NUMERICAL APPROXIMATION OF THE GENERAL COMPRESSIBLE STOKES PROBLEM

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Abstract. In this paper, we propose a discretization for the compressible Stokes problem with an equation of state of the form $p = \varphi(\rho)$ (where p stands for the pressure, ρ for the density and φ is a superlinear nondecreasing function from $\mathbb{R}$ to $\mathbb{R}$). This scheme is based on Crouzeix-Raviart approximation spaces. The discretization of the momentum balance is obtained by the usual finite element technique. The discrete mass balance is obtained by a finite volume scheme, with an upwinding of the density, and two additional terms. We prove the existence of a discrete solution and the convergence of this approximate solution to a solution of the continuous problem.

1. introduction

Let Ω be a bounded open set of $\mathbb{R}^d$, polygonal if $d = 2$ and polyhedral if $d = 3$. Let $\varphi \in C(\mathbb{R}, \mathbb{R})$ be a convex nondecreasing function such that:

$$\varphi(0) = 0,\ \varphi \text{ is } C^1 \text{ on } \mathbb{R}_+^\star$$

and

$$\forall a \in \mathbb{R},\ \exists b > 0 \text{ such that: } \varphi(s) \geq as - b,\ \forall s \in \mathbb{R}_+. \tag{1.1}$$

For M, $\mu > 0$, $\boldsymbol{f} \in \mathrm{L}^2(\Omega)^d$ and $\boldsymbol{g} \in L^\infty(\Omega)^d$, we consider the following problem:

$$-\mu\Delta\boldsymbol{u} - \frac{\mu}{3}\boldsymbol{\nabla}(\mathrm{div}\boldsymbol{u}) + \boldsymbol{\nabla}p = \boldsymbol{f} + \rho\boldsymbol{g} \text{ in } \Omega,\ \ \boldsymbol{u} = 0 \text{ on } \partial\Omega, \tag{1.2a}$$

$$\mathrm{div}(\rho\boldsymbol{u}) = 0 \ \text{ in } \Omega,\ \rho \geq 0 \ \text{ in } \Omega,\ \int_\Omega \rho(x)\,\mathrm{d}\boldsymbol{x} = M, \tag{1.2b}$$

$$p = \varphi(\rho) \text{ in } \Omega. \tag{1.2c}$$

Remark 1.1.

- We assume that the function φ is convex, but not necessarily strictly convex. We also assume that φ is nondecreasing but it can be constant on an interval (in fact, since φ is convex, the function φ is, at least for m large enough, increasing on $[m, +\infty)$).
- The condition (1.1) is equivalent to the following one:

$$\liminf_{s\to+\infty} \varphi(s)/s = +\infty$$

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- The fact that $\varphi(0) = 0$ is not a restriction since p can be replaced by $(p - \varphi(0))$ in the momemtum equation and the EOS (namely the equation (1.2c)) can be written as $p - \varphi(0) = \varphi(\rho) - \varphi(0)$.
- The convexity of the function φ can be replaced by the following condition: there exist $a, \tilde{a}, b, \tilde{b} > 0$ and $\gamma > 1$ such that:

$$\forall s \in \mathbb{R}_+,\ as^\gamma - b \le \varphi(s) \le \tilde{a}s^{2\gamma-1} + \tilde{b}. \tag{1.3}$$

  Here also the function φ is assumed to be nondecreasing but not necessarily increasing.
- The coefficient $\mu/3$ in the second term of the Left Hand Side of (1.2a) is natural from the physical point of view. From the mathematical point of view, it is easy to replace it by $\bar{\mu}$, as long as $\bar{\mu} \ge 0$.

Definition 1.2. Let $\boldsymbol{f} \in \mathrm{L}^2(\Omega)^d$, $\boldsymbol{g} \in L^\infty(\Omega)^d$ and $M > 0$. A weak solution of Problem (1.2) is a function $(\boldsymbol{u}, p, \rho) \in \mathrm{H}_0^1(\Omega)^d \times \mathrm{L}^2(\Omega) \times \mathrm{L}^2(\Omega)$ satisfying:

$$\begin{aligned}\mu \int_\Omega \boldsymbol{\nabla u} : \boldsymbol{\nabla v}\,\mathrm{d}\boldsymbol{x} + \frac{\mu}{3}\int_\Omega \mathrm{div}(\boldsymbol{u})\mathrm{div}(\boldsymbol{v})\,\mathrm{d}\boldsymbol{x} - \int_\Omega p\ \mathrm{div}(\boldsymbol{v})\,\mathrm{d}\boldsymbol{x} \\ = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}\,\mathrm{d}\boldsymbol{x} + \int_\Omega \rho\boldsymbol{g}\cdot\boldsymbol{v}\,\mathrm{d}\boldsymbol{x} \text{ for all } \boldsymbol{v} \in \mathrm{H}_0^1(\Omega)^d,\end{aligned} \tag{1.4a}$$

$$\int_\Omega \rho\boldsymbol{u}\cdot\boldsymbol{\nabla}\varphi\,\mathrm{d}\boldsymbol{x} = 0 \text{ for all } \varphi \in \mathrm{W}^{1,\infty}(\Omega), \tag{1.4b}$$

$$\rho \ge 0 \text{ a.e. in } \Omega,\ \int_\Omega \rho\,\mathrm{d}\boldsymbol{x} = M,\ p = \varphi(\rho) \text{ a.e. in } \Omega. \tag{1.4c}$$

The main objective of this paper is to present a numerical scheme for the computation of an approximate solution of Problem (1.2) and to prove the convergence (up to a subsequence, since, up to now, no uniqueness result is available for the solution of (1.2)) of this approximate solution towards a weak solution of (1.2) (*i.e.* a solution of (1.4)) as the mesh size goes to 0. The present paper follows a previous paper [6] where a similar result was presented in the case $\varphi(\rho) = \rho^\gamma$, $\gamma > 1$ (see also [11]). We present here a discretization with the so called Crouziex-Raviart element, as in [6]. However, it could be possible also, without additional difficulties, to use a MAC scheme, as in [7]. The fact to consider a general EOS (instead of $p = \rho^\gamma$) induces some additional difficulties with respect to the previous papers [6] and [7]. In particular for the estimates on the discrete solutions (Section 3.2 and Appendix A) and for passing to the limit in the EOS (Section 3.3 and Appendix B). For passing in the limit in the EOS, we mimic some ideas which were developped for the study of the Navier-Stokes equations, see [12], [8] or [13]. A part of the results given in this paper was presented in the FVCA6 workshop (Prague, 2011) and in a short paper (containing few proofs) in the proceedings of this workshop, see [9]. The present paper is more general. In particular, it considers more general EOS and it includes the gravity effects (two improvements which induce the need of non trivial developments, for instance for obtaining estimates on u an p and for passing to the limit in the EOS). Furthermore, the present paper contains complete proofs and an appendix with lemmas interesting for their own sake.

Remark 1.3. In the spirit of [12], [8] or [13] (which are devoted to the study of the compressible Navier-Stokes equations, but not on the discretization point of view), it is worth noticing that if $(\rho, u) \in L^2(\Omega) \times \mathrm{H}^1_0(\Omega)$ satisfies (1.4b), then, it is known that (ρ, u) is a renormalized solution of $\mathrm{div}(\rho u) = 0$ in the sense of [4], that is

$$(\rho\phi'(\rho) - \phi(\rho))\mathrm{div}(u) + \mathrm{div}(\phi(\rho)u) = 0 \text{ in } \mathcal{D}'(\mathbb{R}^d),$$

for any C^1-function ϕ from $\mathbb{R}$ to $\mathbb{R}$ such that ϕ' is bounded (in order to give a sense to the preceding equation, we set $u = 0$ in $\mathbb{R}^d \setminus \Omega$, so that $u \in H^1(\mathbb{R}^d)$). This is explained in Remark B.3.

2. Discrete spaces and numerical scheme

Let $\mathcal{T}$ be a decomposition of the domain Ω in simplices, which we call herafter a triangulation of Ω, regardless of the space dimension. By $\mathcal{E}(K)$, we denote the set of the edges ($d = 2$) or faces ($d = 3$) σ of the element $K \in \mathcal{T}$; for short, each edge or face will be called an edge hereafter. The set of all edges of the mesh is denoted by $\mathcal{E}$; the set of edges included in the boundary of Ω is denoted by $\mathcal{E}_{\mathrm{ext}}$ and the set of internal edges (*i.e.* $\mathcal{E} \setminus \mathcal{E}_{\mathrm{ext}}$) is denoted by $\mathcal{E}_{\mathrm{int}}$. The decomposition $\mathcal{T}$ is assumed to be regular in the usual sense of the finite element literature (*e.g.* [2]), and, in particular, $\mathcal{T}$ satisfies the following properties: $\bar{\Omega} = \bigcup_{K \in \mathcal{T}} \bar{K}$; if $K,\, L \in \mathcal{T}$, then $\bar{K} \cap \bar{L} = \emptyset$, $\bar{K} \cap \bar{L}$ is a vertex or $\bar{K} \cap \bar{L}$ is a common edge of K and L, which is denoted by $K|L$. For each internal edge of the mesh $\sigma = K|L$, $\boldsymbol{n}_{KL}$ stands for the normal vector of σ, oriented from K to L (so that $\boldsymbol{n}_{KL} = -\boldsymbol{n}_{LK}$). By $|K|$ and $|\sigma|$ we denote the (d and $d-1$ dimensional) measure, respectively, of an element K and of an edge σ, and h_K and h_σ stand for the diameter of K and σ, respectively. We measure the regularity of the mesh through the parameter θ defined by:

$$\theta = \inf \{ \frac{\xi_K}{h_K},\ K \in \mathcal{T}\} \tag{2.1}$$

where ξ_K stands for the diameter of the largest ball included in K. Note that for all $\sigma \in \mathcal{E}_{\mathrm{int}}$, $\sigma = K|L$, we have $h_\sigma \geq \xi_K \geq \theta\, h_K$ and $h_\sigma \leq h_L$ and so $\theta\, h_K \leq h_L \leq \theta^{-1} h_K$. Note also that for all $K \in \mathcal{T}$ and for all $\sigma \in \mathcal{E}(K)$, the inequality $h_\sigma\, |\sigma| \leq 2\, \theta^{-d}\, |K|$ holds ([10, relation (2.2)]) and if $\sigma = K|L$ a rough estimate gives $|K| \leq (2/\theta)^{2d} |L|$. These relations will be used throughout this paper. Finally, as usual, we denote by h the quantity $\max_{K \in \mathcal{T}} h_K$.

The space discretization relies on the Crouzeix-Raviart element (see [3] for the seminal paper and, for instance, [5, pp. 199–201] for a synthetic presentation). The reference element is the unit d-simplex and the discrete functional space is the space P_1 of affine polynomials. The degrees of freedom are determined by the following set of edge functionals:

$$\{F_\sigma,\ \sigma \in \mathcal{E}(K)\}, \qquad F_\sigma(v) = |\sigma|^{-1} \int_\sigma v \,\mathrm{d}\gamma. \tag{2.2}$$

The mapping from the reference element to the actual one is the standard affine mapping. Finally, the continuity of the average value of a discrete functions v across each edge of the mesh, $F_\sigma(v)$, is required, thus the discrete space V_h is defined as follows:

$$\begin{aligned} V_h = \{\, v \in L^2(\Omega)\ :\ \forall K \in \mathcal{T},\ v|_K \in P_1(K)\,; \\ \forall \sigma \in \mathcal{E}_{\mathrm{int}},\ \sigma = K|L,\ \ F_\sigma(v|_K) = F_\sigma(v|_L);\ \forall \sigma \in \mathcal{E}_{\mathrm{ext}},\ F_\sigma(v) = 0\}. \end{aligned} \tag{2.3}$$

Indeed, this space V_h should be denoted by $V_{\mathcal{T}}$ since it depends on $\mathcal{T}$ and not only on h (which is given by $\mathcal{T}$) but this (somewhat incorrect) notation is commonly used.

The space of approximation for the velocity is the space $\boldsymbol{W}_h$ of vector-valued functions each component of which belongs to V_h: $\boldsymbol{W}_h = (V_h)^d$. The pressure and the density are approximated by the space L_h of piecewise constant functions:

$$L_h = \left\{ q \in L^2(\Omega) \,:\, q|_K = \text{ constant}, \forall K \in \mathcal{T} \right\}.$$

Since only the continuity of the integral over each edge of the mesh is imposed, the functions of V_h are discontinuous through each edge; the discretization is thus nonconforming in $H^1(\Omega)^d$. We then define, for $1 \leq i \leq d$ and $\boldsymbol{u} \in V_h$, $\partial_{h,i}\, \boldsymbol{u}$ as the function of $\mathrm{L}^2(\Omega)$ which is equal to the derivative of u with respect to the i^{th} space variable almost everywhere. This notation allows to define the discrete gradient, denoted by $\boldsymbol{\nabla}_h$, for both scalar and vector-valued discrete functions and the discrete divergence of vector-valued discrete functions, denoted by div_h.

The Crouzeix-Raviart pair of approximation spaces for the velocity and the pressure is *inf-sup* stable, in the usual sense for "piecewise H^1" discrete velocities, *i.e.* there exists $c_{\mathrm{i}} > 0$ only depending on Ω and, in a non-increasing way, on θ, such that:

$$\forall p \in L_h, \qquad \sup_{\boldsymbol{v} \in \boldsymbol{W}_h} \frac{\displaystyle\int_\Omega p \, \mathrm{div}_h(\boldsymbol{v}) \, \mathrm{d}\boldsymbol{x}}{\|\boldsymbol{v}\|_{1,b}} \geq c_{\mathrm{i}} \, \|p - m(p)\|_{\mathrm{L}^2(\Omega)} \,,$$

where $m(p)$ is the value of p over Ω and $\|\cdot\|_{1,b}$ stands for the broken Sobolev H^1 semi-norm, which is defined for scalar as well as for vector-valued functions by:

$$\|v\|_{1,b}^2 \;= \sum_{K \in \mathcal{T}} \int_K |\boldsymbol{\nabla} v|^2 \, \mathrm{d}\boldsymbol{x} = \int_\Omega |\boldsymbol{\nabla}_h v|^2 \, \mathrm{d}\boldsymbol{x}.$$

This norm is known to control the L^2 norm by a Poincaré inequality (*e.g.* [5, lemma 3.31]). We also define a discrete semi-norm on L_h, similar to the usual H^1 semi-norm used in the finite volume context:

$$\forall \rho \in L_h, \qquad |\rho|_{\mathcal{T}}^2 \;= \sum_{\substack{\sigma \in \mathcal{E}_{\mathrm{int}}, \\ \sigma = K|L}} \frac{|\sigma|}{h_\sigma} \, (\rho_K - \rho_L)^2.$$

From the definition (2.2), each velocity degree of freedom may be indexed by the number of the component and the associated edge, thus the set of velocity degrees of freedom reads:

$$\{ v_{\sigma,i}, \ \sigma \in \mathcal{E}_{\mathrm{int}}, \ 1 \leq i \leq d \}.$$

We denote by e_σ the usual Crouzeix-Raviart shape function associated to σ, *i.e.* the scalar function of V_h such that $F_\sigma(e_\sigma) = 1$ and $F_{\sigma'}(e_\sigma) = 0$, for all $\sigma' \in \mathcal{E} \setminus \{\sigma\}$.

Similarly, each degree of freedom for the pressure is associated to a cell K, and the set of pressure degrees of freedom is denoted by $\{p_K, \ K \in \mathcal{T}\}$.

We define by r_h the following interpolation operator:

$$\text{(2.4)} \qquad r_h : \quad \left| \begin{array}{lll} \mathrm{H}_0^1(\Omega) & \longrightarrow & V_h \\ u & \mapsto & r_h u = \displaystyle\sum_{\sigma \in \mathcal{E}} F_\sigma(u) \, e_\sigma = \sum_{\sigma \in \mathcal{E}} |\sigma|^{-1} \left(\int_\sigma v \, \mathrm{d}\gamma \right) e_\sigma. \end{array} \right.$$

This operator naturally extends to vector-valued functions (*i.e.* to perform the interpolation from $\mathrm{H}^1_0(\Omega)^d$ to $\boldsymbol{W}_h$) and we keep the same notation r_h for both the scalar and vector case. The properties of r_h are gathered in the following lemma. They are proven in [3].

Theorem 2.1. *Let $\theta_0 > 0$ and let $\mathcal{T}$ be a triangulation of the computational domain Ω such that $\theta \geq \theta_0$, where θ is defined by* (2.1)*. The interpolation operator r_h enjoys the following properties:*

(1) *preservation of the divergence:*

$$\forall \boldsymbol{v} \in \mathrm{H}^1_0(\Omega)^d,\ \forall q \in L_h, \qquad \int_\Omega q\ \mathrm{div}_h(r_h \boldsymbol{v})\,\mathrm{d}\boldsymbol{x} = \int_\Omega q\ \mathrm{div}(\boldsymbol{v})\,\mathrm{d}\boldsymbol{x},$$

(2) *stability:*

$$\forall v \in \mathrm{H}^1_0(\Omega), \qquad \|r_h v\|_{1,b} \leq c_1(\theta_0)\,|v|_{\mathrm{H}^1(\Omega)}\,,$$

(3) *approximation properties:*

$$\forall v \in \mathrm{H}^2(\Omega) \cap \mathrm{H}^1_0(\Omega),\ \forall K \in \mathcal{T},$$
$$\|v - r_h v\|_{\mathrm{L}^2(K)} + h_K\,\|\boldsymbol{\nabla}_h(v - r_h v)\|_{\mathrm{L}^2(K)} \leq c_2(\theta_0)\,h_K^2\,|v|_{\mathrm{H}^2(K)}\,.$$

In both above inequalities, the notation $c_i(\theta_0)$ means that the real number c_i only depends on θ_0 and Ω, and, in particular, does not depend on the parameter h characterizing the size of the cells; this notation will be kept throughout the paper.

The following compactness result was proven in [10, Theorem 3.3].

Theorem 2.2. *Let $(v_n)_{n\in\mathbb{N}}$ be a sequence of functions satisfying the following assumptions:*

(1) *$\forall n \in \mathbb{N}$, there exists a triangulation of the domain $\mathcal{T}_n$ such that $v_n \in V_{h_n}$, where V_{h_n} is the space of Crouzeix-Raviart discrete functions associated to $\mathcal{T}_n$ (and h_n given by $\mathcal{T}_n$), as defined by* (2.3)*, and the parameter θ_n characterizing the regularity of $\mathcal{T}_n$ is bounded away from zero independently of n,*
(2) *the sequence $(v_n)_{n\in\mathbb{N}}$ is uniformly bounded with respect to the broken Sobolev H^1 semi-norm,* i.e.*:*

$$\forall n \in \mathbb{N}, \qquad \|v_n\|_{1,b} \leq C,$$

where C is a constant real number and $\|\cdot\|_{1,b}$ stands for the broken Sobolev H^1 semi-norm associated to $\mathcal{T}_n$ (with a slight abuse of notation, namely dropping, for short, the index n pointing the dependence of the norm with respect to the mesh).

Then, when $n \to \infty$, possibly up to the extraction of a subsequence, the sequence $(v_n)_{n\in\mathbb{N}}$ converges (strongly) in $\mathrm{L}^2(\Omega)$ to a limit $\bar{v}$ such that $\bar{v} \in \mathrm{H}^1_0(\Omega)$.

We now present the numerical scheme we use. Let ρ^* be the mean density, *i.e.* $\rho^* = M/|\Omega|$ where $|\Omega|$ stands for the measure of the domain Ω. Let also α and ξ be given, with $\alpha > 0$ and $0 < \xi < 2$. Let $\mathcal{T}$ be a (regular) decomposition of the domain Ω in simplices. The discrete unknowns are $\boldsymbol{u}$, p and ρ, with $\boldsymbol{u} \in \boldsymbol{W}_h$ and $p, \rho \in L_h$. Using the notations previously introduced, we consider the following numerical scheme for the discretization of Problem (1.2):

$$\mu \int_\Omega \boldsymbol{\nabla}_h \boldsymbol{u} : \boldsymbol{\nabla}_h \boldsymbol{v} \,\mathrm{d}\boldsymbol{x} + \frac{\mu}{3} \int_\Omega \mathrm{div}_h(\boldsymbol{u}) \mathrm{div}_h(\boldsymbol{v}) \,\mathrm{d}\boldsymbol{x} - \int_\Omega p \; \mathrm{div}_h(\boldsymbol{v}) \,\mathrm{d}\boldsymbol{x} = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} \,\mathrm{d}\boldsymbol{x} + \int_\Omega \rho \boldsymbol{g} \cdot \boldsymbol{v} \,\mathrm{d}\boldsymbol{x} \text{ for all } \boldsymbol{v} \in \boldsymbol{W}_h, \tag{2.5a}$$

$$\sum_{\sigma=K|L} \left(|\sigma| \boldsymbol{u}^+_{K,\sigma}\, \rho_K - |\sigma| \boldsymbol{u}^-_{K,\sigma}\, \rho_L \right) + M_K + T_K = 0 \text{ for all } K \in \mathcal{T}, \tag{2.5b}$$

$$p_K = \varphi(\rho_K) \text{ for all } K \in \mathcal{T}. \tag{2.5c}$$

The quantity $\boldsymbol{u}_{K,\sigma}$ is defined by

$$\boldsymbol{u}_{K,\sigma} = |\sigma|^{-1} \int_\sigma \boldsymbol{u} \,\mathrm{d}\gamma \cdot \boldsymbol{n}_{KL}.$$

As usual, $\boldsymbol{u}^+_{K,\sigma} = \max(\boldsymbol{u}_{K,\sigma},\ 0)$ and $\boldsymbol{u}^-_{K,\sigma} = -\min(\boldsymbol{u}_{K,\sigma},\ 0)$, so that $\boldsymbol{u}_{K,\sigma} = \boldsymbol{u}^+_{K,\sigma} - \boldsymbol{u}^-_{K,\sigma}$. The terms M_K and T_K read:

$$M_K = h^\alpha \, |K| \, (\rho_K - \rho^*) \,, \tag{2.6a}$$

$$T_K = \sum_{\sigma=K|L} h^\xi \, \frac{|\sigma|}{h_\sigma} \, (|\rho_K| + |\rho_L|) \, (\rho_K - \rho_L) \,. \tag{2.6b}$$

3. Existence and convergence of approximate solutions

3.1. **Existence of a solution.** Let $\mathcal{T}$ be a (regular) decomposition of the domain Ω in simplices. We prove in this section the existence of a discrete solution, that the existence of a solution to (2.5), by using the Brouwer fixed point theorem to a convenient application T from $\mathbb{R}^N$ to $\mathbb{R}^N$ where N =card($\mathcal{T}$). We first define T.

Let $\tilde\rho = (\tilde\rho_K)_{K\in\mathcal{T}}$. Choosing the elements of $\mathcal{T}$ in an arbitrary order, we then have $\tilde\rho \in \mathbb{R}^N$. We calculate p by the following relation: $p_K = \varphi(\tilde\rho^+_K)$ for all $K \in \mathcal{T}$.

We now compute $\boldsymbol{u}$ as the unique solution (in $\boldsymbol{W}_h$) of (2.5a) with $\tilde\rho$ instead of ρ in the Right Hand Side of (2.5a) (and p given by $p_K = \varphi(\tilde\rho^+_K)$ for all $K \in \mathcal{T}$). The existence and uniqueness of $\boldsymbol{u}$ is an easy consequence of the coercivity in $\boldsymbol{W}_h$ of the bilinear form

$$(u, v) \mapsto \mu \int_\Omega \boldsymbol{\nabla}_h \boldsymbol{u} : \boldsymbol{\nabla}_h \boldsymbol{v} \,\mathrm{d}\boldsymbol{x}.$$

Furthermore, the solution $\boldsymbol{u}$ continuously depends on $\tilde\rho$ (since φ is continuous).

We have now to define ρ (and we will set $T(\tilde\rho) = \rho$). We change a little bit the term T_K. Instead of (2.6b), we take

$$T_K = \sum_{\sigma=K|L} h^\xi \, \frac{|\sigma|}{h_\sigma} \, (|\tilde\rho_K| + |\tilde\rho_L|) \, (\rho_K - \rho_L) \,.$$

With this choice of T_K, the set of Equations (2.5b) leads to the linear system of N equations with N unknowns (which are ρ_K for $K \in \mathcal{T}$). The equations of this system may be written as:

$$\sum_{L\in\mathcal{T}} a_{K,L} \rho_L = b_K \text{ for all } K \in \mathcal{T}, \tag{3.1}$$

with

$$a_{K,K} = h^\alpha |K| + \sum_{\sigma=K|L} \big(|\sigma| u^+_{K,\sigma} + h^\xi \, \frac{|\sigma|}{h_\sigma} \, (|\tilde\rho_K| + |\tilde\rho_L|) \big),$$

$$a_{K,L} = -|\sigma| u^-_{K,\sigma} - h^\xi \, \frac{|\sigma|}{h_\sigma} \, (|\tilde\rho_K| + |\tilde\rho_L|) \text{ if } \sigma = K|L,$$

$$a_{K,L} = 0 \text{ if } K \text{ and } L \text{ do not share an interface.}$$

$$b_K = h^\alpha |K| \rho^\star.$$

Using the fact that $u^-_{L,\sigma} = u^+_{K,\sigma}$ (for $\sigma = K|L$), one has, for all $K \in \mathcal{T}$,

$$\sum_{L \in \mathcal{T}} a_{K,L} > 0$$

and, for all $K, L \in \mathcal{T}$, $K \neq L$,

$$a_{K,L} \leq 0.$$

With these properties, it is quite easy to show thet the system (3.1) has a unique solution. Furthermore, since $b_K > 0$ for all $K \in \mathcal{T}$ the solution ρ satisfy $\rho_K > 0$ for all $K \in \mathcal{T}$ (see Lemma C.4). Finally, since the coeffcients $a_{K,L}$ and b_K depend continuously of $\tilde\rho$ (and since the application $A \mapsto A^{-1}$ is continuous on the set of invertible $N \times N$ matrix), the solution ρ of (3.1) continuously depends on $\tilde\rho$.

We define now (as we said before) the map T from $\mathbb{R}^N$ to $\mathbb{R}^N$ setting $T(\tilde\rho) = \rho$. The map T is continuous.

If $\rho \in \mathrm{Im}(T)$, we also showed that $\rho_K > 0$ for all $K \in \mathcal{T}$. Futhermore summing for $K \in \mathcal{T}$ the equations (3.1) we obtain

$$\sum_{K \in \mathcal{T}} h^\alpha |K] \rho_K = \sum_{K \in \mathcal{T}} b_K = \sum_{K \in \mathcal{T}} h^\alpha |K| \rho^\star.$$

With the definition of $\rho^\star$, this gives $\sum_{K \in \mathcal{T}} |K| \rho_K = M$. Since $\rho \mapsto \sum_{K \in \mathcal{T}} |K| |\rho_K|$ is a norm on $\mathbb{R}^N$, this proves that the whole set $\mathrm{Im}(T)$ is included in a fixed ball of $\mathbb{R}^N$. Then, we can apply the Brouwer fixed point theorem. It gives the existence of $\rho \in \mathbb{R}^N$ such that $T(\rho) = \rho$. This gives the existence of a solution $(\boldsymbol{u}, p, \rho)$ to (2.5).

We conclude this section by remarking that if $(\boldsymbol{u}, p, \rho)$ is a solution to (2.5), we necessarily have $T(\rho) = \rho$ and this show that

$$\rho_K > 0 \text{ for all } K \in \mathcal{T} \text{ and } \sum_{K \in \mathcal{T}} |K| \rho_K = M.$$

3.2. **Estimates on the discrete solution.**

Lemma 3.1. *Let $\mathcal{T}$ be a triangulation of the computational domain Ω and Φ a nondecreasing function in $C^1(\mathbb{R}^+_*)$. Let $(\boldsymbol{u}, \rho) \in \boldsymbol{W}_h \times L_h$ satisfy the second equation of the scheme,* i.e. *Equation* (2.5b)*. Then, $\rho_K > 0$ for all $K \in \mathcal{T}$ and:*

$$\int_\Omega \Phi(\rho) \mathrm{div}_h(\boldsymbol{u}) \, \mathrm{d}\boldsymbol{x} \leq 0.$$

Proof. We fist remark that ρ is solution of (3.1) with

$$a_{K,K} = h^\alpha |K| + \sum_{\sigma=K|L} \big(|\sigma| u^+_{K,\sigma} + h^\xi \, \frac{|\sigma|}{h_\sigma} \, (|\rho_K| + |\rho_L|) \big),$$

$$a_{K,L} = -|\sigma| u^-_{K,\sigma} - h^\xi \, \frac{|\sigma|}{h_\sigma} \, (|\rho_K| + |\rho_L|) \text{ if } \sigma = K|L,$$

$$a_{K,L} = 0 \text{ if } K \text{ and } L \text{ do not share an interface.}$$

$$b_K = h^\alpha |K| \rho^\star.$$

Then, since $b_K > 0$ for all $K \in \mathcal{T}$, one has $\rho_K > 0$ for all $K \in \mathcal{T}$ (see Lemma C.4). Let the function $\psi \in C^1(\mathbb{R}^\star_+)$ satisfying $\psi'(s) = \frac{\Phi'(s)}{s}$ for all $s > 0$ (ψ is nondecreasing). Multiplying (2.5b) by $\psi(\rho_K)$ and summing over $K \in \mathcal{T}$ yields $T_1 + T_2 + T_3 = 0$ with:

$$T_1 = \sum_{K\in\mathcal{T}} \psi(\rho_K) \sum_{\sigma=K|L} |\sigma| \rho_\sigma \, \boldsymbol{u}_\sigma \cdot \boldsymbol{n}_{KL},$$

$$T_2 = \sum_{K\in\mathcal{T}} h^\alpha \, |K| \, \psi(\rho_K) \, (\rho_K - \rho^*) \, ,$$

$$T_3 = \sum_{K\in\mathcal{T}} \psi(\rho_K) \sum_{\sigma=K|L} (h_K + h_L)^\xi \, \frac{|\sigma|}{h_\sigma} \, (\rho_K + \rho_L) \, (\rho_K - \rho_L) \, .$$

Let:

$$T_4 = \sum_{K\in\mathcal{T}} \int_K \Phi(\rho_K) \mathrm{div}(\boldsymbol{u}) = \sum_{\sigma=K|L} |\sigma| \boldsymbol{u}_\sigma \cdot \boldsymbol{n}_{KL} (\Phi(\rho_K) - \Phi(\rho_L))$$

We have: $T_4 = T_4 - T_1 - T_2 - T_3$

$$= \sum_{\sigma=K|L} |\sigma| \boldsymbol{u}_\sigma \cdot \boldsymbol{n}_{KL} [\Phi(\rho_K) - \Phi(\rho_L) - \rho_\sigma (\psi(\rho_K) - \psi(\rho_L))] - T_2 - T_3,$$

with $\rho_\sigma = \rho_K$ if $\boldsymbol{u}_\sigma \cdot \boldsymbol{n}_{KL} > 0$ and $\rho_\sigma = \rho_L$ if $\boldsymbol{u}_\sigma \cdot \boldsymbol{n}_{KL} < 0$.

The fact that ψ is nondecreasing yields:
$\star$ $T_2 \geq \sum_{K\in\mathcal{T}} h^\alpha |K| \psi(\rho^*) \, (\rho_K - \rho^*) = 0$,
$\star$ $T_3 = \sum_{\sigma=K|L} (h_K + h_L)^\xi \, \frac{|\sigma|}{h_\sigma} \, (\rho_K + \rho_L) \, (\rho_K - \rho_L) \, (\psi(\rho_K) - \psi(\rho_L)) \geq 0$.

For $\alpha > 0$, we define Φ_α on $\mathbb{R}^\star_+$ by $\Phi_\alpha(s) = \Phi(\alpha) - \Phi(s) - \alpha(\psi(\alpha) - \psi(s))$. Since Φ is nondecreasing (and $s\psi'(s) = \Phi'(s)$), one has $\Phi_\alpha(s) \leq 0$ for all $s \in \mathbb{R}^\star_+$. Then, thanks to the choice of ρ_σ, one has

$$\sum_{\sigma=K|L} |\sigma| \boldsymbol{u}_\sigma \cdot \boldsymbol{n}_{KL} [\Phi(\rho_K) - \Phi(\rho_L) - \rho_\sigma (\psi(\rho_K) - \psi(\rho_L))] \leq 0$$

which gives:

$$T_4 = \int_\Omega \Phi(\rho) \mathrm{div}_h(\boldsymbol{u}) \, \mathrm{d}\boldsymbol{x} \leq 0.$$

□

Proposition 3.2. *Let $\theta_0 > 0$ and let $\mathcal{T}$ be a triangulation of the computational domain Ω such that $\theta \geq \theta_0$, where θ is defined by (2.1). Let $(\boldsymbol{u}, p, \rho) \in \boldsymbol{W}_h \times L_h \times L_h$ be a solution of (2.5). Then there exists C, only depending on the data of the problem Ω, $\boldsymbol{f}$, $\boldsymbol{g}$, μ, φ, M and on θ_0, such that:*

$$\|\boldsymbol{u}\|_{1,b} \le C,\ \|p\|_{\mathrm{L}^2(\Omega)} \le C \text{ and } \|\rho\|_{\mathrm{L}^2(\Omega)} \le C. \tag{3.2}$$

Proof. Let $(\boldsymbol{u}, p, \rho)$ be a solution of (2.5) .Taking $\boldsymbol{u}$ as test function in (2.5a) yields:

$$\mu \|\boldsymbol{u}\|_{1,b}^2 + \frac{\mu}{3}\int_\Omega \mathrm{div}_h^2(\boldsymbol{u})\,\mathrm{d}\boldsymbol{x} - \int_\Omega p\ \mathrm{div}_h(\boldsymbol{u})\,\mathrm{d}\boldsymbol{x} = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{u}\,\mathrm{d}\boldsymbol{x} + \int_\Omega \rho\boldsymbol{g}\cdot\boldsymbol{u}\,\mathrm{d}\boldsymbol{x}. \tag{3.3}$$

Using Lemma 3.1, a (well known) discrete Poincaré Inequality and the Hölder Inequality, one obtains the existence of C_1 only depending on Ω, $\boldsymbol{f}$, μ, $\boldsymbol{g}$ such that

$$\|\boldsymbol{u}\|_{1,b} \le C_1(1 + \|\rho\|_{\mathrm{L}^2(\Omega)}). \tag{3.4}$$

Since $p = \varphi(\rho)$, using (1.1), for all $\varepsilon > 0$ there exists C_ε (only depending on ε, φ and Ω) such that:

$$\|\rho\|_{\mathrm{L}^2(\Omega)} \le C_\varepsilon + \varepsilon\,\|p\|_{\mathrm{L}^2(\Omega)}. \tag{3.5}$$

Then, with (3.4), for all $\varepsilon > 0$, there exists $\bar{C}_\varepsilon$, only depending on Ω, $\boldsymbol{f}$, μ, $\boldsymbol{g}$, φ and ε such that

$$\|\boldsymbol{u}\|_{1,b} \le \bar{C}_\varepsilon + \varepsilon\,\|p\|_{\mathrm{L}^2(\Omega)}. \tag{3.6}$$

We now use Lemma C.2. There exists $\boldsymbol{w} \in \mathrm{H}_0^1(\Omega)^d$ such that $\mathrm{div}(\boldsymbol{w}) = p - m(p)$ a.e. in Ω and $\|\boldsymbol{w}\|_{\mathrm{H}^1(\Omega)^d} \le c_2\,\|p - m(p)\|_{\mathrm{L}^2(\Omega)}$ where c_2 only depends on Ω.

Taking $\boldsymbol{v} = r_h\boldsymbol{w}$ as test function in (2.5a) yields:

$$\int_\Omega p\ \mathrm{div}_h(\boldsymbol{v})\,\mathrm{d}\boldsymbol{x} = \mu\int_\Omega \boldsymbol{\nabla}_h\boldsymbol{u} : \boldsymbol{\nabla}_h\boldsymbol{v}\,\mathrm{d}\boldsymbol{x} + \frac{\mu}{3}\int_\Omega \mathrm{div}_h(\boldsymbol{u})\mathrm{div}_h(\boldsymbol{v})\,\mathrm{d}\boldsymbol{x} - \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}\,\mathrm{d}\boldsymbol{x} - \int_\Omega \rho\boldsymbol{g}\cdot\boldsymbol{v}\,\mathrm{d}\boldsymbol{x}.$$

Since $\int_\Omega \mathrm{div}_h(\boldsymbol{v})\,\mathrm{d}\boldsymbol{x} = 0$, this gives also

$$\int_\Omega [p - m(p)]\ \mathrm{div}_h(\boldsymbol{v})\,\mathrm{d}\boldsymbol{x} = \mu\int_\Omega \boldsymbol{\nabla}_h\boldsymbol{u} : \boldsymbol{\nabla}_h\boldsymbol{v}\,\mathrm{d}\boldsymbol{x} + \frac{\mu}{3}\int_\Omega \mathrm{div}_h(\boldsymbol{u})\mathrm{div}_h(\boldsymbol{v})\,\mathrm{d}\boldsymbol{x} - \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}\,\mathrm{d}\boldsymbol{x} - \int_\Omega \rho\boldsymbol{g}\cdot\boldsymbol{v}\,\mathrm{d}\boldsymbol{x}$$

and then

$$\int_\Omega [p - m(p)]^2\,\mathrm{d}\boldsymbol{x} = \mu\int_\Omega \boldsymbol{\nabla}_h\boldsymbol{u} : \boldsymbol{\nabla}_h\boldsymbol{v}\,\mathrm{d}\boldsymbol{x} + \frac{\mu}{3}\int_\Omega \mathrm{div}_h(\boldsymbol{u})\mathrm{div}_h(\boldsymbol{v})\,\mathrm{d}\boldsymbol{x} - \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}\,\mathrm{d}\boldsymbol{x} - \int_\Omega \rho\boldsymbol{g}\cdot\boldsymbol{v}\,\mathrm{d}\boldsymbol{x}.$$

Using theorem 2.1, lemma C.2 and the inequalities (3.5) and (3.6) we get for all $\varepsilon > 0$, the existence of D_ε, only depending on Ω, $\boldsymbol{f}$, μ, $\boldsymbol{g}$, φ, θ_0 and ε such that

$$\|p - m(p)\|_{\mathrm{L}^2(\Omega)} \le D_\varepsilon + \varepsilon\,\|p\|_{\mathrm{L}^2(\Omega)}.$$

In order to obtain an estimate on $\|p\|_{L^2}$, we now use the fact that $\int_\Omega \rho dx = M$ (and we will deduce an estimate on $\|p\|_{L^2}$ in term of Ω, $\boldsymbol{f}$, μ, $\boldsymbol{g}$, θ_0, φ and M).

We first modify a little bit the function φ (which is only nondecreasing) in order to obtain a function $\bar{\varphi}$ continuous and one-to-one from $\mathbb{R}_+$ onto $\mathbb{R}_+$, so as to be able to use its inverse function. Let $s_0 > 0$ such that $\varphi(s_0) = 1$. We define the increasing function $\bar{\varphi}$ from $\mathbb{R}_+$ to $\mathbb{R}_+$ by

$$\bar{\varphi}(s) = \frac{s}{s_0} \text{ if } 0 \le s \le s_0,$$
$$\bar{\varphi}(s) = s \max_{s\in[s_0,s]} \frac{\varphi(t)}{t} \text{ if } s_0 < s.$$

The function $\bar{\varphi}$ is a continuous increasing and one-to-one function from $\mathbb{R}_+$ onto $\mathbb{R}_+$. Then, there exists ψ (continuous increasing and one-to-one) from $\mathbb{R}_+$ onto $\mathbb{R}_+$ such that

$$\psi(\bar{\varphi}(s)) = \bar{\varphi}(\psi(s)) = s \text{ for all } s \in \mathbb{R}_+.$$

Since $Im(\psi) = \mathbb{R}_+$, we have $\lim_{s\to+\infty} \psi(s) = +\infty$.
We also remark that for all $s \ge 0$ one has for $s \ge s_0$, $\bar{\varphi}(s) \ge \varphi(s)$ and then, a.e. in Ω,

$$\psi(p) = \psi(\varphi(\rho)) \le \psi(\bar{\varphi}(\rho)) + \varphi(s_0) = \rho + 1.$$

This gives $\int_\Omega \psi(p)dx \le M + \lambda_d(\Omega)$.

We now use Lemma A.1. It gives the existence of $\bar{C}$, only depending on Ω, $\boldsymbol{f}$, μ, $\boldsymbol{g}$, θ_0, φ and M such that

$$\|p\|_{L^2} \le \bar{C}. \tag{3.7}$$

Using (3.7) in (3.4) we thus get the estimate on $\|\boldsymbol{u}\|_{1,b}$.

Finally, thanks to $p = \varphi(\rho)$ and (1.1), the estimate on ρ follows. □

Lemma 3.3. *Let $\theta_0 > 0$ and let $\mathcal{T}$ be a triangulation of the computational domain Ω such that $\theta \ge \theta_0$, where θ is defined by* (2.1)*. Let $(\boldsymbol{u}, p, \rho) \in \boldsymbol{W}_h \times L_h \times L_h$ be a solution of* (2.5)*. Then, there exists $\bar{C}$ only depending on Ω, $\boldsymbol{f}$, $\boldsymbol{g}$, μ, φ, M and θ_0 such that*

$$h^\xi\, |\rho|^2_{\mathcal{T}} \le \bar{C} \text{ and } E(\rho) \le \bar{C}$$

where $E(\rho) = \displaystyle\sum_{\sigma=K|L} min(\frac{1}{\rho_K}, \frac{1}{\rho_L})(\rho_K - \rho_L)^2 |\sigma||\boldsymbol{u}_{K,\sigma}|.$

Proof. We recall that $\rho_K > 0$ for all $K \in \mathcal{T}$. Multiplying Equation (2.5b) by $\ln(\rho(K))$ and summing over $K \in \mathcal{T}$, we thus obtain:

$$\sum_{K\in\mathcal{T}} \ln(\rho_K) \sum_{\sigma=K|L} |\sigma| \boldsymbol{u}_{K,\sigma}\rho_\sigma + \sum_{K\in\mathcal{T}} \ln(\rho_K) M_K + \sum_{K\in\mathcal{T}} \ln(\rho_K) T_K = 0,$$

with $\rho_\sigma = \rho_K$ if $\boldsymbol{u}_{K,\sigma} > 0$ and $\rho_\sigma = \rho_L$ if $\boldsymbol{u}_{K,\sigma} < 0$.
The fact that the function $s \in \mathbb{R}_+^* \to \ln(s)$ is increasing yields:

$$\sum_{K\in\mathcal{T}} \ln(\rho_K) \sum_{\sigma=K|L} |\sigma| \boldsymbol{u}_{K,\sigma}\rho_\sigma + \sum_{K\in\mathcal{T}} \ln(\rho_K) T_K \le 0 \tag{3.8}$$

Reordering the summations in the second term yields:

$$\sum_{K\in\mathcal{T}} \ln(\rho_K) T_K = \sum_{\sigma=K|L} h^\xi\, \frac{|\sigma|}{h_\sigma}\, (\rho_K + \rho_L)\, (\ln(\rho_K) - \ln(\rho_L))\, (\rho_K - \rho_L)\,.$$

Then, using the mean value theorem, for all $\sigma = K|L$ there exists $\tilde{\rho}_\sigma$ between ρ_K and ρ_L such that

$$\sum_{K\in\mathcal{T}} \ln(\rho_K) T_K = \sum_{\sigma=K|L} h^\xi \, \frac{|\sigma|}{h_\sigma} \, (\rho_K - \rho_L)^2 \, \frac{(\rho_K + \rho_L)}{\tilde{\rho_\sigma}}, \; (\, \tilde{\rho_\sigma} \in (\rho_K, \rho_L))$$

and this gives $\sum_{K\in\mathcal{T}} \ln(\rho_K) T_K \geq \sum_{\sigma=K|L} h^\xi \, \frac{|\sigma|}{h_\sigma} \, (\rho_K - \rho_L)^2$.

Using this inequality in (3.8) we get

$$\sum_{K\in\mathcal{T}} \ln(\rho_K) \sum_{\sigma=K|L} |\sigma| \boldsymbol{u}_{K,\sigma} \rho_\sigma + \sum_{\sigma=K|L} h^\xi \, \frac{|\sigma|}{h_\sigma} \, (\rho_K - \rho_L)^2 \leq 0$$

which be rewritten as

$$\sum_{\sigma=K|L} |\sigma| \boldsymbol{u}_{K,\sigma} \rho_\sigma (\ln(\rho_K) - \ln(\rho_L)) + \sum_{\sigma=K|L} h^\xi \, \frac{|\sigma|}{h_\sigma} \, (\rho_K - \rho_L)^2 \leq 0.$$

If $\sigma = K|L$, we now choose for K the cell satisfying $\boldsymbol{u}_{K,\sigma} \geq 0$. We thus obtain

$$\sum_{\sigma=K|L} |\sigma| \boldsymbol{u}_{K,\sigma} \rho_K (\ln(\rho_K) - \ln(\rho_L)) + \sum_{\sigma=K|L} h^\xi \, \frac{|\sigma|}{h_\sigma} \, (\rho_K - \rho_L)^2 \leq 0.$$

Adding and substracting the quantity $\sum_{\sigma=K|L} |\sigma| \boldsymbol{u}_{K,\sigma} (\rho_K - \rho_L)$, we then get

$$\sum_{\sigma=K|L} |\sigma| \boldsymbol{u}_{K,\sigma} [\rho_K (\ln(\rho_K) - \ln(\rho_L)) - (\rho_K - \rho_L)] + \sum_{\sigma=K|L} h^\xi \, \frac{|\sigma|}{h_\sigma} \, (\rho_K - \rho_L)^2$$
$$\leq - \sum_{\sigma=K|L} |\sigma| \boldsymbol{u}_{K,\sigma} (\rho_K - \rho_L) = - \int_\Omega \rho \, \mathrm{div}_h \boldsymbol{u} \leq \; \|\rho\|_{\mathrm{L}^2(\Omega)} \; \|\boldsymbol{u}\|_{1,b} \, .$$

Since we have $\|\rho\|_{\mathrm{L}^2(\Omega)} \leq C$ and $\|\boldsymbol{u}\|_{1,b} \leq C$ where C is given by Proposition 3.2, we obtain

$$\begin{aligned} &\sum_{\sigma=K|L} |\sigma| \boldsymbol{u}_{K,\sigma} [\rho_K (\ln(\rho_K) - \ln(\rho_L)) - (\rho_K - \rho_L)] \\ &\qquad + \sum_{\sigma=K|L} h^\xi \, \frac{|\sigma|}{h_\sigma} \, (\rho_K - \rho_L)^2 \leq C^2. \end{aligned} \tag{3.9}$$

We now use Lemma C.5 with $\psi(s) = \ln(s)$. We obtain the existence for $\sigma = KlL$ of $\tilde{\rho_\sigma}$ between ρ_K and ρ_L such that

$$\sum_{\sigma=K|L} |\sigma| \boldsymbol{u}_{K,\sigma} [\rho_K (\ln(\rho_K) - \ln(\rho_L)) - (\rho_K - \rho_L)] = \sum_{\sigma=K|L} \frac{1}{2} |\sigma| u_{K,\sigma} (\rho_K - \rho_L)^2 \tilde{\rho}_\sigma^{-1}.$$

Using this equality in (3.9), we get:

$$\underbrace{\sum_{\sigma\in\mathcal{E}_{int}} \frac{1}{2} |\sigma| u_{K,\sigma} (\rho_K - \rho_L)^2 \tilde{\rho}_\sigma^{-1}}_{S_1} + \underbrace{\sum_{\sigma=K|L} h^\xi \, \frac{|\sigma|}{h_\sigma} \, (\rho_K - \rho_L)^2}_{S_2} \leq C^2.$$

This gives $S_1 \leq C^2$ and $S_2 \leq C^2$ and concludes the proof since $S_2 = h^\xi \, |\rho|^2_{\mathcal{T}}$ and $E(\rho) \leq S_1$. □

3.3. **Passing to the limit in the discrete problem.**

Theorem 3.4. *Let $\alpha > 0$ and $0 < \xi < 2$. Let a sequence of triangulations $(\mathcal{T}_n)_{n\in\mathbb{N}}$ of Ω be given. We assume that h_n (given by $\mathcal{T}_n$) tends to zero when $n \to \infty$. In addition, we assume that the sequence of discretizations is regular, in the sense that $\theta_n \geq \theta_0 > 0$ for all $n \in \mathbb{N}$. For $n \in \mathbb{N}$, we denote by $\boldsymbol{W}_{h_n}$ and L_{h_n} the discrete spaces (for velocity, pressure and density) associated to $\mathcal{T}_n$ and by $(\boldsymbol{u}_n, p_n, \rho_n) \in \boldsymbol{W}_{h_n} \times L_{h_n} \times L_{h_n}$ a corresponding solution to the discrete problem (2.5). Then, up to the extraction of a subsequence, when $n \to \infty$:*

1. *The sequence $(\boldsymbol{u}_n)_{n\in\mathbb{N}}$ (strongly) converges in $\mathrm{L}^2(\Omega)^d$ to a limit $\boldsymbol{u} \in \mathrm{H}^1_0(\Omega)^d$ and $(p_n)_{n\in\mathbb{N}}$ and $(\rho_n)_{n\in\mathbb{N}}$ converge weakly in $\mathrm{L}^2(\Omega)$ to p, ρ respectively;*
2. *$(\boldsymbol{u}, p, \rho)$ is a solution to Problem (1.4).*

Furthermore, if φ is increasing, the sequences $(p_n)_{n\in\mathbb{N}}$ and $(\rho_n)_{n\in\mathbb{N}}$ converge in $L^p(\Omega)$ for $1 \leq p < 2$ (up to a subsequence).

Proof. The proof is devided in four steps:

• Step 1. Existence of a limit

The convergence (up to the extraction of a subsequence) of the sequence $(\boldsymbol{u}_n, p_n, \rho_n)$ is a consequence of the uniform (with respect to n) estimates of Proposition 3.2 (applying Theorem 2.2 to each component of $\boldsymbol{u}_n$). Then (up to an extraction) the sequence $(\boldsymbol{u}_n)_{n\in\mathbb{N}}$ (strongly) converges in $\mathrm{L}^2(\Omega)^d$ to a limit $\boldsymbol{u} \in \mathrm{H}^1_0(\Omega)^d$ and $(p_n)_{n\in\mathbb{N}}$ and $(\rho_n)_{n\in\mathbb{N}}$ converge weakly in $\mathrm{L}^2(\Omega)$ to p and ρ.

Since $\rho_n > 0$ and $\int_\Omega \rho_n \,\mathrm{d}\boldsymbol{x} = M$, we obtain, passing to the limit as $n \to \infty$, $\rho \geq 0$ a.e. and $\int_\Omega \rho \,\mathrm{d}\boldsymbol{x} = M$.

We now have to prove that (u, p) satisfies (1.4a) (this is proven in Step 2), that (u, ρ) statisfies (1.4b) (Step 3) and that $p = \varphi(\rho)$ a.e. (Step 4). Step 4 will also gives the strong convergence of ρ and p if φ is increasing.

• Step 2. Passing to the limit in (2.5a)

Let ψ be a function of $C_c^\infty(\Omega)^d$. We denote by ψ_n the interpolant of ψ in $\boldsymbol{W}_{h_n}$, *i.e.* $\psi_n = r_{h_n}(\psi)$. Taking $v = \psi_n$ in (2.5a), we obtain:

$$\begin{aligned} \mu \int_\Omega \boldsymbol{\nabla}_{h_n} \boldsymbol{u}_n : \boldsymbol{\nabla}_{h_n} \psi_n \,\mathrm{d}\boldsymbol{x} &+ \frac{\mu}{3} \int_\Omega \mathrm{div}_{h_n}(\boldsymbol{u}_n) \mathrm{div}_{h_n}(\psi_n) \,\mathrm{d}\boldsymbol{x} \\ - \int_\Omega p_n \, \mathrm{div}_{h_n}(\psi_n) \,\mathrm{d}\boldsymbol{x} &= \int_\Omega \boldsymbol{f} \cdot \psi_n \,\mathrm{d}\boldsymbol{x} + \int_\Omega \rho_n \boldsymbol{g} \cdot \psi_n \,\mathrm{d}\boldsymbol{x}. \end{aligned} \tag{3.10}$$

We now write $\int_\Omega \boldsymbol{\nabla}_{h_n} \boldsymbol{u}_n : \boldsymbol{\nabla}_{h_n} \psi_n \,\mathrm{d}\boldsymbol{x} = T_1 + T_2$ with

$$T_1 = \int_\Omega \boldsymbol{\nabla}_{h_n} \boldsymbol{u}_n : \boldsymbol{\nabla}_{h_n} (\psi_n - \psi) \,\mathrm{d}\boldsymbol{x} \text{ and } T_2 = \int_\Omega \boldsymbol{\nabla}_{h_n} \boldsymbol{u}_n : \boldsymbol{\nabla}_{h_n} \psi \,\mathrm{d}\boldsymbol{x}.$$

Using the third property of the interpolation operator given in theorem 2.1, we get, with $c(\theta_0)$ only depending on Ω and θ_0,

$$|T_1| \leq \|\boldsymbol{u}_n\|_{1,b} \ \|(\psi_n - \psi)\|_{1,b} \leq c(\theta_0) h_n \|\boldsymbol{u}_n\|_{1,b} \, |\psi|_{\mathrm{H}^2(\Omega)}$$

and thus T_1 tends to zero as n tends to $+\infty$. Integrating by parts over each control volume, the term T_2 reads:

$$T_2 = -\int_\Omega \boldsymbol{u}_n \cdot \Delta\psi \,\mathrm{d}\boldsymbol{x} + \sum_{\sigma \in \mathcal{E}_{\mathrm{int}}} \int_\sigma [\boldsymbol{u}_n]_\sigma : \boldsymbol{\nabla}\psi \,\mathrm{d}\gamma,$$

where $[\boldsymbol{u}_n]_\sigma = (\boldsymbol{u}_K \otimes \boldsymbol{n}_K + \boldsymbol{u}_L \otimes \boldsymbol{n}_L)$ if $\sigma = K|L$ (for all $K \in \mathcal{T}_n$, $\boldsymbol{u}_K$ is the value of $\boldsymbol{u}_n$ in K, and $\boldsymbol{n}_K$ is the normal vector to ∂K exterior to K). We omit the dependance of $\mathcal{E}_{\rm int}$ with respect to n. Noticing that $\boldsymbol{n}_L = -\boldsymbol{n}_K$ and applying Lemma 2.4 in [10], we get, again with $c(\theta_0)$ only depending on Ω and θ_0,

$$\Big| \sum_{\sigma \in \mathcal{E}_{\rm int}} \int_\sigma [\boldsymbol{u}_n] : \boldsymbol{\nabla}\psi \; n_\sigma \,{\rm d}\gamma \Big| \le c(\theta_0) h_n \, \|\boldsymbol{u}_n\|_{1,b} \, |\psi|_{{\rm H}^2(\Omega)}$$

and thus tends to zero as n tends to $+\infty$. On the other hand we have:

$$\begin{aligned} -\int_\Omega \boldsymbol{u}_n \cdot \Delta\psi \,{\rm d}\boldsymbol{x} &\to -\int_\Omega \boldsymbol{u} \cdot \Delta\psi \,{\rm d}\boldsymbol{x} \text{ as } n \to +\infty \\ &= \int_\Omega \boldsymbol{\nabla}\boldsymbol{u} : \boldsymbol{\nabla}\psi \,{\rm d}\boldsymbol{x} \text{ since } \boldsymbol{u} \in {\rm H}^1_0(\Omega). \end{aligned}$$

Then, the first term of the Left Hand Side of (3.10) converges to $\int_\Omega \boldsymbol{\nabla}\boldsymbol{u} : \boldsymbol{\nabla}\psi \,{\rm d}\boldsymbol{x}$ as $n \to \infty$. For the second term of (3.10), using the first property of the interpolation operator in theorem 2.1, we get, with $[\boldsymbol{u}_n \cdot \boldsymbol{n}]_\sigma = \boldsymbol{u}_K \cdot \boldsymbol{n}_K + \boldsymbol{u}_L \cdot \boldsymbol{n}_L$,

$$\begin{aligned} &\int_\Omega {\rm div}_{h_n}(\boldsymbol{u}_n) {\rm div}_{h_n}(\psi_n) \,{\rm d}\boldsymbol{x} = \int_\Omega {\rm div}_{h_n}(\boldsymbol{u}_n) {\rm div}(\psi) \,{\rm d}\boldsymbol{x} \\ &= \sum_{K \in \mathcal{T}_n} \sum_{i \le d} \sum_{j \le d} \int_K (\boldsymbol{u}_n)_i \frac{\partial^2 \psi_j}{\partial x_i \partial x_j} \,{\rm d}\boldsymbol{x} + \sum_{K \in \mathcal{T}_n} \int_{\partial K} \boldsymbol{u}_n {\rm div}\psi . n_K \,{\rm d}\gamma \\ &= \sum_{K \in \mathcal{T}_n} \sum_{i \le d} \sum_{j \le d} \int_K (\boldsymbol{u}_n)_i \frac{\partial^2 \psi_j}{\partial x_i \partial x_j} \,{\rm d}\boldsymbol{x} + \sum_{\sigma \in \mathcal{E}_{\rm int}} \int_\sigma [\boldsymbol{u}_n \cdot \boldsymbol{n}]_\sigma {\rm div}\psi \,{\rm d}\gamma \\ &\qquad = T_{2,1} + T_{2,2}. \end{aligned}$$

Applying Lemma 2.4 in [10], we get, with $c(\theta_0)$ only depending on Ω and θ_0,

$$|T_{2,2}| = | \sum_{\sigma \in \mathcal{E}_{\rm int}} \int_\sigma [\boldsymbol{u}_n \cdot \boldsymbol{n}]_\sigma {\rm div}\psi \,{\rm d}\gamma | \le c(\theta_0) h_n \, \|\boldsymbol{u}_n\|_{1,b} \, |{\rm div}\psi|_{{\rm H}^1(\Omega)}$$

and thus $T_{2,2}$ tends to zero as n tends to $+\infty$. Then, the second term of (3.10) has the same limit as $T_{2,1}$ and this limit is $\int_\Omega {\rm div}\boldsymbol{u} \; {\rm div}\psi \,{\rm d}\boldsymbol{x}$.

For the third term of (3.10), we use, once again, Theorem 2.1 which yields:

$$\int_\Omega p_n \; {\rm div}_{h_n}(\psi_n) \,{\rm d}\boldsymbol{x} = \int_\Omega p_n \; {\rm div}(\psi) \,{\rm d}\boldsymbol{x} \to \int_\Omega p \; {\rm div}(\psi) \,{\rm d}\boldsymbol{x} \text{ as } n \to +\infty.$$

We now consider the Right Hand Side of (3.10). Since $\psi_n \to \psi$ in $L^2(\Omega)^d$ we obtain

$$\int_\Omega \boldsymbol{f} \cdot \psi_n \,{\rm d}\boldsymbol{x} \to \int_\Omega \boldsymbol{f} \cdot \psi \,{\rm d}\boldsymbol{x} \text{ as } n \to +\infty.$$

For the last term of (3.10), we use, once again, the $(L^2)^d$ convergence of ψ_n to ψ and we use the weak-L^2 convergence of ρ_n to ρ. We obtain

$$\int_\Omega \rho_n \boldsymbol{g} \cdot \psi_n \,{\rm d}\boldsymbol{x} \to \int_\Omega \rho \boldsymbol{g} \cdot \psi \,{\rm d}\boldsymbol{x} \text{ as } n \to +\infty.$$

Finally, we can pass to limit in (3.10) as $n \to \infty$ and we get (1.4a) for all $\psi \in C_c^\infty(\Omega)^d$ (and then, by density, for all $\psi \in H_0^1(\Omega)^d$), namely:

$$\begin{aligned} \mu \int_\Omega \boldsymbol{\nabla}\boldsymbol{u} : \boldsymbol{\nabla}\psi \,{\rm d}\boldsymbol{x} + \frac{\mu}{3} \int_\Omega {\rm div}(\boldsymbol{u}) {\rm div}(\psi) \,{\rm d}\boldsymbol{x} - \int_\Omega p \; {\rm div}(\psi) \,{\rm d}\boldsymbol{x} \\ = \int_\Omega \boldsymbol{f} \cdot \psi \,{\rm d}\boldsymbol{x} + \int_\Omega \rho \boldsymbol{g} \cdot \psi \,{\rm d}\boldsymbol{x}. \end{aligned}$$

• **Step 3. Passing to the limit in** (2.5b)

Let ψ be a function of $C_c^\infty(\Omega)^d$.Multiplying (2.5b) by $\psi_K = \psi(x_K)$ and summing over $K \in \mathcal{T}_n$ we obtain:

$$T_1 + T_2 + T_3 = \sum_{K\in\mathcal{T}_n} \psi_K \sum_{\sigma\in\mathcal{E}(K)} |\sigma| \boldsymbol{u}_{K,\sigma}\rho_\sigma + \sum_{K\in\mathcal{T}_n} h_n^\alpha |K| \psi_K(\rho_K - \rho^*)$$
$$+ \sum_{K\in\mathcal{T}_n} \psi_K \sum_{\sigma\in\mathcal{E}(K)} h_n^\xi \, \frac{|\sigma|}{h_\sigma} \, (\rho_K + \rho_L)\, (\rho_K - \rho_L) = 0. \tag{3.11}$$

The first term T_1 reads, with $\psi_\sigma = \psi(x_\sigma)$,

$$T_1 = \sum_{K\in\mathcal{T}_n} \sum_{\sigma\in\mathcal{E}(K)} |\sigma| \boldsymbol{u}_{K,\sigma}\rho_\sigma \psi_K = \sum_{K\in\mathcal{T}_n} \sum_{\sigma\in\mathcal{E}(K)} |\sigma| \boldsymbol{u}_{K,\sigma}\rho_\sigma (\psi_K - \psi_\sigma)$$
$$= \sum_{K\in\mathcal{T}_n} \sum_{\sigma\in\mathcal{E}(K)} |\sigma| \boldsymbol{u}_{K,\sigma}\rho_K (\psi_K - \psi_\sigma) + R_1$$
$$\text{with } R_1 = -\sum_{K\in\mathcal{T}_n} \sum_{\sigma\in\mathcal{E}(K)} |\sigma| \boldsymbol{u}_{K,\sigma}(\rho_K - \rho_\sigma)(\psi_K - \psi_\sigma).$$

Then,

$$T_1 = \sum_{K\in\mathcal{T}_n} \rho_K \psi_K \sum_{\sigma\in\mathcal{E}(K)} |\sigma| \boldsymbol{u}_{K,\sigma} - \sum_{K\in\mathcal{T}_n} \rho_K \sum_{\sigma\in\mathcal{E}(K)} |\sigma| \boldsymbol{u}_{K,\sigma}\psi_\sigma + R_1$$
$$= \sum_{K\in\mathcal{T}_n} \rho_K \int_K \psi \mathrm{div} u_n \,\mathrm{d}\boldsymbol{x} - \sum_{K\in\mathcal{T}_n} \rho_K \int_K \mathrm{div}(\psi u_n)\,\mathrm{d}\boldsymbol{x} + R_1 + R_2 + R_3.$$
$$\text{with } R_2 = -\sum_{K\in\mathcal{T}_n} \rho_K \int_K (\psi - \psi_K)\mathrm{div}\boldsymbol{u}_n \,\mathrm{d}\boldsymbol{x}$$
$$\text{and } R_3 = \sum_{K\in\mathcal{T}_n} \rho_K \sum_{\sigma\in\mathcal{E}(K)} \int_\sigma (\psi - \psi_\sigma)\boldsymbol{u}_n \cdot \boldsymbol{n}_{K,\sigma}\,\mathrm{d}\gamma$$
$$T_1 = -\int_\Omega \rho_n \boldsymbol{u}_n \cdot \boldsymbol{\nabla}\psi + R_1 + R_2 + R_3.$$

Let us now prove that the terms $R_1, R_2, R_3 \to 0$ as n $\to +\infty$. We begin with R_1. One has, with $C_\psi = \||\boldsymbol{\nabla}\psi|\|_{L^\infty(\Omega)}$,

$$|R_1| = |\sum_{K\in\mathcal{T}_n} \sum_{\sigma\in\mathcal{E}(K)} |\sigma| \boldsymbol{u}_{K,\sigma}(\rho_K - \rho_\sigma)(\psi_K - \psi_\sigma)|$$
$$\leq C_\psi \sum_{\sigma=K|L} (h_K + h_L)|\rho_K - \rho_L||\sigma||\boldsymbol{u}_{K,\sigma}|.$$

This gives, with the Cauchy-Schwarz inequality,

$$|R_1| \leq C_\psi E(\rho_n)\Big(\sum_{\sigma=K|L} \frac{(h_K + h_L)^2}{min(\frac{1}{\rho_K}, \frac{1}{\rho_L})}|\sigma||\boldsymbol{u}_{K,\sigma}|\Big)^{\frac{1}{2}}.$$

Then,

$$|R_1| \leq C_\psi E(\rho_n)(\underbrace{\sum_{\sigma=K|L} (h_K + h_L)^2(\rho_K + \rho_L)|\sigma||\boldsymbol{u}_{K,\sigma}|)^{\frac{1}{2}}}_{S2}.$$

Using again the Cauchy Schwarz inequality we thus obtain:

$$S_2 \le \Big(\sum_{\sigma=K|L} (h_K+h_L)|\sigma|(\rho_K+\rho_L)^2\Big)^{1/2}\Big(\sum_{\sigma=K|L}(h_K+h_L)^3|\sigma||\boldsymbol{u}_{K,\sigma}|^2\Big)^{1/2}$$

The properties of the scheme given in section 2 and Hölder's Inequality yields, with $C_1(\theta_0)$ and $C_2(\theta_0)$ only depending on Ω and θ_0,

$$\begin{aligned} S_2 &\le C_1(\theta_0)\Big(\sum_{\sigma=K|L}(|K|+|L|)(\rho_K+\rho_L)^2\Big)^{1/2}\Big(\sum_{\sigma=K|L}(h_K+h_L)^3\|\boldsymbol{u}_n\|^2_{L^2(\sigma)}\Big)^{1/2} \\ &\le C_2(\theta_0)\Big(\sum_{K\in\mathcal{T}_n}|K|\rho_K^2\Big)^{1/2}\Big(\sum_{\sigma=K|L}h_\sigma^3\|\boldsymbol{u}_n\|^2_{L^2(\sigma)}\Big)^{1/2}. \end{aligned}$$

The estimate on ρ_n in $L^2(\Omega)$ gives the existence of C_3, only depending on the L^2-bound on ρ_n and on $C_2(\theta_0)$ such that:

$$S_2 \le C_3\Big(\sum_{\sigma=K|L}h_\sigma^3\|\boldsymbol{u}_n\|^2_{L^2(\sigma)}\Big)^{1/2}.$$

By Lemma 2.3 in [10], we have:

$$\|\boldsymbol{u}_n\|_{L^2(\sigma)} \le \Big(d\frac{|\sigma|}{|K|}\Big)^{1/2}\big(\|\boldsymbol{u}_n\|_{L^2(K)}+h_K\|\boldsymbol{\nabla}\boldsymbol{u}_n\|_{L^2(K)}\big).$$

We thus obtain, with some C_4 and C_5 only depending on the L^2-bound on ρ_n, Ω and θ_0,

$$\begin{aligned} S_2 &\le C_4\Big(\sum_{K\in\mathcal{T}_n}h_K^2\big(\|\boldsymbol{u}_n\|^2_{L^2(K)}+h_K^2\|\boldsymbol{\nabla}\boldsymbol{u}_n\|^2_{L^2(K)}\big)\Big)^{1/2} \\ &\le C_5 h_n\big(\|\boldsymbol{u}_n\|^2_{L^2(\Omega)}+\|\boldsymbol{u}_n\|^2_{1,b}\big)^{1/2}. \end{aligned}$$

Finally, thanks to the bound on u_n (Proposition 3.2) we get $\lim_{n\to\infty} S_2 = 0$ and thanks to the bound on $E(\rho_n)$ (Lemma 3.3) we conclude that $\lim_{n\to\infty} R_1 = 0$.

We now come to R_2. One has

$$|R_2| \le C_\psi h_n\,\|\rho_n\|_{\mathrm{L}^2(\Omega)}\,\|\mathrm{div}_{h_n}(\boldsymbol{u}_n)\|_{\mathrm{L}^2(\Omega)} \le C_\psi h_n\,\|\rho_n\|_{\mathrm{L}^2(\Omega)}\,\|\boldsymbol{u}_n\|_{1,b}\,,$$

which tends to 0 as $n\to+\infty$.

It remains to treat R_3. One has

$$\begin{aligned} |R_3| &= \Big|\sum_{K\in\mathcal{T}_n}\rho_K\sum_{\sigma\in\mathcal{E}(K)}\int_\sigma(\psi-\psi_\sigma)\boldsymbol{u}_n\cdot\boldsymbol{n}_{K,\sigma}\,\mathrm{d}\gamma\Big| \\ &= \Big|\sum_{\sigma=K|L}(\rho_K-\rho_L)\int_\sigma(\psi-\psi_\sigma)\boldsymbol{u}_n\cdot\boldsymbol{n}_{K,\sigma}\,\mathrm{d}\gamma\Big| \end{aligned}$$

In order to prove that $\lim_{n\to\infty} R_3 = 0$, we treat separetely, the interfaces σ where the sign of $\boldsymbol{u}_n\cdot\boldsymbol{n}_{K,\sigma}$ is constant or not (for σ between K and L, it can be different for K and L). For the interfaces where the sign of $\boldsymbol{u}_n\cdot\boldsymbol{n}_{K,\sigma}$ is constant, we use the same arguments as for the first term R_1 (bound on $\boldsymbol{u}_n$ and bound on $E(\rho_n)$) and we get a bound in $\sqrt{h_n}$ for the sum of these terms. For the interfaces where the sign of $\boldsymbol{u}_n\cdot\boldsymbol{n}_{K,\sigma}$ is not constant, we use a bound (only depending of the regularity of the mesh, that is θ) of $\|\boldsymbol{u}_n\cdot\boldsymbol{n}_{K,\sigma}\|_{L^1(\sigma)}$ by $\|\boldsymbol{\nabla}\boldsymbol{u}_n\|_{L^1(K)}$ (this bound uses the fact that $\boldsymbol{u}_n \cdot \boldsymbol{n}_{K,\sigma}$ vanishes at a point of σ). Then, thanks to the bound on $\|\boldsymbol{u}_n\|_{1,b}$ and $\|\rho_n\|_{L^2(\Omega)}$, we get a bound in h_n for the sum of these terms.

Finally, since $\lim_{n\to\infty} R_i = 0$ for $i = 1, 2, 3$, one has

$$\lim_{n\to\infty} T_1 = -\lim_{n\to\infty} \int_\Omega \rho_n u_n \cdot \boldsymbol{\nabla}\psi \,\mathrm{d}\boldsymbol{x}.$$

Using the $L^2(\Omega)$ convergence of $\boldsymbol{u}_n$ and the $L^2(\Omega)$-weak convergence of ρ_n, we conclude that

$$\lim_{n\to\infty} T_1 = -\int_\Omega \rho \boldsymbol{u} \cdot \boldsymbol{\nabla}\psi \,\mathrm{d}\boldsymbol{x}.$$

We now prove that T_2 and T_3 tend to 0 as $n \to \infty$. We remark that

$$| T_2 |=| \sum_{K\in\mathcal{T}_n} h_K^\alpha \,|K|\, (\rho_K - \rho^*)\, \psi_K \,|\leq h_n^\alpha 2M\, \|\psi\|_{L^\infty(\Omega)} \to 0 \text{ as } n \to +\infty$$

and

$$\begin{aligned} | T_3 | &=| \sum_{K\in\mathcal{T}_n} \sum_{\sigma\in\mathcal{E}(K)} h_n^\xi \,\frac{|\sigma|}{h_\sigma}\, (\rho_K + \rho_L)\, (\rho_K - \rho_L)\, \psi_K \,| \\ &=| \,h_n^\xi \sum_{\sigma=K|L} \frac{|\sigma|}{h_\sigma}\, (\rho_K + \rho_L)\, (\rho_K - \rho_L)\, (\psi_K - \psi_L)\,| \\ &\leq C_\psi\, h_n^\xi \sum_{\sigma=K|L} (h_K + h_L)\, \frac{|\sigma|}{h_\sigma}\, (\rho_K + \rho_L)\, |\, \rho_K - \rho_L \,|\,. \end{aligned}$$

We now use the Cauchy-Schwarz Inequality to obtain, with C_1 only depending on ψ and the bound on $h_n^\xi|\rho_n|^2$ given by Lemma 3.3,

$$\begin{aligned} |T_3| &\leq C_\psi\, h_n^\xi \,|\, \rho_n \,|_\mathcal{T} \,\Big(\sum_{\sigma=K|L} (h_K + h_L)^2 \,\frac{|\sigma|}{h_\sigma}\, (\rho_K + \rho_L)^2 \Big)^{\frac{1}{2}} \\ &\leq C_1\, h_n^{\xi/2} \Big(\sum_{\sigma=K|L} (h_K + h_L)^2 \,\frac{|\sigma|}{h_\sigma}\, (\rho_K + \rho_L)^2 \Big)^{\frac{1}{2}}. \end{aligned}$$

The properties of the mesh given in section 2 yield the existence of $c(\theta_0)$ only depending on Ω and θ_0 such that

$$\frac{|\sigma|}{h_\sigma} \leq c(\theta_0) \frac{|K| + |L|}{(h_K + h_L)^2}.$$

We thus obtain $|\, T_3 \,|\leq C_1 \sqrt{c(\theta_0)}\, h_n^{\xi/2} (\sum_{\sigma=K|L} (|K| + |L|)\, (\rho_K + \rho_L)^2)^{\frac{1}{2}}$. Thanks to the L^2-estimate on ρ_n, we then conclude that $\lim_{n\to\infty} T_3 = 0$.

Finally, we can pass to the limit in (3.11) as $n \to \infty$ and we obtain (1.4b) for all $\psi \in C_c^\infty(\Omega)$. This gives also (1.4b) for all $\psi \in W^{1,\infty}(\Omega)$ thanks to Lemma B.6 (since $u \in H_0^1(\Omega)$ and $\rho \in L^2(\Omega)$).

• Step 4. Passing to the limit in the Equation Of State

In order to conclude the proof of Theorem 3.4, it remains to prove that the equation of state is satisfied, that is $p = \varphi(\rho)$ a.e. in Ω. This is a tricky part of the proof.

Let $(q_n)_{n\in\mathbb{N}}$ be a sequence such that $q_n \in L_{h_n}$ for all $n \in \mathbb{N}$. We assume that the sequence $(q_n)_{n\in\mathbb{N}}$ weakly converges in $\mathrm{L}^2(\Omega)$ to $q \in \mathrm{L}^2(\Omega)$ and satisfies

$$|q_n|_\mathcal{T} \leq c\, h_n^{-\eta},$$

where c is a positive real number and η is such that $\eta < 1$. Then one has:

$$\forall \psi \in \mathrm{C}_c^\infty(\Omega), \quad \lim_{n\to\infty} \int_\Omega \big(\mathrm{div}_{h_n}(\boldsymbol{u}_n) - p_n\big)\, q_n \psi \,\mathrm{d}\boldsymbol{x} = \int_\Omega \big(\mathrm{div}(\boldsymbol{u}) - p\big)\, q \psi \,\mathrm{d}\boldsymbol{x}. \tag{3.12}$$

This result is proven in [6], Proposition 5.9. Indeed, in Proposition 5.9 of [6] the hypothesis on ρ is $\rho \in L^{2\gamma}(\Omega)$, $\gamma > 1$, and the sequence $(\rho_n)_{n\in\mathbb{N}}$ converges to ρ weakly in $L^{2\gamma}(\Omega)$, but the proof given in [6] is also true for $\gamma = 1$.

Taking $q_n = \rho_n$ in (3.12) (which is possible with $\eta = \xi/2$, thanks to Lemma 3.3), one obtains

$$\forall \psi \in \mathrm{C}_c^\infty(\Omega), \quad \lim_{n\to\infty} \int_\Omega \big(\mathrm{div}_{h_n}(\boldsymbol{u}_n) - p_n\big)\, \rho_n \psi \,\mathrm{d}\boldsymbol{x} = \int_\Omega \big(\mathrm{div}(\boldsymbol{u}) - p\big)\, \rho \psi \,\mathrm{d}\boldsymbol{x}. \tag{3.13}$$

We now want to prove (3.13) with $\psi = 1$ a.e. on Ω. This is possible, thanks to Lemma C.1, if the sequence $((\mathrm{div}_{h_n}\boldsymbol{u}_n - p_n)\rho_n)_{n\in\mathbb{N}}$ is equi-integrable. The condition (1.1) on φ, and the L^2-bound on $\mathrm{div}_{h_n}\boldsymbol{u}_n$ and on p_n will give this equi-integrability. Let $a > 0$ and $b > 0$ given by (1.1). One has a.e. on Ω,

$$a\rho_n \le \varphi(\rho_n) + b = p_n + b,$$

so that

$$\rho_n^2 \le \frac{2p_n^2}{a^2} + \frac{2b^2}{a^2}.$$

If C is a bound for the L^2-norm of p_n (such a bound is given by Proposition 3.2), one obtains for any borelian subset A of Ω,

$$\int_A \rho_n^2 dx \le \frac{2C^2}{a^2} + \frac{2b^2}{a^2}|A|.$$

Let $\varepsilon > 0$, we then take $a^2 = 2C^2/\varepsilon$ which yields:

$$\int_A \rho_n^2 dx \le \varepsilon + \frac{2b^2}{a^2}|A|.$$

and then, with $\delta = \dfrac{\varepsilon a^2}{2b^2}$,

$$|A| \le \delta \Rightarrow \int_A \rho_n^2 dx \le 2\varepsilon.$$

This proves the equi-integrability of the sequence $(\rho_n^2)_{n\in\mathbb{N}}$. Since the sequence $((\mathrm{div}_{h_n}\boldsymbol{u}_n - p_n))_{n\in\mathbb{N}}$ is bounded in $\mathrm{L}^2(\Omega)$, we then easily conclude (with the Cauchy-Scwarz inequality) that the sequence $((\mathrm{div}_{h_n}\boldsymbol{u}_n - p_n)\rho_n)_{n\in\mathbb{N}}$ is equi-integrable. Thus Lemma C.1 yields the conclusion, namely (3.13) is true for $\psi = 1$ a.e. on Ω:

$$\lim_{n\to\infty} \int_\Omega \big(\mathrm{div}_{h_n}(\boldsymbol{u}_n) - p_n\big)\, \rho_n \,\mathrm{d}\boldsymbol{x} = \int_\Omega \big(\mathrm{div}(\boldsymbol{u}) - p\big)\, \rho \,\mathrm{d}\boldsymbol{x}. \tag{3.14}$$

We now want to get rid of $\int_\Omega \rho \mathrm{div}(\boldsymbol{u})\,\mathrm{d}\boldsymbol{x}$ and $\int_\Omega \rho_n \mathrm{div}(\boldsymbol{u}_n)\,\mathrm{d}\boldsymbol{x}$ in (3.14).

Since $\rho \in L^2(\Omega)$, $\rho \ge 0$ a.e. in Ω, $u \in H_0^1(\Omega)^d$ and (ρ, u) satisfies (1.4b), we can use Lemma B.1. It gives

$$\int_\Omega \rho \;\mathrm{div}(\boldsymbol{u})\,\mathrm{d}\boldsymbol{x} = 0. \tag{3.15}$$

Then, using (3.15) in (3.14) we get:

$$\lim_{n\to\infty}\int_\Omega \big(p_n - \mathrm{div}_{h_n}(\boldsymbol{u}_n)\big)\,\rho_n\,\mathrm{d}\boldsymbol{x} - \int_\Omega p\,\rho\,\mathrm{d}\boldsymbol{x} = 0.$$

By Lemma 3.1 we also have $\int_\Omega \rho_n \mathrm{div}_{h_n}(\boldsymbol{u}_n)\ \mathrm{d}\boldsymbol{x} \le 0$. Hence:

$$\limsup_{n\to\infty}\int_\Omega p_n\,\rho_n\,\mathrm{d}\boldsymbol{x} \le \int_\Omega p\,\rho\,\mathrm{d}\boldsymbol{x}. \tag{3.16}$$

To conclude the proof of $p=\varphi(\rho)$, we will now use the so called Minty trick. Let $\bar\rho \in L^2(\Omega)$ such that $\varphi(\bar\rho)\in L^2(\Omega)$. We define for $n\in\mathbb{N}$ the function G_n by

$$G_n = (\varphi(\rho_n)-\varphi(\bar\rho))(\rho_n-\bar\rho) = (p_n-\varphi(\bar\rho))(\rho_n-\bar\rho).$$

One has $G_n \in L^1(\Omega)$, $G_n\ge 0$ a.e. in Ω (since φ is nondecreasing) and

$$0 \le \int_\Omega G_n\,dx = \int_\Omega (p_n\rho_n - p_n\bar\rho - \varphi(\bar\rho)\rho_n + \varphi(\bar\rho)\bar\rho)\,\mathrm{d}\boldsymbol{x}. \tag{3.17}$$

Using (3.16) and the weak convergences of p_n to p and ρ_n to ρ in $L^2(\Omega)$, we obtain:

$$0 \le \limsup_{n\to\infty}\int_\Omega G_n\,dx \le \int_\Omega (p-\varphi(\bar\rho))(\rho-\bar\rho)\,\mathrm{d}\boldsymbol{x}.$$

We have thus proven that for all $\bar\rho\in L^2(\Omega)$ such that $\varphi(\bar\rho)\in L^2(\Omega)$ one has

$$\int_\Omega (p-\varphi(\bar\rho))(\rho-\bar\rho)\,\mathrm{d}\boldsymbol{x} \ge 0. \tag{3.18}$$

We now have to choose $\bar\rho$ conveniently to deduce $p=\varphi(\rho)$ a.e. on Ω from (3.18). The idea of the Minty trick is to take $\bar\rho = \rho + (1/k)\psi$ with $\psi\in C_c^\infty(\Omega)$, $k\in\mathbb{N}^\star$ and to let k goes to $+\infty$. Unfortunately, $\varphi(\rho+(1/k)\psi)$ is not necessarily in $L^2(\Omega)$. then, such a choice for $\bar\rho$ is not possible. We will use here (and only here) the convexity of φ. Since $(\rho_n)_n$ weakly converges in $L^2(\Omega)$ to ρ and since the sequence $(\varphi(\rho_n))_{n\in\mathbb{N}}$ is bounded in $L^2(\Omega)$, we deduce, using the convexity of φ, that $\varphi(\rho)\in L^2(\Omega)$. This is proven in Lemma B.8. This allows us a convenient choice for $\bar\rho$.

Let $\psi\in C_c^\infty(\Omega,\mathbb{R})$. For $k,m\in\mathbb{N}^\star$, we set

$$\rho_{k,m} = \rho + \frac{1}{k}\psi 1_{\rho\le m}.$$

Since $\rho\in L^2(\Omega)$, one has $\rho_{k,m}\in L^2(\Omega)$. Using the fact that φ is nondecreasing (and nonnegative), we have, with $M=\|\psi\|_{L^\infty(\Omega)}$,

$$\varphi(\rho_{k,m}) \le \varphi(\rho) + \varphi(m+M),$$

so that $\varphi(\rho_{k,m})\in L^2(\Omega)$ (since $\varphi(\rho)\in L^2(\Omega)$). Then, since $\rho_{k,m}$ and $\varphi(\rho_{k,m})$ belong to $L^2(\Omega)$, we can choose $\bar\rho=\rho_{k,m}$ in (3.18). We obtain

$$\int_\Omega (p-\varphi(\rho+\frac{1}{k}\psi 1_{\rho\le m}))\psi 1_{\rho\le m} \le 0.$$

Fixing m in $\mathbb{N}^\star$, we use the Dominated Convergence theorem on the sequence $(g_k)_{k\in\mathbb{N}^\star}$ with $g_k = (p-\varphi(\rho+\frac{1}{k}\psi 1_{\rho\le m}))\psi 1_{\rho\le m}$. Indeed, the continuity of φ gives $g_k \to (p-\varphi(\rho))\psi 1_{\rho\le m}$ a.e. in Ω. Furthermore, since φ is nondecreasing, one has, for all $n\in\mathbb{N}^\star$,

$$|g_k| \le H = [p+\varphi(\rho)+\varphi(m+M)]|\psi| \text{ a.e. in } \Omega,$$

and $H \in L^1(\Omega)$. Then, the Dominated Convergence theorem yields

$$\int_\Omega (p - \varphi(\rho))\psi 1_{\rho \le m} \le 0.$$

Changing ψ in $-\psi$, we conclude that $\int_\Omega (p - \varphi(\rho))\psi 1_{\rho \le m} = 0$ for all $\psi \in C_c^\infty(\Omega, \mathbb{R})$.

Once again by the Dominated Convergence Theorem, as m to $+\infty$ we get: $\int_\Omega (p - \varphi(\rho))\psi = 0$ for all $\psi \in C_c^\infty(\Omega)$ This gives $p = \varphi(\rho)$ a.e. in Ω.

The hypothesis of convexity of the function φ is only used to get that the four terms of the Right Hand Side of (3.17) are in $L^1(\Omega)$. If the hypothesis of convexity for φ is replaced by the hypothesis (1.3), the proof is a little simpler. In this case, the L^2-bound of p_n gives a $L^{2\gamma}$-bound on ρ_n (since $a\rho_n^\gamma \le \varphi(\rho_n) + b = p_n + b$). Then one has $\rho_n \to \rho$ weakly in $L^{2\gamma}(\Omega)$ and we can use G_n with $\bar\rho \in L^{2\gamma}(\Omega)$ such that $\varphi(\bar\rho) \in L^{2\gamma/(2\gamma-1)}(\Omega)$ (which is the dual space to $L^{2\gamma}(\Omega)$). With such a $\bar\rho$, the four terms in the Right Hand Side of (3.17) are in $L^1(\Omega)$) and we obtain (3.18). For $\psi \in C_c^\infty(\Omega)$ and $k > 0$, we take $\bar\rho = \rho + (1/k)\psi$ (so that $\bar\rho \in L^{2\gamma}(\Omega)$ and $\varphi(\bar\rho) \in L^{2\gamma/(2\gamma-1)}(\Omega)$). Passing to the limit as $k \to +\infty$ in (3.18) leads to

$$\int_\Omega (p - \varphi(\rho))\psi \,\mathrm{d}\boldsymbol{x} \le 0.$$

With this inequality, we conclude, as before, that $p = \varphi(\rho)$ a.e. in Ω.

In both cases (φ convex or φ satisfies (1.3)), if φ is increasing, we can obtain a strong convergence of ρ_n and p_n, as in [6]. We take directly $\bar\rho = \rho$ in the definition of G_n. We obtain that $G_n = (\varphi(\rho_n) - \varphi(\rho))(\rho_n - \rho) \to 0$ in $L^1(\Omega)$ as $n \to \infty$. Then, up to a subsequence, one has $G_n \to 0$ a.e. in Ω. Since φ is increasing, we finally deduce that $\rho_n \to \rho$ a.e.. This yields also $\rho_n \to \rho$ in $L^q(\Omega)$ for all $q \in [1, 2[$, $p = \varphi(\rho)$ a.e. in Ω and $p_n \to p$ in $L^q(\Omega)$ for all $q \in [1, 2[$.

The proof of Theorem 3.4 is now complete. □

Conclusion

We gave a scheme for the discretization of the compressible Stokes problem with a general EOS and we proved the existence of a solution of the scheme along with the convergence of the approximate solution to an exact solution (up to a subsequence) as the mesh size goes to zero. A first difficulty of the paper is to get some estimates on the approximate solution (in particular with the dependancy of the forcing term with the density). A second complication is in the passage to the limit in the EOS. This difficulty is due to the nonlinearity of the EOS and the fact that the estimates on pressure and density only lead to weak convergences. It will be now interesting to consider the Navier-Stokes problem along with the evolution problem.

Appendix A. Estimate on p

Lemma A.1. *Let Ω be a bounded set of $\mathbb{R}^d$ ($d \ge 1$) and $p \in L^2(\Omega)$, $p \ge 0$ a.e.. We assume that there exist $a < 1$ and $b \in \mathbb{R}$ such that*

$$\|p - m\|_{L^2} \le a\|p\|_{L^2} + b,$$

where m is the mean value of p. Furthermore, we assume that there exist $A \in \mathbb{R}$ and a continuous function ψ from $\mathbb{R}_+$ to $\mathbb{R}_+$ such that $\int_\Omega \psi(p)dx \le A$ and $\lim_{s\to\infty} \psi(s) = +\infty$. Then, there exists C only depending on Ω, a, b A and ψ such that $\|p\|_{L^2} \le C$.

Proof of Lemma A.1
We first modify the function ψ. Let $s_0 \in \mathbb{R}_+$ such that $\psi(s_0) > 0$. We define $\bar\psi$ by

$$\bar\psi(s) = \psi(s_0) \text{ if } 0 \le s \le s_0,$$
$$\bar\psi(s) = s \min_{t\in[s_0,s]} \frac{\psi(t)}{t} \text{ if } s_0 < s.$$

We remark that $\bar\psi(s) \le \psi(s)$ for $s \ge s_0$, so that $\int_\Omega \bar\psi(p)dx \le \bar A = A + \psi(s_0)\lambda_d(\Omega)$. Furthermore, one has $\lim_{s\to+\infty} \bar\psi(s) = +\infty$. In order to prove this result, let $(s_n)_{n\in\mathbb{N}}$ be an incresing sequence such that $\lim_{n\to\infty} s_n = +\infty$. For $n \in \mathbb{N}$ let $t_n \in [s_0, s_n]$ such that $\bar\psi(s_n) = (\psi(t_n)/t_n)s_n$. For any converging (in $\mathbb{R}_+ \cup \{+\infty\}$) subsequence of the sequence $(t_n)_{n\in\mathbb{N}}$, still denoted $(t_n)_{n\in\mathbb{N}}$, we have two possible cases,
First case. $\lim_{n\to\infty} t_n = t \in \mathbb{R}_+$. Then $\lim_{n\to\infty} \bar\psi(s_n) = +\infty$
(since $\psi(t)/t > 0$)
Second case. $\lim_{n\to\infty} t_n = +\infty$. Then $\lim_{n\to\infty} \bar\psi(s_n) = +\infty$
since $\bar\psi(s_n) \ge \psi(t_n)$.
We then conclude that $\lim_{s\to+\infty} \bar\psi(s) = +\infty$. Finally we also remark that the function $s \mapsto \frac{\bar\psi(s)}{s}$ is nonincreasing on $\mathbb{R}_+$.
We now prove the bound on $\|p\|_{L^2}$. Let $N > 0$, one has

$$\int_\Omega p(x)dx = \int_{p\ge N} p(x)dx + \int_{p<N} p(x)dx \le \frac{1}{N}\int_\Omega p^2(x)dx + \frac{N}{\bar\psi(N)}\int_\Omega \bar\psi(p(x))dx.$$

This gives $m\lambda_d(\Omega) \le \frac{1}{N}\|p\|_{L^2}{}^2 + \frac{N}{\bar\psi(N)}\bar A$. We now use the bound on $\|p-m\|_{L^2}$, it leads to

$$\|p\|_{L^2} \le \|p-m\|_{L^2} + m\lambda_d(\Omega)^{1/2}$$
$$\le a\|p\|_{L^2} + b + \frac{1}{N\lambda_d(\Omega)^{1/2}}\|p\|_{L^2}{}^2 + \frac{N}{\bar\psi(N)\lambda_d(\Omega)^{1/2}}\bar A.$$

If $\|p\|_{L^2} \ne 0$, we now choose N such that $\frac{1}{N\lambda_d(\Omega)^{1/2}} = \frac{1-a}{2\|p\|_{L^2}}$, that is $N = \frac{2\|p\|_{L^2}}{(1-a)\lambda_d(\Omega)^{1/2}}$, we obtain

$$\frac{1-a}{2}\|p\|_{L^2} \le b + \frac{2\bar A}{\bar\psi(N)(1-a)\lambda_d(\Omega)}\|p\|_{L^2}.$$

Since $\lim_{s\to\infty} \bar\psi(s) = +\infty$, there exists C_1 such that

$$N \ge C_1 \Rightarrow \frac{2\bar A}{\bar\psi(N)(1-a)\lambda_d(\Omega)} \le \frac{1-a}{4}.$$

Then, with C_2 such that $\frac{2C_2}{(1-a)\lambda_d(\Omega)^{1/2}} = C_1$, ona has

$$\|p\|_{L^2} \ge C_2 \Rightarrow \frac{2\bar A}{\bar\psi(N)(1-a)\lambda_d(\Omega)} \le \frac{1-a}{4}.$$

Therefore

$$\|p\|_{L^2} \ge C_2 \Rightarrow \|p\|_{L^2} \le \frac{4b}{1-a}.$$

Then, we conclude that $\|p\|_{L^2} \le C = \max\{C_2, \frac{4b}{1-a}\}$.

Appendix B. Passing to the limit in the EOS

Lemma B.1. *Let Ω be a bounded open set of $\mathbb{R}^d$. Let $\rho \in L^2(\Omega)$, $\rho \geq 0$ a.e. in Ω and $u \in H_0^1(\Omega)^d$. Assume that (ρ, u) satisfies:*

$$\int_\Omega \rho\, \boldsymbol{u} \cdot \boldsymbol{\nabla}\varphi dx = 0 \text{ for all } \varphi \in W^{1,\infty}(\Omega). \tag{B.1}$$

Then,

$$\int_\Omega \rho \operatorname{div}(\boldsymbol{u})\, \mathrm{d}\boldsymbol{x} = 0. \tag{B.2}$$

Remark B.2. Before giving the proof of Lemma B.1, we want to point out the following remark. In the case of a regular function ρ, say $\rho \in C^1(\bar{\Omega})$, and assuming that $\rho > 0$ in Ω, the proof is very easy. We take $\varphi = \ln(\rho)$ in (B.1) which yields, since $\boldsymbol{\nabla}\varphi = \frac{1}{\rho}\boldsymbol{\nabla}\rho$,

$$\int_\Omega u \cdot \boldsymbol{\nabla}\rho\, \mathrm{d}\boldsymbol{x} = 0.$$

But, for any $v \in C_c^\infty(\Omega)^d$ one has $\int_\Omega v \cdot \boldsymbol{\nabla}\rho dx = -\int_\Omega \rho \operatorname{div}(v)\, \mathrm{d}\boldsymbol{x}$. Then, the density of $C_c^\infty(\Omega)^d$ in $H_0^1(\Omega)^d$ yields $\int_\Omega v \cdot \boldsymbol{\nabla}\rho dx = -\int_\Omega \rho \operatorname{div}(v)\, \mathrm{d}\boldsymbol{x}$ for $v \in H_0^1(\Omega)^d$. This gives (B.2).

This proof is interesting because it suggests the proof of an equivalent result in the case of a discrete version (using a convenient numerical scheme) of $\operatorname{div}(\rho u) = 0$ (see Lemma 3.1). In other words, working on a numerical scheme is quite similar of working on the continuous equation with a regular solution.

Proof. We now prove Lemma B.1. (without assuming $\rho \in C^1(\bar{\Omega})$ and $\rho > 0$).

We set $u = 0$ in $\mathbb{R}^d \setminus \Omega$ and $\rho = 0$ in $\mathbb{R}^d \setminus \Omega$, we have $\rho \in L^2(\mathbb{R}^d)$ and $u \in H^1(\mathbb{R}^d)^d$. We also deduce from (B.1):

$$\int_{\mathbb{R}^d} \rho u \cdot \boldsymbol{\nabla}\varphi dx = 0 \text{ for all } \varphi \in C^1(\mathbb{R}^d). \tag{B.3}$$

Let $(r_n)_{n\in\mathbb{N}^\star}$ be a sequence of mollifiers, that is:

$$r \in C_c^\infty(\mathbb{R}^d, \mathbb{R}), \int_{\mathbb{R}^d} r dx = 1,\ r \geq 0 \text{ in } \mathbb{R}^d \\ \text{and, for } n \in \mathbb{N}^\star,\ x \in \mathbb{R}^d,\ r_n(x) = n^d r(nx). \tag{B.4}$$

For $n \in \mathbb{N}^\star$, we set $\rho_n = \rho \star r_n$ and $(\rho u)_n = (\rho u) \star r_n$. Thanks to (B.3), we have $\operatorname{div}((\rho u)_n) = 0$ in $\mathbb{R}^d$. Since $u \in H^1(\mathbb{R}^d)^d$ and $\rho \in L^2(\mathbb{R}^d)$, we will prove in Lemma B.4 that $\operatorname{div}((\rho u)_n - \rho_n u) \to 0$ in $L^1(\mathbb{R}^d)$ as $n \to \infty$. Then, if $(q_n)_{n\in\mathbb{N}^\star}$ is a bounded sequence in $L^\infty(\mathbb{R}^d)$ which converges a.e. to q, we have:

$$-\int_{\mathbb{R}^d} \operatorname{div}(\rho_n u)\, q_n dx = \int_{\mathbb{R}^d} \operatorname{div}((\rho u)_n - \rho_n u)\, q_n dx \to 0 \text{ as } n \to \infty. \tag{B.5}$$

Let ψ be a bounded and C^1 function from $\mathbb{R}$ to $\mathbb{R}$, taking $q_n = \psi(\rho_n)$ in (B.5) (which converges a.e. to $\psi(\rho)$, at least up to a subsequence) we obtain

$$-\int_{\mathbb{R}^d} \operatorname{div}(\rho_n u)\, \psi(\rho_n) dx \to 0 \text{ as } n \to \infty.$$

We now define θ by $\theta(s) = \int_0^s t\psi'(t)dt$ for $s \in \mathbb{R}$ and we obtain

$$\int_{\mathbb{R}^d} \theta(\rho_n)\mathrm{div}(u)dx = \int_{\mathbb{R}^d} \rho_n\psi'(\rho_n)u \cdot \boldsymbol{\nabla}\rho_n dx = \int_{\mathbb{R}^d} \rho_n u \cdot \boldsymbol{\nabla}\psi(\rho_n)dx$$
$$= -\int_{\mathbb{R}^d} \mathrm{div}(\rho_n u)\, \psi(\rho_n)\, \mathrm{d}\boldsymbol{x},$$

and then $\displaystyle\int_{\mathbb{R}^d} \theta(\rho)\mathrm{div}(u)dx = \lim_{n\to\infty} \int_{\mathbb{R}^d} \theta(\rho_n)\mathrm{div}(u)\, \mathrm{d}\boldsymbol{x} = 0.$

It is now quite easy to construct a sequence of functions $(\psi_n)_{n\in\mathbb{N}}$ such that $0 \le \theta_n(s) \le s$ for all $s \in \mathbb{R}_+$ and $\lim_{n\to\infty} \theta_n(s) = s$ for all $s \in \mathbb{R}_+$. With the Dominated Convergence Theorem we then conclude that $\displaystyle\int_{\mathbb{R}^d} \rho\mathrm{div}(u)\, \mathrm{d}\boldsymbol{x} = 0.$ □

Remark B.3. Under the hypothesis of Lemma B.1, a quick look on the proof of this lemma shows that it is also possible to prove

$$\int_\Omega \psi(\rho)\mathrm{div}(\boldsymbol{u})\, \mathrm{d}\boldsymbol{x} = 0,$$

for any continuous function ψ (from $\mathbb{R}$ to $\mathbb{R}$) “at most linear”, that is such that

$$\limsup_{s\to+\infty} \frac{|\psi(s)|}{s} < +\infty.$$

It is also possible (as it was said in Remark 1.3) to prove that (ρ, u) is a renormalized solution to $\mathrm{div}(\rho u) = 0$ in $\mathbb{R}^d$.
Indeed, let ψ be a bounded and C^1 function from $\mathbb{R}$ to $\mathbb{R}$ and $\varphi \in C_c^\infty(\Omega)$. Taking $q_n = \psi(\rho_n)\varphi$ in (B.5) (which converges a.e. to $\psi(\rho)\varphi$, at least up to a subsequence) we obtain

$$-\int_{\mathbb{R}^d} \mathrm{div}(\rho_n u)\, \psi(\rho_n)\varphi\, \mathrm{d}\boldsymbol{x} \to 0 \text{ as } n \to \infty.$$

Taking, for $s \in \mathbb{R}$, $\bar{\psi}(s) = \int_0^s \psi(t)dt$ and $\theta(s) = \int_0^s t\psi'(t)dt = s\psi(s) - \bar{\psi}(s)$, we obtain, after some integrations by parts and passing to the limit as $n \to \infty$,

$$\int_{\mathbb{R}^d} (\rho\bar{\psi}'(\rho) - \bar{\psi}(\rho))(\mathrm{div}u)\varphi\, \mathrm{d}\boldsymbol{x} - \int_{\mathbb{R}^d} \bar{\psi}(\rho)u \cdot \boldsymbol{\nabla}\varphi\, \mathrm{d}\boldsymbol{x} = 0.$$

Then, it is easy to see that this equality also holds if $\bar{\psi}$ is a C^1 fonction form $\mathbb{R}$ to $\mathbb{R}$ with a bounded derivative. This proves that (ρ, u) is a renormalized solution to $\mathrm{div}(\rho u) = 0$ in $\mathbb{R}^d$.

Lemma B.4. *Let $\rho \in L^2(\mathbb{R}^d)$ and $u \in H^1(\mathbb{R}^d)^d$. Let $(r_n)_{n\in\mathbb{N}^\star}$ be a sequence of mollifiers as given by* (B.4) *and, for $n \in \mathbb{N}^\star$, $\rho_n = \rho \star r_n$ and $(\rho u)_n = (\rho u) \star r_n$. Then,*

$$\boldsymbol{\nabla}((\rho u)_n - \rho_n u) \to 0 \textit{ in } L^1(\mathbb{R}^d)^{d\times d},$$

and then,

$$\mathrm{div}((\rho u)_n - \rho_n u) \to 0 \textit{ in } L^1(\mathbb{R}^d).$$

Proof. Let $i, j \in \{1, \dots, d\}$. Denoting by $u_1, \dots, u_d$ the components of u and by ∂_i the derivative with respect to x_i, we have to prove that the sequence $(\partial_i[(\rho u_j)_n - \rho_n u_j])_{n\in\mathbb{N}^\star}$ converges to 0 in $L^1(\mathbb{R}^d)$. (As a consequence, taking $i = j$ and summing on i, we obtain that $\mathrm{div}((\rho u)_n - \rho_n u) \to 0$ in $L^1(\mathbb{R}^d)$.)
We have

$$\partial_i[(\rho u_j)_n - \rho_n u_j] = (\rho u_j) \star \partial_i r_n - (\rho \star \partial_i r_n)\ u_j - \rho_n\partial_i u_j = F_n - G_n,$$

with

$$F_n = (\rho u_j) \star \partial_i r_n - (\rho \star \partial_i r_n)\ u_j - \rho(\partial_i u_j \star r_n)$$

and

$$G_n = \rho_n \partial_i u_j - \rho(\partial_i u_j \star r_n).$$

Since $\rho_n \to \rho$ in $L^2(\mathbb{R}^d)$ and $\partial_i u_j \star r_n \to \partial_i u_j$ in $L^2(\mathbb{R}^d)$ (as $n \to \infty$), the two parts of G_n converges in $L^1(\mathbb{R}^d)$ (as $n \to \infty$) to $\rho \partial_i u_j$. Then, the sequence $(G_n)_n$ converges in $L^1(\mathbb{R}^d)$ (as $n \to \infty$) to 0. It remains to show that $F_n \to 0$ in $L^1(\mathbb{R}^d)$.

Using the fact that $\partial_i u_j \star r_n = u_j \star \partial_i r_n$ and the fact that r_n has a compact support, we have, for a.e. $x \in \mathbb{R}^d$,

$$\begin{aligned} F_n(x) &= \int_{\mathbb{R}^d} (\rho(x-y) - \rho(x))\ \big(u_j(x-y) - u_j(x)\big)\ \partial_i r_n(y) dy \\ &= \int_B (\rho(x - \frac{z}{n}) - \rho(x))\ \big(u_j(x - \frac{z}{n}) - u_j(x)\big)\ n\ \partial_i r(z) dz, \end{aligned}$$

where B is a ball of center 0 and radius R containing the support of r. Then, we get:

$$|F_n(x)| \le n \int_B |(\rho(x - \frac{z}{n}) - \rho(x))\ \big(u_j(x - \frac{z}{n}) - u_j(x)\big)|\ |\partial_i r(z)| dz.$$

We integrate over $\mathbb{R}$ the preceding inequality and we use the Fubini-Tonelli Theorem,

$$\begin{aligned} &\int_{\mathbb{R}^d} |F_n(x)| dx \le \\ &n \int_B \left[\int_{\mathbb{R}^d} |(\rho(x - \frac{z}{n}) - \rho(x))(u_j(x - \frac{z}{n}) - u_j(x))| dx \right] |\partial_i r(z)| dz. \end{aligned} \tag{B.6}$$

Using the Cauchy-Schwarz Inequality, we have for $z \in B$,

$$\begin{aligned} &\int_{\mathbb{R}^d} |(\rho(x - \frac{z}{n}) - \rho(x))(u_j(x - \frac{z}{n}) - u_j(x))| dx \\ &\le \left[\int_{\mathbb{R}^d} |\rho(x - \frac{z}{n}) - \rho(x)|^2 dx \right]^{1/2} \left[\int_{\mathbb{R}^d} |u_j(x - \frac{z}{n}) - u_j(x)|^2 dx \right]^{1/2}. \end{aligned}$$

For all $z \in B$ (see Lemma B.5) we have

$$\int_{\mathbb{R}^d} |u_j(x - \frac{z}{n}) - u_j(x)|^2 dx \le (\frac{R}{n})^2 \|u\|^2_{H^1(\mathbb{R}^d)^d}.$$

Then,

$$\begin{aligned} &\int_{\mathbb{R}^d} |(\rho(x - \frac{z}{n}) - \rho(x))(u_j(x - \frac{z}{n}) - u_j(x))| dx \\ &\le \frac{R}{n} \|u\|_{H^1(\mathbb{R}^d)^d} \left[\int_{\mathbb{R}^d} |\rho(x - \frac{z}{n}) - \rho(x)|^2 dx \right]^{1/2}. \end{aligned} \tag{B.7}$$

Let $\varepsilon > 0$. Since $\rho \in L^2(\mathbb{R}^d)$, there exists $\delta > 0$ such that

$$h \in \mathbb{R}^d,\ |h| \le \delta \Rightarrow \|\rho(\cdot + h) - \rho\|_{L^2(\mathbb{R}^d)} \le \varepsilon.$$

With (B.7), this gives if $n \ge R/\delta$ and $z \in B$,

$$\int_{\mathbb{R}^d} |(\rho(x - \frac{z}{n}) - \rho(x))(u_j(x - \frac{z}{n}) - u_j(x))| dx \le \varepsilon \frac{R}{n} \|u\|_{H^1(\mathbb{R}^d)^d}.$$

Coming back to (B.6), we obtain, if $n \geq R/\delta$,

$$\int_{\mathbb{R}^d} |F_n(x)|dx \leq n\frac{R}{n}\varepsilon \left\| u \right\|_{H^1(\mathbb{R}^d)^d} \int_B |\partial_i r(z)|dz = \varepsilon\, R \left\| u \right\|_{H^1(\mathbb{R}^d)^d} \int_B |\partial_i r(z)|dz.$$

This proves that $F_n \to 0$ in $L^1(\mathbb{R}^d)$ as $n \to \infty$ and concludes the proof of Lemma B.4. □

Lemma B.5. *Let $w \in H^1(\mathbb{R}^d)$. Then, for $h \in \mathbb{R}^d$,*

$$\left\| w(\cdot + h) - w \right\|_{L^2(\mathbb{R}^d)} \leq |h| \left\| w \right\|_{H^1(\mathbb{R}^d)}, \tag{B.8}$$

where $|h|$ is the Euclidean norm of h.

Lemma B.5 is well-known. A proof is given, for instance, in [6].
The following lemma (Lemma B.6) proves that (for regular enough set Ω) in Lemma B.1, $W^{1,\infty}(\Omega)$ can be replaced by $C_c^\infty(\Omega)$. That is to say that B.1 is true with $\varphi \in W^{1,\infty}(\Omega)$ if (and only if) it is true with the weaker assumption $\varphi \in C_c^\infty(\Omega)$. Lemma B.6 is given with $\rho \in L^2(\Omega)$ and $u \in (H_0^1(\Omega))^d$, which is the case needed for the present paper (and allows a nice proof using the Hardy inequality). Similar results are possible with different assumptions on u and ρ (for instance, $\rho \in L^\infty(\Omega)$ and $u \in W_0^{1,1}(\Omega)$). However, the fact that $\rho u \in L^1(\Omega)$ is obviously not sufficient to ensure that (B.1) is true with $\varphi \in W^{1,\infty}(\Omega)$ as long as it is true for $\varphi \in C_c^\infty(\Omega)$. In a following paper, dealing with the Navier-Stokes equations, we will give the same lemma with a weaker assumption on ρ (since $\rho \notin L^2(\Omega)$ in the case of the Navier-Stokes equations, when $d = 3$ and $\gamma < \frac{5}{3}$). In this case, the proof will use a different argument, slightly more complicated.

Lemma B.6. *Let Ω be a bounded open set of $\mathbb{R}^d$, with a Lipschitz continuous boundary. Let $u \in (H_0^1(\Omega))^d$ and $\rho \in L^2(\Omega)$ such that, for all $\varphi \in C_c^\infty(\Omega)$,*

$$\int_\Omega \rho u \cdot \nabla \varphi dx = 0. \tag{B.9}$$

Then (B.9) *holds for all $\varphi \in W^{1,\infty}(\Omega)$.*

The proof of this lemma is given in [7] (Lemma A.1).

Remark B.7. The hypothesis $\rho \in L^2(\Omega)$ is sharp in Lemma B.6, as we will see now. Let $d > 1$and $2d/(d+2) < q < 2$. We give here an example for which (B.9) holds for all $\varphi \in C_c^\infty(\Omega)$ but does not hold for some $\varphi \in W^{1,\infty}(\Omega)$. In this example, one has $\rho \in L^q(\Omega)$ and $u \in (H_0^1(\Omega))^d$ (so that $\rho u \in L^1(\mathbb{R})^d)$. Let us assume that $\Omega =]0, 2[\times] - 1, 1[^{d-1}$. Let $\alpha \in]\frac{1}{2}, \frac{1}{q}[$. We define ρ and $u = (u_1, \ldots, u_d)^t$ as follows:

$$u_1(x) = x_1^\alpha \prod_{i=2}^d (1 - |x_i|) \text{ if } x \in \Omega,\ x_1 \leq 1,$$

$$u_1(x) = (2 - x_1)^\alpha \prod_{i=2}^d (1 - |x_i|) \text{ if } x \in \Omega,\ x_1 > 1,$$

$$u_2 = \ldots = u_d = 0,$$

$$\rho(x) = \frac{1}{x_1^\alpha} \text{ if } x \in \Omega,\ x_1 \leq 1,$$

$$\rho(x) = \frac{1}{(2 - x_1)^\alpha} \text{ if } x \in \Omega,\ x_1 > 1.$$

We have $\rho \in L^q(\Omega)$ (thanks to $\alpha q < 1$) and $u \in (H_0^1(\Omega))^d$ (thanks to $2\alpha > 1$). Since ρu_1 does not depend on x_1, if is easy to see (integrating by parts) that (B.9) holds for all $\varphi \in C_c^\infty(\Omega)$. Taking now $\varphi \in C_c^\infty(\mathbb{R}^d)$ with, for instance $\varphi = 0$ outside $]-1,1[\times]-\frac{1}{2},\frac{1}{2}[^{d-1}$, one has

$$\int_\Omega \rho u \cdot \nabla \varphi dx = -\int_{]-\frac{1}{2},\frac{1}{2}[^{d-1}} \prod_{i=2}^{d} (1-|x_i|)\varphi(0,y)dy,$$

where $y = (x_2, \ldots, x_n)$. It is possible to choose φ such that $\varphi(0,y) > 0$ for all $y \in]-\frac{1}{2},\frac{1}{2}[^{d-1}$. This gives $\int_0 \rho u \cdot \nabla \varphi dx < 0$ and proves that (B.1) does not hold for this choice of φ (which belongs to $W^{1,\infty}(\Omega)$).

Lemma B.8. *Let φ be a convex function from $\mathbb{R}_+$ to $\mathbb{R}_+$ and $(\rho_n)_{n\in\mathbb{N}}$ be a sequence of nonnegative functions of $L^2(\Omega)$ weakly converging in $L^2(\Omega)$ to ρ. We assume that the sequence $(\varphi(\rho_n))_{n\in\mathbb{N}}$ is bounded in $L^2(\Omega)$. Then, $\varphi(\rho) \in L^2(\Omega)$.*

Proof. Since $\rho_n \geq 0$ a.e. (for all $n \in \mathbb{N}$), one has also $\rho \geq 0$ a.e..

Since the sequence $(\rho_n)_{n\in\mathbb{N}}$ weakly converge in $L^2(\Omega)$ to ρ, there exists a sequence $(\tilde\rho_n)_{n\in\mathbb{N}}$ converging (strongly) in $L^2(\Omega)$ to ρ and such that $\tilde\rho_n$ is (for all $n \in \mathbb{N}$) a convex combination of $\{\rho_k,\ k \geq n\}$ (this result is known as the Mazur lemma). Then, for all $n \in \mathbb{N}$, there exists $q_n \in \mathbb{N}$ and $\alpha_{n,0}, \ldots, \alpha_{n,q_n}$ such that

$$\tilde\rho_n = \sum_{i=0}^{q_n} \alpha_{n,i}\rho_{n+i},\ \sum_{i=0}^{q_n} \alpha_{n,i} = 1 \text{ and } \alpha_{n,i} \geq 0 \text{ for } i = 0, \ldots, q_n.$$

Let $M = \sup\{\ \|\varphi(\rho_n)\|_{L^2(\Omega)}\ \}$. Using the convexity of φ (and the fact that φ take its values in $\mathbb{R}_+$) we have, for all $n \in \mathbb{N}$,

$$0 \leq \varphi(\tilde\rho_n) \leq \sum_{i=0}^{q_n} \alpha_{n,i}\varphi(\rho_{n+i}) \text{ a.e.},$$

and then

$$\|\varphi(\tilde\rho_n)\|_{L^2(\Omega)}\ \leq \sum_{i=0}^{q_n} \alpha_{n,i}\,\|\varphi(\rho_{n+i})\|_{L^2(\Omega)}\ \leq M.$$

Up to a subsequence, one has $\tilde\rho_n \to \tilde\rho$ a.e. and then, using the continuity of the function φ, $\varphi^2(\tilde\rho_n) \to \varphi^2(\rho)$ a.e on Ω. Then, using Fatou Lemma, we thus get $\varphi(\rho) \in L^2(\Omega)$ (and $\|\varphi(\rho)\|_{L^2(\Omega)}\ \leq M$). □

Appendix C. general lemmas

Lemma C.1. *Let $(F_n)_{n\in\mathbb{N}} \subset \mathrm{L}^1(\Omega)$ be an equi-integrable sequence, and F be a function of $\mathrm{L}^1(\Omega)$. We assume that:*

$$\lim_{n\to\infty} \int_\Omega F_n\varphi \,\mathrm{d}\boldsymbol{x} = \int_\Omega F\varphi \,\mathrm{d}\boldsymbol{x} \textit{ for all } \varphi \in \mathrm{C}_c^\infty(\Omega). \tag{C.1}$$

Then:

$$\lim_{n\to\infty} \int_\Omega F_n \,\mathrm{d}\boldsymbol{x} = \int_\Omega F \,\mathrm{d}\boldsymbol{x}.$$

Lemma C.1 is well-known. A proof is given, for instance, in [6]. The following lemma is also well-known. A simple proof of this result is given in [1].

Lemma C.2. *Let $q \in \mathrm{L}^2(\Omega)$ such that $\int_\Omega q \,\mathrm{d}\boldsymbol{x} = 0$. Then, there exists $\boldsymbol{w} \in (\mathrm{H}^1_0(\Omega))^d$ such that $\mathrm{div}(\boldsymbol{w}) = q$ a.e. in Ω and $\|\boldsymbol{w}\|_{\mathrm{H}^1(\Omega)^d} \le c_2 \|q\|_{\mathrm{L}^2(\Omega)}$ where c_2 only depends on Ω.*

We now give two simple lemmas related to the so-called "M-matrices". We recall that for a vector x of $\mathbb{R}^n$, the fact that all the components of x are nonnegative is denoted by $x \ge 0$. Similarly the fact that all the components of x are positive is denoted by $x > 0$.

Lemma C.3. *Let $n \in \mathbb{N}^\star$ and A be a $n\times n$ matrix with real entries (these entries are denoted by $a_{i,j}$, i, $j = 1,\dots,n$). We asume that A satisfies the following properties:*

$$\begin{cases} a_{i,j} \le 0 \text{ for all } i,j \in \{1,\dots,n\},\ i \ne j, \\ a_{i,i} + \sum_{j\ne i} a_{i,j} > 0 \text{ for all } i \in \{1,\dots,n\}. \end{cases}$$

then,

$$x \in \mathbb{R}^n,\ A\,x \ge 0 \Rightarrow x \ge 0, \tag{C.2}$$

which is equivalent to say that A is invertible and that all the entries of A^{-1} are nonnegatives. Futhermore, one also has

$$x \in \mathbb{R}^n,\ A\,x > 0 \Rightarrow x > 0, \tag{C.3}$$

Proof. The proof of (C.2) is very classical. We can do it, for instance, by contradiction. Let $x \in \mathbb{R}^n$ such that $Ax \ge 0$. We assume that $\alpha = \min\{x_i,\ i = 1,\dots,n\} < 0$ (where the x_i are the components of x) and we choose $i_0 \in \{1,\dots,n\}$ such that $x_{i_0} = \alpha$.

Since the i_0-component of Ax is nonnegative and since $x_{i_0} \le x_i$ for all i, one has, thanks to the properties of A,

$$x_{i_0}(a_{i_0,i_0} + \sum_{j\ne i_0} a_{i_0,j}) \ge 0,$$

Which gives $x_{i_0} \ge 0$, in contradiction with $x_{i_0} = \alpha < 0$. This proves (C.2).

In order to prove (C.3). Let e be the vector of $\mathbb{R}^n$ whose all components are equal to 1. let $x \in \mathbb{R}^n$ such $Ax > 0$. Then, for $\varepsilon > 0$ small enough, one has $A(x-\varepsilon e) = Ax - \varepsilon Ae \ge 0$. Thanks to (C.2), one deduces $x - \varepsilon e \ge 0$ and this gives $x > 0$. □

The second lemma is a little bit less classical but is a very simple consequence of the first one.

Lemma C.4. *Let $n \in \mathbb{N}^\star$ and A be a $n\times n$ matrix with real entries (these entries are denoted by $a_{i,j}$, i, $j = 1,\dots,n$). We asume that A satisfies the following properties:*

$$\begin{cases} a_{i,j} \le 0 \text{ for all } i,j \in \{1,\dots,n\},\ i \ne j, \\ a_{i,i} + \sum_{j\ne i} a_{j,i} > 0 \text{ for all } i \in \{1,\dots,n\}. \end{cases}$$

then,

$$x \in \mathbb{R}^n,\ A\,x \ge 0 \Rightarrow x \ge 0 \tag{C.4}$$

and

$$x \in \mathbb{R}^n,\ A\,x > 0 \Rightarrow x > 0, \tag{C.5}$$

Proof. The matrix A^t satisfies the properties of lemma C.3. Then A^t is invertible and $(A^t)^{-1}$ has all its entries nonnegative. This gives that A is also invertible and has all its entries nonnegative since $(A^t)^{-1} = (A^{-1})^t$. This gives that A satisfies (C.4)

The proof of (C.5) is the same as the proof of (C.3) in lemma C.3. □

Lemma C.5. *Let φ be a function of class C^1 from $\mathbb{R}_+^\star$ to $\mathbb{R}$. Let ψ from $\mathbb{R}_+^\star$ to $\mathbb{R}$ such that $s\psi'(s) = \varphi'(s)$ for all $s \in \mathbb{R}_+^\star$. Let $a, b \in \mathbb{R}_+^\star$, $a \neq b$. Then, there exists c between a et b such that*

$$(\psi(b) - \psi(a))b - (\varphi(b) - \varphi(a)) = \frac{1}{2}(b-a)^2\psi'(c).$$

Proof. One has

$$(\psi(b) - \psi(a))b - (\varphi(b) - \varphi(a)) = b\int_a^b \psi'(s)ds - \int_a^b \varphi'(s)ds = \int_a^b (b-s)\psi'(s)ds.$$

But,

$$\min_{t\in[a,b]} \psi'(t)\int_a^b (b-s)ds \leq \int_a^b (b-s)\psi'(s)ds \leq \max_{t\in[a,b]} \psi'(t)\int_a^b (b-s)ds.$$

Then, since ψ' is continuous on $[a,b]$, there exists $c \in [a,b]$ such that

$$\int_a^b (b-s)\psi'(s)ds = \psi'(c)\int_a^b (b-s)ds.$$

Noticing that $\int_a^b (b-s)ds = \frac{1}{2}(b-a)^2$, we obtain the desired equality. □

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