \tag{5.10}$$

where $d_{KL} = d(\boldsymbol{x}_K, \boldsymbol{x}_L)$ and $d_{K,\sigma} = d(\boldsymbol{x}_K, \sigma)$. We then introduce the following bilinear form on $L_{\mathcal{M}}$:

$$\begin{aligned} &\forall (p,q) \in L^2_{\mathcal{M}}, \qquad \int_\Omega (-\Delta_{\mathcal{M}} p)\, q \,\mathrm{d}\boldsymbol{x} = [p,q]_{1,\mathcal{M}}, \\ &\text{with } [p,q]_{1,\mathcal{M}} = \sum_{\sigma\in\mathcal{E}_{\rm int}, \sigma=K|L} \frac{|\sigma|}{d_{KL}} (p_K - p_L)\,(q_K - q_L) + \sum_{\sigma\in\mathcal{E}_{\rm int}, \sigma\subset K} \frac{|\sigma|}{d_{K,\sigma}}\, p_K\, q_K. \end{aligned} \tag{5.11}$$

The bilinear form $\left| \begin{array}{l} L_{\mathcal{M}} \times L_{\mathcal{M}} \to \mathbb{R} \\ (p,q) \mapsto [p,q]_{1,\mathcal{M}} \end{array} \right.$ is an inner product on $L_{\mathcal{M}}$ which induces the following norms:

$$\|q\|^2_{1,\mathcal{M}} = [q,q]_{1,\mathcal{M}} = \sum_{\sigma\in\mathcal{E}_{\rm int}, \sigma=K|L} \frac{|\sigma|}{d_{KL}} (q_K - q_L)^2 + \sum_{\sigma\in\mathcal{E}_{\rm int}, \sigma\subset K} \frac{|\sigma|}{d_{K,\sigma}}\, q_K^2. \tag{5.12}$$

Similarly to (3.25) we introduce the following norm

$$q \in L_{\mathcal{M}} \mapsto \|q\|_{-1,\mathcal{M}} = \max\{\left|\int_\Omega q w \,\mathrm{d}\boldsymbol{x}\right| \;;\; w \in L_{\mathcal{M}} \text{ and } \|w\|_{1,\mathcal{M}} \le 1\}. \tag{5.13}$$

The following Proposition states the existence of a solution of the discrete Laplace equation supplemented with the discrete Dirichlet boundary condition.

Proposition 3. *Let $\mathcal{D} = (\mathcal{M}, \mathcal{E})$ be a MAC grid. Then for any $\varrho \in L_{\mathcal{M}}$ there exists only one $w \in L_{\mathcal{M}}$ such that*

$$-\Delta_{\mathcal{M}} w = \varrho. \tag{5.14}$$

Moreover

$$\|w\|_{1,\mathcal{M}} \le \|\varrho\|_{-1,\mathcal{M}}. \tag{5.15}$$

The following Lemma is used to pass to the limit in the nonlinearity (3.5*c*).

Lemma 10. *Let $w \in L_{\mathcal{M}}$. Let $\boldsymbol{v} = -\overline{\nabla}_{\mathcal{E}} w \in \mathbf{H}_{\mathcal{E}}$ be defined by (5.5). Then, with the discrete* curl *operator denoted* $\mathrm{curl}_{\mathcal{M}}$ *defined by (5.7), we have* $\mathrm{curl}_{\mathcal{M}} \boldsymbol{v} = 0$.
Furthermore, if w satisfies $-\Delta_{\mathcal{M}} w = \varrho$ then $\mathrm{div}_{\mathcal{M}} \boldsymbol{v} = \varrho$.

In the following we define an approximation $\varphi_{\mathcal{M}} \in L_{\mathcal{M}}$ of a function $\varphi \in \mathrm{C}^\infty_c(\Omega)$ defined by:

$$\varphi_{\mathcal{M}}(\boldsymbol{x}) = \varphi(\boldsymbol{x}_K) \text{ for all } \boldsymbol{x} \in K, \tag{5.16}$$

and consider for $w \in L_{\mathcal{M}}$, the gradient of the function $w\varphi_{\mathcal{M}} \in L_{\mathcal{M}}$ as defined in (5.5), that is $\overline{\nabla}_{\mathcal{E}}(w\varphi_{\mathcal{M}}) = (\overline{\eth}_1(w\varphi_{\mathcal{M}}), ..., \overline{\eth}_d(w\varphi_{\mathcal{M}}))^t$.

The following Lemma deals with the discrete $H^2_{\rm loc}$ estimates of the solution of (5.14).

Lemma 11. *Let $\mathcal{D} = (\mathcal{M}, \mathcal{E})$ be a MAC grid. Let $\eta > 0$ such that $\eta \le \eta_{\mathcal{M}}$ where $\eta_{\mathcal{M}}$ is defined by (3.3). let $\varrho \in L_{\mathcal{M}}$ and let $w \in L_{\mathcal{M}}$ be the finite volume solution of $-\Delta_{\mathcal{M}} w = \varrho$, with an homogeneous Dirichlet boundary condition,* i.e. *let w be the solution to (5.14). Let $\varphi \in \mathrm{C}^\infty_c(\Omega)$. Then, there exists C_φ only depending on φ, $\eta_{\mathcal{M}}$ and Ω, such that $\|\overline{\nabla}_{\mathcal{E}}(w\varphi_{\mathcal{M}})\|_{1,\mathcal{E},0} \le C_\varphi \|\varrho\|_{\mathrm{L}^2(\Omega)}$ where $\|\cdot\|_{1,\mathcal{E},0}$ is defined in (3.21b).*

Proposition 4 (Weak convergence of the effective viscous flux)**.** *Under the assumptions of Proposition 2, we have for any $(\varphi, \psi) \in \mathrm{C}_c^\infty(\Omega) \times \mathrm{C}_c^\infty(0, T)$,*

$$\lim_{n\to\infty} \int_0^T \int_\Omega (p_n - (2\mu + \lambda)\mathrm{div}_{\mathcal{M}_n} \boldsymbol{u}_n)\varrho_n \varphi\psi \,\mathrm{d}\boldsymbol{x}\,\mathrm{dt} = \int_0^T \int_\Omega (p - (2\mu + \lambda)\mathrm{div}\, \boldsymbol{u})\varrho\varphi\psi \,\mathrm{d}\boldsymbol{x}\,\mathrm{dt}, \tag{5.17}$$

up to a subsequence, if necessary.

Proof. To prove Proposition 4 we proceed in several steps.

Let $\varphi \in \mathrm{C}_c^\infty(\Omega)$. For a MAC grid $\mathcal{D} = (\mathcal{M}, \mathcal{E})$, we define $\varphi_{\mathcal{M}} \in L_{\mathcal{M}}$, $\varphi_{\mathcal{E}}^{(i)} \in H_{\mathcal{E},0}^{(i)}$ by:

$$\begin{cases} \varphi_{\mathcal{M}}(\boldsymbol{x}) = \varphi(\boldsymbol{x}_K), \forall \boldsymbol{x} \in K,\ \forall K \in \mathcal{M}, \\ \\ \varphi_{\mathcal{E}}^{(i)}(\boldsymbol{x}) = \varphi(\boldsymbol{x}_\sigma), \forall \boldsymbol{x} \in D_\sigma,\ \forall \sigma \in \mathcal{E}_{\mathrm{int}}^{(i)}. \end{cases}$$

For a sequence of grids $\mathcal{M}_n$, for short we shall denote $\varphi_n = \varphi_{\mathcal{M}_n}$. We define $w_n(t)$ with (5.14) (with $\mathcal{M}_n$ and $\varrho_n(t)$ instead of $\mathcal{M}$ and ϱ) and $\boldsymbol{v}_n(t)$ with $\boldsymbol{v}_n(t) = -\overline{\nabla}_{\mathcal{E}_n}(w_n(t))$.

Step 1 : $\boldsymbol{v}_n \to \boldsymbol{v}$ in $\mathrm{L}^2(0, T; \mathrm{L}^2_{\mathrm{loc}}(\Omega)^3)$ where $\boldsymbol{v} \in \mathrm{L}^2(0, T; \mathrm{H}^1_{\mathrm{loc}}(\Omega)^3)$.

Denoting $\boldsymbol{v}_n = (v_{n,1}, v_{n,2}, v_{n,3})$ and setting $\boldsymbol{V}_n^\varphi = (v_{n,1}\varphi_{\mathcal{E}_n}^{(1)}, v_{n,2}\varphi_{\mathcal{E}_n}^{(2)}, v_{n,3}\varphi_{\mathcal{E}_n}^{(3)}) \in \boldsymbol{X}_{\mathcal{E}_n,\delta t_n}$, it is sufficient to prove that the sequence $(\boldsymbol{V}_n^\varphi)_{n\in\mathbb{N}}$ converges in $\mathrm{L}^2(0, T; \mathrm{L}^2(\Omega)^3)$ for any $\varphi \in \mathrm{C}_c^\infty(\Omega)$. We will use Theorem 5 with $B = \mathrm{L}^2(\Omega)^3$, $X_n = \mathbf{H}_{\mathcal{E}_n,0}$ and $Y_n = \mathrm{L}^2(\Omega)^3$. Clearly, thanks to (5.15) and the fact that ϱ_n is bounded in $\mathrm{L}^2((0, T) \times \Omega)$, the sequence $(\boldsymbol{V}_n^\varphi)_{n\in\mathbb{N}}$ is bounded in $\mathrm{L}^2(0, T; \mathrm{L}^2(\Omega)^3)$. From Lemma 11 we infer that the sequence $(\|\boldsymbol{V}_n^\varphi\|_{\mathrm{L}^2(0,T;\mathbf{H}_{\mathcal{E}_n,0})})_{n\in\mathbb{N}}$ is bounded.

Now we can write $\eth_t \boldsymbol{V}_n^\varphi = (\varphi_{\mathcal{E}_n}^{(1)}\eth_t v_{n,1}, \varphi_{\mathcal{E}_n}^{(2)}\eth_t v_{n,2}, \varphi_{\mathcal{E}_n}^{(3)}\eth_t v_{n,3})$. Consequently

$$\|\eth \boldsymbol{V}_n^\varphi(t)\|_{Y_n} \le C_\varphi \,\|\eth_t \boldsymbol{v}_n(t)\|_{Y_n} \le C_\varphi \|\overline{\nabla}_{\mathcal{E}_n}(\eth w_n(t))\|_{Y_n}$$

Finally by virtue of (5.15) and (4.15) we obtain the existence of C only depending on such that, for any $n \in \mathbb{N}$,

$$\|\eth_t \boldsymbol{V}_n^\varphi\|_{\mathrm{L}^1(0,T;Y_n)} \le C. \tag{5.18}$$

Noting that the sequence $(\mathbf{H}_{\mathcal{E}_n,0})_{n\in\mathbb{N}}$ is $\mathrm{L}^2(\Omega)^d$-limit included in $\mathrm{H}^1_0(\Omega)^d$ in the sense of Definition 5, thanks to the discrete Aubin-Simon theorem 5 and to the regularity of the limit, Theorem 6, there exists $\boldsymbol{v} \in \mathrm{L}^2(0, T; \mathrm{H}^1_{\mathrm{loc}}(\Omega)^3)$ such that $\boldsymbol{v}_n \to \boldsymbol{v}$ in $\mathrm{L}^2(0, T; \mathrm{L}^2_{\mathrm{loc}}(\Omega)^3)$.

Step 2 : Conclusion of the proof of Proposition 4.

Since $\boldsymbol{V}_n^\varphi(t) \in \mathbf{H}_{\mathcal{E},0}$, it is possible to take $\boldsymbol{v} = \psi(t)\boldsymbol{V}_n^\varphi(t)$ in (3.26):

$$(2\mu + \lambda) \int_\Omega \psi(t)\mathrm{div}_{\mathcal{M}_n} [\boldsymbol{u}_n(t)]\, \mathrm{div}_{\mathcal{M}_n} [\boldsymbol{V}_n^\varphi(t)]\, \mathrm{d}\boldsymbol{x} + \mu \int_\Omega \psi(t)\mathrm{curl}_{\mathcal{M}_n}[\boldsymbol{u}_n(t)]\, \mathrm{curl}_{\mathcal{M}_n} [\boldsymbol{V}_n^\varphi(t)]\, \mathrm{d}\boldsymbol{x} \\ - \int_\Omega \psi(t)p_n(t)\mathrm{div}_{\mathcal{M}} [\boldsymbol{V}_n^\varphi(t)]\, \mathrm{d}\boldsymbol{x} = 0. \tag{5.19}$$

Since $\mathrm{div}_{\mathcal{M}_n}[\boldsymbol{v}_n(t)] = \varrho_n(t)$, we first remark that:

$$\int_\Omega \psi(t)\mathrm{div}_{\mathcal{M}_n}[\boldsymbol{u}_n(t)]\,\mathrm{div}_{\mathcal{M}_n}[\boldsymbol{V}_n^\varphi(t)]\,\mathrm{d}\boldsymbol{x} = \int_\Omega \psi(t)\mathrm{div}_{\mathcal{M}_n}[\boldsymbol{u}_n(t)]\varrho_n(t)\varphi\,\mathrm{d}\boldsymbol{x} + \int_\Omega \psi(t)\mathrm{div}_{\mathcal{M}_n}[\boldsymbol{u}_n(t)]\boldsymbol{v}_n(t)\cdot\boldsymbol{\nabla}\varphi\,\mathrm{d}\boldsymbol{x} + R_{1,n}(t), \quad (5.20)$$

where $\lim_{n\to\infty}\|R_{1,n}\|_{L^1(0,T)} = 0$, thanks to the discrete $L^2(0,T;\mathrm{H}^1(\Omega))$-estimate on $\boldsymbol{u}_n$ and the $L^2(0,T;\mathrm{L}^2_{loc}(\Omega))$ estimate on $\boldsymbol{v}_n$. Replacing the quantity $\mathrm{div}_{\mathcal{M}_n}[\boldsymbol{u}_n(t)]$ by $p_n(t)$, the same computation gives:

$$\int_\Omega \psi(t)p_n(t)\,\mathrm{div}_{\mathcal{M}_n}[\boldsymbol{V}_n^\varphi(t)]\,\mathrm{d}\boldsymbol{x} = \int_\Omega \psi(t)p_n(t)\rho_n(t)\varphi\,\mathrm{d}\boldsymbol{x} + \int_\Omega \psi(t)p_n(t)\boldsymbol{v}_n(t)\cdot\boldsymbol{\nabla}\varphi\,\mathrm{d}\boldsymbol{x} + R_{2,n}(t), \quad (5.21)$$

where $\lim_{n\to\infty}\|R_{2,n}\|_{L^1(0,T)} = 0$. Following [11], the second term of (5.19) can be transformed as follows:

$$\int_\Omega \psi(t)\mathrm{curl}_{\mathcal{M}_n}[\boldsymbol{u}_n(t)]\,\mathrm{curl}_{\mathcal{M}_n}[\boldsymbol{V}_n^\varphi(t)]\,\mathrm{d}\boldsymbol{x} = \int_\Omega \psi(t)\mathrm{curl}_{\mathcal{M}_n}[\boldsymbol{u}_n(t)]\mathrm{curl}_{\mathcal{M}_n}[\boldsymbol{v}_n(t)]\varphi\,\mathrm{d}\boldsymbol{x} + \int_\Omega \psi(t)\mathrm{curl}_{\mathcal{M}_n}[\boldsymbol{u}_n(t)]\cdot L(\varphi)\overline{\boldsymbol{v}}_n\,\mathrm{d}\boldsymbol{x} + R_{3,n}(t)$$

where: .1cm

$L(\varphi)$ is the same matrix involving the first order derivatives

$\overline{\boldsymbol{v}}_n$ is a convex combination of the values of $\boldsymbol{v}_n$ and is such that $\overline{\boldsymbol{v}}_n \to \boldsymbol{v}$ in $L^2(0,T;L^2_{\mathrm{loc}}(\Omega))$. Note that the coefficients in the convex combination involve the ratios of cell diameters, and the proof of convergence requires the assumption $\eta \le \eta_{\mathcal{M}_n}$, see [11, Proposition 7.4] for the complete expression and proof.

$\lim_{n\to\infty}\|R_{3,n}\|_{L^1(0,T)} = 0$ (for the same reason as $R_{1,n}$).

By Lemma 10, $\mathrm{curl}_{\mathcal{M}_n}[\boldsymbol{v}_n(t)] = 0$ we obtain

$$\int_\Omega \psi(t)\mathrm{curl}_{\mathcal{M}_n}[\boldsymbol{u}_n(t)]\,\mathrm{curl}_{\mathcal{M}_n}[\tilde{\boldsymbol{v}}_n(t)]\,\mathrm{d}\boldsymbol{x} = \int_\Omega \psi(t)\mathrm{curl}_{\mathcal{M}_n}[\boldsymbol{u}_n(t)]\cdot L(\varphi)\overline{\boldsymbol{v}}_n(t)\,\mathrm{d}\boldsymbol{x} + R_{3,n}(t).$$

Note that the expression of $\overline{\boldsymbol{v}}_n$ can be found in [11] and the convergence of $\overline{\boldsymbol{v}}_n$ previously stated is a consequence of the fact that $\eta \le \eta_{\mathcal{M}_n}$ (see also [11]).

Combining the previous estimate and integrating over the time we obtain

$$\int_0^T\int_\Omega \psi(t)\varrho_n(t)\varphi\Big((2\mu+\lambda)\mathrm{div}_{\mathcal{M}_n}[\boldsymbol{u}_n(t)] - p_n(t)\Big)\,\mathrm{d}\boldsymbol{x}\,\mathrm{dt} + \int_0^T\int_\Omega \psi(t)\Big((2\mu+\lambda)\mathrm{div}_{\mathcal{M}_n}[\boldsymbol{u}_n(t)] - p_n(t)\Big)\boldsymbol{v}_n(t)\cdot\boldsymbol{\nabla}\varphi\,\mathrm{d}\boldsymbol{x}\,\mathrm{dt} + \mu\int_0^T\int_\Omega \psi(t)\mathrm{curl}_{\mathcal{M}_n}[\boldsymbol{u}_n(t)]L(\varphi)\overline{\boldsymbol{v}}_n(t)\,\mathrm{d}\boldsymbol{x}\,\mathrm{dt} + \mathcal{R}_n = 0, \quad (5.22)$$

where $\lim_{n\to+\infty}\mathcal{R}_n = 0$. Passing to the limit in (5.22) we obtain

$$\lim_{n\to+\infty}\int_0^T\int_\Omega \psi(t)\varrho_n(t)\varphi\Big((2\mu+\lambda)\mathrm{div}_{\mathcal{M}_n}[\boldsymbol{u}_n(t)] - p_n(t)\Big)\,\mathrm{d}\boldsymbol{x}\,\mathrm{dt} = \int_0^T\int_\Omega \psi(t)\Big(p(t) - (2\mu+\lambda)\mathrm{div}[\boldsymbol{u}(t)]\Big)\boldsymbol{v}(t)\cdot\boldsymbol{\nabla}\varphi\,\mathrm{d}\boldsymbol{x}\,\mathrm{dt} - \int_0^T\int_\Omega \mu\psi(t)\mathrm{curl}[\boldsymbol{u}(t)]\cdot L(\varphi)\boldsymbol{v}(t)\,\mathrm{d}\boldsymbol{x}\,\mathrm{dt}\,. \quad (5.23)$$

Since $(\boldsymbol{u}, p, \varrho)$ satisfy the momentum equation (see Proposition 2) we infer, since $(\boldsymbol{u}, \boldsymbol{v}) \in L^2(0,T; H^1_0(\Omega)^3) \times L^2(0,T; H^1_{\rm loc}(\Omega)^3)$,

$$\int_0^T \int_\Omega \psi\Big((2\mu+\lambda)\mathrm{div}\boldsymbol{u} - p\Big)\,\mathrm{div}(\varphi\boldsymbol{v})\,\mathrm{d}\boldsymbol{x}\,\mathrm{dt} \;+\mu \int_0^T \int_\Omega \psi\mathrm{curl}(\boldsymbol{u})\cdot\mathrm{curl}(\varphi\boldsymbol{v})\,\mathrm{d}\boldsymbol{x}\,\mathrm{dt} = 0.$$

But thanks to the discrete $L^2(0,T; H^1_{\rm loc}(\Omega)^3)$ on $\boldsymbol{v}_n$, it is quite easy to prove that $\mathrm{div}_{\mathcal{M}_n}[\boldsymbol{v}_n]$ and $\mathrm{curl}_{\mathcal{M}_n}[\boldsymbol{v}_n]$ converge weakly in $L^2(0,T; L^2_{\rm loc}(\Omega))$ towards $\mathrm{div}\boldsymbol{u}$ and $\mathrm{curl}\boldsymbol{v}$. This gives $\mathrm{div}[\boldsymbol{v}(t)] = \varrho(t)$ and $\mathrm{curl}[\boldsymbol{v}(t)] = 0$ and therefore

$$\begin{aligned}\int_0^T \int_\Omega \psi\Big(p - (2\mu+\lambda)\mathrm{div}\boldsymbol{u}\Big)\boldsymbol{v}\cdot\nabla\varphi\,\mathrm{d}\boldsymbol{x}\,\mathrm{dt}& \\ = \int_0^T \int_\Omega ((2\mu+\lambda)\mathrm{div}\boldsymbol{u} - p\Big)\psi\varphi\varrho\,\mathrm{d}\boldsymbol{x}\,\mathrm{dt} &+\mu \int_0^T \int_\Omega \psi\mathrm{curl}[\boldsymbol{u}]L(\varphi)\boldsymbol{v}\,\mathrm{d}\boldsymbol{x}\,\mathrm{dt}\,. \end{aligned} \tag{5.24}$$

We obtain the expected result by combining (5.23) and (5.24).

□

5.1.3. A.e. and strong convergence of ϱ_n and p_n

Let us now prove the a.e. convergence of ϱ_n and p_n. First of all we respectively denote $\overline{\varrho^{\gamma+1}}$, $\overline{\varrho^\gamma}$, $\overline{\varrho\ln\varrho}$ and $\overline{\varrho\,\mathrm{div}\,\boldsymbol{u}}$ the weak limits of the sequences $(\varrho_n^{\gamma+1})_{n\in\mathbb{N}}$, $(\varrho_n^{\gamma})_{n\in\mathbb{N}}$, $(\varrho_n\ln\varrho_n)_{n\in\mathbb{N}}$ and $(\varrho_n\,\mathrm{div}_{\mathcal{M}_n}\,\boldsymbol{u}_n)_{n\in\mathbb{N}}$ in a suitable $\mathrm{L}^p((0,T)\times\Omega)$ space where $p > 1$, passing to subsequences if necessary. Using [12, Lemma 2.1], one has for any $s \in (0,T)$:

$$\lim_{n\to\infty} \int_0^s \int_\Omega (p_n - (2\mu+\lambda)\,\mathrm{div}_{\mathcal{M}_n}\,\boldsymbol{u}_n)\varrho_n\,\mathrm{d}\boldsymbol{x}\,\mathrm{dt} = \int_0^s \int_\Omega (p - (2\mu+\lambda)\,\mathrm{div}\,\boldsymbol{u})\varrho\,\mathrm{d}\boldsymbol{x}\,\mathrm{dt}$$

More precisely, with the notation introduced above we have for any $s \in (0,T)$

$$\int_0^s \int_\Omega \overline{\varrho^{\gamma+1}} - (2\mu+\lambda)\overline{\varrho\,\mathrm{div}\,\boldsymbol{u}}\,\mathrm{d}\boldsymbol{x}\,\mathrm{dt} = \int_0^s \int_\Omega \overline{\varrho^\gamma}\varrho - (2\mu+\lambda)\varrho\,\mathrm{div}\,\boldsymbol{u}\,\mathrm{d}\boldsymbol{x}\,\mathrm{dt}\,. \tag{5.25}$$

As the result of the DiPerna-Lions theory (see [42, Lemma 6.9]), $(\varrho, \boldsymbol{u})$ extended by zero outside Ω satisfies

$$\partial_t(\varrho\ln(\varrho)) + \mathrm{div}(\varrho\ln(\varrho)\boldsymbol{u}) + \varrho\,\mathrm{div}\,\boldsymbol{u} = 0 \text{ in } \mathcal{D}'((0,T)\times\mathbb{R}^3).$$

Consequently the function $t \to \varrho(t)\ln(\varrho(t))$ is continuous with values in some Lebesgue space equipped with the weak topology and we can use the previous equation to obtain for almost all $s \in (0,T)$:

$$\int_\Omega (\varrho\ln(\varrho))(s)\,\mathrm{d}\boldsymbol{x} - \int_\Omega \varrho_0\ln(\varrho_0)\,\mathrm{d}\boldsymbol{x} + \int_0^s \int_\Omega \varrho\,\mathrm{div}\,\boldsymbol{u}\,\mathrm{d}\boldsymbol{x}\,\mathrm{dt} = 0 \tag{5.26}$$

Moreover from (4.5) with $B(t) = t\ln(t)$ we obtain for almost all $s \in (0,T)$:

$$\int_\Omega (\varrho_n\ln(\varrho_n))(s)\,\mathrm{d}\boldsymbol{x} - \int_\Omega \varrho_0\ln(\varrho_0)\,\mathrm{d}\boldsymbol{x} + \int_0^s \int_\Omega \varrho_n\,\mathrm{div}_{\mathcal{M}_n}\,\boldsymbol{u}_n\,\mathrm{d}\boldsymbol{x}\,\mathrm{dt} \le 0.$$

Passing to the limit in the previous inequation we infer that

$$\int_\Omega \overline{\varrho\ln(\varrho)}(s)\,\mathrm{d}\boldsymbol{x} - \int_\Omega \varrho_0\ln(\varrho_0)\,\mathrm{d}\boldsymbol{x} + \int_0^s \int_\Omega \overline{\varrho\,\mathrm{div}\,\boldsymbol{u}}\,\mathrm{d}\boldsymbol{x}\,\mathrm{dt} \le 0. \tag{5.27}$$

Substracting (5.26) from (5.27) gives

$$\int_\Omega (\varrho \ln(\varrho))(s) - \overline{\varrho \ln(\varrho)}(s)\, \mathrm{d}\boldsymbol{x} \leq - \int_0^s \int_\Omega \overline{\varrho \operatorname{div} \boldsymbol{u}} - \varrho \operatorname{div} \boldsymbol{u} \, \mathrm{d}\boldsymbol{x} \, \mathrm{dt} \tag{5.28}$$

But by virtue of 5.25 we have

$$\int_0^s \int_\Omega \overline{\varrho \operatorname{div} \boldsymbol{u}} - \varrho \operatorname{div} \boldsymbol{u} \, \mathrm{d}\boldsymbol{x} \, \mathrm{dt} = \frac{1}{2\mu + \lambda} \int_0^s \int_\Omega \overline{\varrho^{\gamma+1}} - \overline{\varrho^\gamma} \varrho \, \mathrm{d}\boldsymbol{x} \, \mathrm{dt} \tag{5.29}$$

where the last inequality follows as in [13], so the following relation holds:

$$\overline{\varrho \ln \varrho} = \varrho \ln \varrho \text{ a.e in } (0, T) \times \Omega.$$

From Lemma 12 we obtain $\varrho_n \to \varrho$ a.e in $\{\varrho > 0\}$. Consequently $\varrho_n \to \varrho$ in $\mathrm{L}^1((0,T) \times \Omega)$ and the following convergences hold

$$\varrho_n \to \varrho \text{ strongly in } \mathrm{L}^q((0,T) \times \Omega), \text{ for any } q \in [1, 2\gamma),$$

$$p_n = \varrho_n^\gamma \to \varrho^\gamma \text{ strongly in } \mathrm{L}^q((0,T) \times \Omega), \text{ for any } q \in [1, 2),$$

which gives $p = \varrho^\gamma$ a.e in $(0, T) \times \Omega$. We have thus proved that the pressure p and the density ϱ satisfy the equation of state (1.1c). To conclude the proof of Theorem 1, there only remains to prove the energy inequality.

5.2. Passing to the limit in the energy inequality

Since $\varrho_n \to \varrho$ in $\mathrm{L}^\gamma((0,T) \times \Omega)$ we infer that

$$\int_\Omega \varrho_n^\gamma(t) \, \mathrm{d}\boldsymbol{x} \to \int_\Omega \varrho^\gamma(t) \, \mathrm{d}\boldsymbol{x}, \text{ a.e on } (0, T). \tag{5.30}$$

Morover $\nabla_{\mathcal{E}_n} \boldsymbol{u}_n$ tends to ∇u weakly in $\mathrm{L}^2((0,T) \times \Omega)^{3\times3}$. Therefore (2.3) is obtained by passing to the limit in (4.4). The proof of Theorem 1 is now complete.

6. Conclusion

In this paper, we considered the MAC scheme for the semi-stationary barotropic compressible Navier-Stokes equations. This scheme, which is very popular in the computational fluid dynamics community, also proved to be quite adapted to a convergence analysis. To our knowledge, the convergence analysis established in this article seems to be the first for this problem (see [37] for an alternative discretization with different boundary conditions). Ongoing work concerns the extension to the stationary and non stationary Navier-Stokes equations in two or three space dimensions.

7. Appendix: some functional and discrete-functional analysis results

The following discrete Sobolev inequality is a direct consequence of [10, Lemma 9.1].

Theorem 2 (Discrete Sobolev inequality)**.** *Let Ω be an open bounded subset of $\mathbb{R}^d$, $d = 2$ or $d = 3$, adapted to the MAC-scheme (that is any finite union of rectangles in 2D or rectangular in 3D), and let $\mathcal{D} = (\mathcal{M}, \mathcal{E})$ be a MAC scheme of Ω. Let $q < +\infty$ if $d = 2$ and $q = 6$ if $d = 3$.Then there exists $C = C(q, \Omega)$ such that, for all $u \in \mathbf{H}_{\mathcal{E},0}$,*

$$\|\boldsymbol{u}\|_{L^q(\Omega)} \leq C\|\boldsymbol{u}\|_{1,\mathcal{E},0}.$$

The following estimate on the translates of a discrete function $\boldsymbol{u}$ as a function of $\|\boldsymbol{u}\|_{1,\mathcal{E},0}$ is a consequence of [10, Lemma 9.3].

Proposition 5 (Estimate on the translates)**.** *Let Ω be an open bounded subset of $\mathbb{R}^d$, $d = 2$ or $d = 3$, adapted to the MAC-scheme (that is any finite union of rectangles in 2D or rectangular in 3D), and let $\mathcal{D} = (\mathcal{M}, \mathcal{E})$ be a MAC grid of Ω of size $h_{\mathcal{M}}$. Let $\boldsymbol{u} \in \mathbf{H}_{\mathcal{E},0}$. We denote by $\tilde{\boldsymbol{u}}$ the extension of $\boldsymbol{u}$ to $\mathbb{R}^d$. Then the following estimate holds:*

$$\forall \boldsymbol{\eta} \in \mathbb{R}^d,\ \|\tilde{\boldsymbol{u}}(\cdot + \boldsymbol{\eta}) - \tilde{\boldsymbol{u}}\|^2_{L^2(\mathbb{R}^d)^d} \leq C\|\boldsymbol{\eta}\|\Big(\|\boldsymbol{\eta}\| + h_{\mathcal{M}}\Big)\|\boldsymbol{u}\|^2_{1,\mathcal{E},0}. \tag{7.1}$$

where $C \geq 0$ depends only on Ω.

8. Some functional analysis results

For the convenience of the reader we list some functional analysis results to be used throughout this article.

The first Lemma is a classical result on weak limits and convexity, see e.g. [16, Lemma 10.20].

Lemma 12. *(Let O be a bounded open subset of $\mathbb{R}^M$, $M \geq 1$. Suppose $g : \mathbb{R} \to (-\infty, \infty]$ is a lower semicontinuous convex function and $(f_n)_{n\in\mathbb{N}}$ is a sequence of functions on O for which $f_n \to f$ weakly in $\mathrm{L}^1(O)$, $g(f_n) \in \mathrm{L}^1(O)$ for each n, $g(f_n) \to \overline{g(f)}$ weakly in $\mathrm{L}^1(O)$. Then $g(f) \leq \overline{g(f)}$ a.e in O, $g(f) \in \mathrm{L}^1(O)$ and $\int_O g(f)\,\mathrm{d}\boldsymbol{x} \leq \liminf_{n\to\infty} \int_O g(f_n)\,\mathrm{d}\boldsymbol{x}$. If in addition g is stricly convex on an open interval $(a, b) \subset \mathbb{R}$ and $g(f) = \overline{g(f)}$ a.e in O, then, passing to a subsequence if necessary, $f_n(y) \to f(y)$ for a.e $y \in \{y \in O, f(y) \in (a, b)\}$.*

The following lemma is used in the passage to the limit in the discrete continuity equation.

Lemma 13. *For $n \in \mathbb{N}$, consider a partition I_n of the time interval $[0, T]$, assumed to be uniform for the sake of simplicity. Let the constant time step δt_n be such that $\delta t_n \to 0$ as $n \to +\infty$. Let $1 < p_1, q_1 < +\infty$, $1 \leq p_2, q_2 \leq +\infty$ such that $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{q_1} + \frac{1}{q_2} = 1$. For any $n \in \mathbb{N}$, let $(f_n, \boldsymbol{g}_n) \in L^{p_1}(0, T; L^{q_1}(\Omega)) \times L^{p_2}(0, T; L^{q_2}(\Omega))$. Assume that:*

1. *f_n and $\boldsymbol{g}_n$ are constant on each interval of the partition I_n.*
2. *The sequences $(f_n)_{n\in\mathbb{N}}$ and $(\boldsymbol{g}_n)_{n\in\mathbb{N}}$ converge weakly respectively to f and $\boldsymbol{g}$ in $\mathrm{L}^{p_1}(0, T; \mathrm{L}^{q_1}(\Omega))$ and $\mathrm{L}^{p_2}(0, T; \mathrm{L}^{q_2}(\Omega)^d)$.*
3. *The sequence $\Big(t \to \frac{f(t,\cdot - \delta t_n) - f(t)}{\delta t_n}\Big)_{n\in\mathbb{N}}$ is bounded in $\mathrm{L}^1(0, T; W^{-1,1}(\Omega))$.*

Then $f_n\boldsymbol{g}_n \to f\boldsymbol{g}$ in the sense of distributions on $(0, T) \times \Omega$.

Proof. We follow the proof of [38, Lemma 2.3]. Let

$$f_n = \sum_{p=1}^{N_n} f_n^p \mathbb{1}_{(t^{p-1},t^p)}(t), \quad \text{and} \quad \boldsymbol{g}_n = \sum_{p=1}^{N_n} \boldsymbol{g}_n^p \mathbb{1}_{(t^{p-1},t^p)}(t), \tag{8.1}$$

we notice that by virtue of item 3,

$$f_n - \tilde{f}_n \to 0 \text{ in } \mathcal{D}'((0,T)\times\Omega), \tag{8.2}$$

where $\tilde{f}_n$ is the piecewise affine function defined by

$$\tilde{f}_n = \sum_{p=1}^{N_n} \Big(f_n^{p-1} + (f_n^p - f_h^{p-1}) \frac{t - t^{p-1}}{\delta t} \Big) \mathbb{1}_{(t^{p-1},t^p)}(t). \tag{8.3}$$

Since the sequence of derivatives $(\partial_t \tilde{f}_n)_{n\in\mathbb{N}}$ is bounded in $\mathrm{L}^1(0,T;W^{-1,1}(\Omega))$ and using item 4, an application of [41, Lemma 5.1] gives

$$\tilde{f}_n \boldsymbol{g}_n \to f\boldsymbol{g} \text{ in } \mathcal{D}'((0,T)\times\Omega)^3. \tag{8.4}$$

It only remains to prove that $(f_n - \tilde{f}_n)\boldsymbol{g}_n \to 0$ in $\mathcal{D}'((0,T)\times\Omega)^3$, statement for which we refer to the proof of Lemma 2.3 in [38]. $\square$

We continue with a weak compactness result of the space $\boldsymbol{X}_{\mathcal{E},\delta t}$ into $\mathrm{L}^2(0,T;\mathrm{H}_0^1(\Omega)^d)$.

Theorem 3. *Consider, for any $n\in\mathbb{N}$, a partition I_n of the time interval $[0,T]$, which, for the sake of simplicity, we suppose uniform. Let δt_n be the constant time step going to 0 as $n\to\infty$.*

Consider a sequence of MAC grids $(\mathcal{D}_n = (\mathcal{M}_n, \mathcal{E}_n))_{n\in\mathbb{N}}$ of Ω with step size $h_{\mathcal{M}_n}$ going to zero as $n\to\infty$. Let us consider for any $n\in\mathbb{N}$ a function $\boldsymbol{u}_n \in \boldsymbol{X}_{\mathcal{E}_n,\delta t_n}$ such that the sequence $(\|\boldsymbol{u}_n\|_{L^2(0,T;\mathbf{H}_{\mathcal{E}_n,0})})_{n\in\mathbb{N}}$ is bounded. Let $1\le q<+\infty$ if $d=2$ and $q=6$ if $d=3$.

Then, passing to subsequences if necessary, the sequence $(\boldsymbol{u}_n)_{n\in\mathbb{N}}$ converges weakly in $\mathrm{L}^2(0,T;\mathrm{L}^q(\Omega)^d)$ to a limit $\boldsymbol{u}$ and this limit satisfies $\boldsymbol{u}\in\mathrm{L}^2(0,T;\mathrm{H}_0^1(\Omega)^d)$.

Furthermore, the sequence of functions $t\in(0,T)\to\nabla_{\mathcal{E}_n}(\boldsymbol{u}_n(t))\in\mathrm{L}^2(\Omega)^{d\times d}$ converges weakly in $\mathrm{L}^2((0,T)\times\Omega)^{d\times d}$ to $t\to\nabla[\boldsymbol{u}(t)]$.

Proof. Let us denote by $\tilde{\boldsymbol{u}}_n$ (respectively $\widetilde{\nabla}_{\mathcal{E}_n}\boldsymbol{u}_n$) the extension of $\boldsymbol{u}_n$ to $(0,T)\times\mathbb{R}^d$ by zero. By virtue of Theorem 2 there exits $(\boldsymbol{u},\boldsymbol{V})\in\mathrm{L}^2(0,T;\mathrm{L}^q(\Omega)^d)\times\mathrm{L}^2((0,T)\times\Omega)^{d\times d}$ such that up to a subsequence

$$\tilde{\boldsymbol{u}}_n \to \tilde{\boldsymbol{u}} \text{ weakly in } \mathrm{L}^2(0,T;\mathrm{L}^q(\Omega)^d), \qquad \widetilde{\nabla}_{\mathcal{E}_n}\boldsymbol{u}_n \to \tilde{\boldsymbol{V}} \text{ weakly in } \mathrm{L}^2((0,T)\times\Omega)^{d\times d}, \tag{8.5}$$

where $(\tilde{\boldsymbol{u}},\tilde{\boldsymbol{V}})$ stands for the extension of $(\boldsymbol{u},\boldsymbol{V})$ to $(0,T)\times\mathbb{R}^d$ by zero. We obtain from Theorem 5(see Appendix) applied to $\boldsymbol{u}_n$ after an integration over $(0,T)$

$$\forall\boldsymbol{\eta}\in\mathbb{R}^d,\ \|\tilde{\boldsymbol{u}}_n(\cdot,\cdot+\boldsymbol{\eta})-\tilde{\boldsymbol{u}}_n\|^2_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{R}^d)^d)} \le C\|\boldsymbol{\eta}\|\big(\|\boldsymbol{\eta}\|+h_{\mathcal{M}}\big). \tag{8.6}$$

Consequently passing to the limit in the previous inequality

$$\forall \boldsymbol{\eta} \in \mathbb{R}^d,\ \|\tilde{\boldsymbol{u}}(\cdot, \cdot + \boldsymbol{\eta}) - \tilde{\boldsymbol{u}}\|^2_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{R}^d)^d)} \leq C\|\boldsymbol{\eta}\|^2 \tag{8.7}$$

and therefore the function $\boldsymbol{u}$ belongs to $\mathrm{L}^2(0, T; \mathrm{H}^1_0(\Omega)^d)$. Moreover for any $\boldsymbol{\psi} \in C_c^\infty((0, T) \times \Omega)^{d\times d}$ there exists $R_{\boldsymbol{\psi}}$ such that

$$\int_0^T \int_\Omega \boldsymbol{u}_n \cdot \operatorname{div} \boldsymbol{\psi} \,\mathrm{d}\boldsymbol{x}\,\mathrm{dt} = -\int_0^T \int_\Omega \nabla_{\mathcal{E}_n} \boldsymbol{u}_n : \cdot \nabla \boldsymbol{\psi} \,\mathrm{d}\boldsymbol{x}\,\mathrm{dt} + R_{\boldsymbol{\psi}} \tag{8.8}$$

where the remainder $R_{\boldsymbol{\psi}}$ satisfies

$$|R_{\boldsymbol{\psi}}| C_{\boldsymbol{\phi}} h_{\mathcal{M}_n}. \tag{8.9}$$

Consequently passing to the limit in we obtain that

$$\int_0^T \int_\Omega \boldsymbol{u} \cdot \operatorname{div} \boldsymbol{\psi} \,\mathrm{d}\boldsymbol{x}\,\mathrm{dt} = -\int_0^T \int_\Omega \boldsymbol{V} : \nabla \boldsymbol{\psi} \,\mathrm{d}\boldsymbol{x}\,\mathrm{dt} \tag{8.10}$$

which gives the expected result. □

The famous Aubin-Simon lemma discusses the compactness in $\mathrm{L}^p(0, T; B)$ ($1 \leq p \leq \infty$), which is widely used in the study of nonlinear evolution partial differential equations. This theorem states as follows

Theorem 4. *Let $1 \leq p < \infty$. Let X, B, Y be three Banach spaces such that*

1. *$X \subset B$ with compact embedding,*

2. *$B \subset Y$ with continuous embedding.*

Let $T > 0$ and $(u_n)_{n\in\mathbb{N}}$ be a sequence of $\mathrm{L}^p(0, T; X)$ such that

1. *$(u_n)_{n\in\mathbb{N}}$ is bounded in $\mathrm{L}^p(0, T; X)$.*

2. *$(\frac{d}{dt}u_n)_{n\in\mathbb{N}}$ is bounded in $\mathrm{L}^1(0, T; Y)$.*

Then there exists $u \in \mathrm{L}^p(0, T; B)$ such that, up to a subsequence, $u_n \to u$ dans $\mathrm{L}^p(0, T; B)$.

In numerical analysis, the spaces X and Y depend on the discretization of the computational domain. Futhermore we have deal with discrete derivatives. We then introduce a discrete version of the above theorem to take account these two points.

We continue with two definitions useful to introduce a discrete version of the Aubin-Simon theorem Theorem 5 below, see [44] for the original result in the continuous setting. We follow here [21].

Definition 3 (Compactly embedded sequence)**.** *Let B be a Banach space and $(X_n)_{n\in\mathbb{N}}$ be a sequence of Banach spaces included in B. We say that the sequence $(X_n)_{n\in\mathbb{N}}$ is compactly embedded in B if any sequence satisfying*

1. *$u_n \in X_n$ for all $n \in \mathbb{N}$.*

2. *the sequence* $(\|u_n\|_{X_n})_{n\in\mathbb{N}}$ *is bounded*

is relatively compact in B.

Remark 5. Given a sequence $(\mathcal{D}_n = (\mathcal{M}_n, \mathcal{E}_n))_{n\in n\in\mathbb{N}}$ of MAC grid of the computational domain Ω, with step size $h_{\mathcal{M}_n}$ going to 0 as $n \to \infty$, it is well known that the sequence $(\mathbf{H}_{\mathcal{E}_n,0})_{n\in n\in\mathbb{N}}$ is compactly embedded in $\mathrm{L}^2(\Omega)^d$.

Definition 4. *Let B a Banach space,* $(X_n)_{n\in\mathbb{N}}$ *be a sequence of Banach spaces included in B and* $(Y_n)_{n\in\mathbb{N}}$ *be a sequence of Banach spaces. We say that the sequence* $(X_n, Y_n)_{n\in\mathbb{N}}$ *is compact-continuous in B if the following conditions are satisfied*

1. *The sequence* $(X_n)_{n\in\mathbb{N}}$ *is compactly embedded in B.*

2. $X_n \subset Y_n$ *(for all* $n \in \mathbb{N}$*) and if the sequence* $(u_n)_{n\in\mathbb{N}}$ *is such that* $u_n \in X_n$ *(for all* $n \in \mathbb{N}$*),* $(\|u_n\|_{X_n})_{n\in\mathbb{N}}$ *bounded and* $\|u_n\|_{Y_n} \to 0$ *(as* $n \to +\infty$*), then any subsequence converging in B converge (in B) to* 0.

Remark 6. In agreement with Remark 5 the sequence $(\mathbf{H}_{\mathcal{E}_n,0}, \mathrm{L}^2(\Omega)^d)_{n\in\mathbb{N}}$ is compact-continuous is $B = \mathrm{L}^2(\Omega)^d$.

Let us now state the discrete version of the Aubin-Simon theorem. This theorem is proved in [7]. It is useful to prove the so-called Effective viscous flux (see Proposition 4).

Theorem 5 (Aubin-Simon Theorem with a sequence of subspaces and a discrete derivative.)**.** *Let* $1 \le p < \infty$*, let B be a Banach space, and let* $(X_n)_{n\in\mathbb{N}}$ *and* $(Y_n)_{n\in\mathbb{N}}$ *be sequences of Banach spaces such that* $X_n \subset B$ *for* $n \in \mathbb{N}$*. We assume that the sequence* $(X_n, Y_n)_{n\in\mathbb{N}}$ *is compact-continuous in B. Let* $T > 0$ *and* $(u_n)_{n\in\mathbb{N}}$ *be a sequence of* $\mathrm{L}^p(0, T; B)$ *satisfying the following conditions:*

- *(H1) the sequence* $(u_n)_{n\in\mathbb{N}}$ *is bounded in* $\mathrm{L}^p(0, T; B)$*.*
- *(H2) the sequence* $(\|u_n\|_{\mathrm{L}^p(0,T;X_n)})_{n\in\mathbb{N}}$ *is bounded.*
- *(H3) the sequence* $(\|\eth_t u_n\|_{\mathrm{L}^1(0,T;Y_n)})_{n\in\mathbb{N}}$ *is bounded.*

Then there exists $u \in \mathrm{L}^p(0, T; B)$ *such that, up to a subsequence,* $u_n \to u$ *in* $\mathrm{L}^p(0, T; B)$*.*

With the hypotheses of Theorem 5, another interesting question is to prove an additional regularity for u, namely $u \in \mathrm{L}^p(0, T; B)$ where X is some space closely related to the X_n (and included in B). In definition 5 we precise the meaning of the sentence "X closely related to the X_n" and we give in Theorem 6 a regularity result.

Definition 5 (B-limit-included)**.** *Let B be a Banach space,* $(X_n)_{n\in\mathbb{N}}$ *be a sequence of Banach spaces included in B and X be a Banach space included in B. The sequence* $(X_n)_{n\in\mathbb{N}}$ *is B-limit-included in X if there exists* $C \in \mathbb{R}$ *such that if u is the limit in B of a subsequence of a sequence* $(u_n)_{n\in\mathbb{N}}$ *verifying* $u_n \in X_n$ *and* $\|u_n\|_{X_n} \le 1$*, then* $u \in X$ *and* $\|u\|_X \le C$*.*

The regularity of a possible limit of approximate solutions may be proved thanks to the theorem which we recall below [22, Theorem B1].

Theorem 6 (Regularity of the limit)**.** *Let $1 \le p < \infty$ and $T > 0$. Let B be a Banach space, $(X_n)_{n\in\mathbb{N}}$ be a sequence of Banach spaces included in B and B-limit-included in X (where X is a Banach space included in B). Let $T > 0$ and, for $m \in \mathbb{N}$, Let $u_n \in \mathrm{L}^p(0, T; X_n)$. We assume that the sequence $(\|u_n\|_{\mathrm{L}^p(0,T;X_n)})_{n\in\mathbb{N}}$ is bounded and that $u_n \to u$ a.e. as $n \to \infty$. Then $u \in \mathrm{L}^p(0, T; X)$.*

Acknowledgements

This work was supported by the MODTERCOM project within the APEX programme of the Provence-Alpes-Côte d'Azur region.

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