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ERROR ESTIMATE FOR APPROXIMATE SOLUTIONS OF A NONLINEAR CONVECTION-DIFFUSION PROBLEM

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Abstract. This paper proves the estimate $\|u_\varepsilon - u\|_{L^1(Q_T)} \le C\varepsilon^{1/5}$, where, for all $\varepsilon > 0$, u_ε is the weak solution of $(u_\varepsilon)_t + \operatorname{div}(\mathbf{q}\ f(u_\varepsilon)) - \Delta(\varphi(u_\varepsilon)+\varepsilon u_\varepsilon) = 0$ with initial and boundary conditions, u is the entropy weak solution of $u_t + \operatorname{div}(\mathbf{q}f(u)) - \Delta(\varphi(u)) = 0$ with the same initial and boundary conditions, and $C > 0$ does not depend on ε. The domain Ω is assumed to be regular and T is a given positive value.

1. Introduction

In the hydrogeological engineering setting, one has to model air-water flows in soils. In several cases, the Richards approximation (in which the air phase is assumed to be at the atmospheric pressure everywhere) cannot be used, and a full two-phase flow model is used, writing both conservation equations of air and water in the soils. Assuming that air and water phases are immiscible and incompressible (the pressure variations are small enough to justify this assumption), one considers for the sake of simplicity a problem in a horizontal domain and one also assumes that flows in soils can be modeled using Darcy's law:

$$u_t(x,t) - \operatorname{div}\left(k_w(u(x,t))\nabla P_w(x,t)\right) = 0, \tag{1.1}$$

$$(1-u)_t(x,t) - \operatorname{div}\left(k_a(u(x,t))\nabla P_a(x,t)\right) = 0, \tag{1.2}$$

$$P_a(x,t) - P_w(x,t) = Pc(u(x,t)), \quad \text{for all } x \in \Omega \text{ and } t \ge 0, \tag{1.3}$$

where

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- $x \in \Omega$ denotes the space variable in the domain Ω and $t \geq 0$ denotes the time variable,
- the index w (resp. a) stands for the water phase (resp. the air phase),
- $P_p(x,t)$ is the pressure of phase p,
- $u(x,t) \in [0,1]$ denotes the saturation of the water phase (i.e., the percentage of porous space occupied by the water phase) and $1 - u(x,t)$ the saturation of the air phase,
- $Pc(u)$ is the capillary pressure ($Pc(u)$ is a regular function which is decreasing and verifies $Pc(1) = 0$),
- $k_p(u)$ is the mobility of phase p. The function $k_w(u)$ is regular, non decreasing and verifies $k_w(0) = 0$, and the function $k_a(u)$ is regular, non increasing and verifies $k_a(1) = 0$.

In several practical applications, one has $u(x,t) \geq U_I > 0$, and one can assume that the water phase mobility is positive everywhere in Ω. On the contrary, the air phase cannot be assumed to be present in the whole domain, and therefore in the general case, there exists $T > 0$ such that, for all $t \in (0,T)$, there exists a subdomain $E_1(t)$ of Ω such that $u(x,t) = 1$ for all $x \in E_1(t)$. In order to explicit the consequences of the existence of such a subdomain, one can exhibit a degenerate parabolic equation the solution of which is u. Indeed, if we introduce the vector field $\mathbf{q}$, defined for all $(x,t) \in \Omega \times (0,+\infty)$ by

$$\mathbf{q}(x,t) = -\big(k_w(u(x,t))\nabla P_w(x,t) + k_a(u(x,t))\nabla P_a(x,t)\big), \tag{1.4}$$

we get, by summing (1.1) and (1.2)

$$\operatorname{div} \mathbf{q}(x,t) = 0.$$

Extracting ∇P_w and ∇P_a from (1.4) and (1.3), we get

$$u_t(x,t) + \operatorname{div}\big(\mathbf{q}\, f(u)\big)(x,t) - \Delta\big(\varphi(u)\big)(x,t) = 0, \tag{1.5}$$

in which the functions φ and f are defined by

$$\varphi(u) = \int_u^1 \frac{k_w(s)k_a(s)}{k_w(s)+k_a(s)} Pc'(s)ds, \quad \text{and} \quad f(u) = \frac{k_w(u)}{k_w(u)+k_a(u)}.$$

The negative function φ is non decreasing, since $Pc' \leq 0$, and for practical data we can observe that φ is equivalent to $-(1-u)^\alpha$ for some $\alpha > 1$ as $u \longrightarrow 1$, which characterizes a degenerate parabolic problem similar to the "porous medium equation". Thus, for $t \in (0,T)$, we have $\varphi'(u) = 0$ in $E_1(t)$, domain whose the boundary is free. It has been shown in [13] for instance that the system of equations (1.1), (1.2) and (1.3) can successfully be approximated using a finite volume scheme, the location of the free boundary simply resulting from the local conservation of the fluid components. However, numerous engineers implied in soil mechanics prefer using some finite element methods (coupled with finite element methods for the mechanical behaviour of the porous skeletton) the convergence of which is only obtained for $\varphi'(u) \geq \varepsilon > 0$ (see [17] for example). Therefore these engineers introduce a function

$$\varphi_\varepsilon(u) = \int_u^1 \frac{k_w(s)k_a(s)}{k_w(s) + k_a(s)} Pc'(s)ds + \varepsilon u,$$

and then they use a finite element method to solve (1.5) with φ_ε instead of φ. It has been shown on a physical example (see [8] or [10]) that the error committed by such a substitution is far from being negligible. It was thus of a large interest to evaluate this error and its order as a function of ε. This is one of the objectives of the present paper.

Another motivation to study such a perturbation of a nonlinear degenerate parabolic equation is the study of the convergence of numerical schemes. Indeed, it is well known that a discretization of a conservation law (hyperbolic or convection dominated parabolic equations) yields a numerical diffusion term which is a discrete analog of a continuous term of the form $-\varepsilon\Delta u$. We were recently able to prove the convergence of finite volume approximations to (1.5) towards an entropy weak solution [13]. However, the rate of convergence is not yet known, and the obtention of such an error estimate is under study. The error estimate in the case of a continuous diffusion perturbation is hoped to shed some light on the means to obtain the discrete error estimate.

2. Mathematical formulation and results

We now complete the mathematical formulation of the problem presented in the previous section. Let Ω be a bounded open subset of $\mathbb{R}^d$, ($d = 1, 2$ or 3) with a regular (C^2) boundary denoted by $\partial\Omega$. Let $T \in \mathbb{R}_+^*$, and $Q_T = \Omega \times (0, T)$. Let u be the entropy weak solution of the following problem :

$$u_t(x,t) + \operatorname{div}\big(\mathbf{q}\ f(u)\big)(x,t) - \Delta\big(\varphi(u)\big)(x,t) = 0, \text{ for a.e. } (x,t) \in Q_T, \quad (2.1)$$

with initial condition:

$$u(x, 0) = u_0(x) \quad \text{for a.e. } x \in \Omega. \quad (2.2)$$

and boundary condition:

$$u(x,t) = \bar{u}(x,t), \quad \text{for a.e. } (x,t) \in \partial\Omega \times (0,T). \quad (2.3)$$

Note that for this mathematical study, we could replace the convective term $\mathbf{q}(x,t)f(u)$ by the more general term $F(u,x,t)$. An advantage of this particular case is that it leads to easier notations though it involves the same tools as the general framework.

One supposes that the following hypotheses, globally referred in the following as hypotheses H, are fulfilled (hypotheses H are satisfied for a large number of problems including the one which is presented in the introduction of this paper, and in particular can apply to purely hyperbolic problems, i.e. $\varphi = 0$, or Stefan-like problems, i.e., $\varphi' = 0$ on some intervals).

Hypotheses H

(H1) The boundary $\partial\Omega$ of Ω is of class C^2,

(H2) The initial condition u_0 belongs to $L^\infty(\Omega)\cap BV(\Omega)$ and the boundary condition $\bar{u}$ belongs to $L^\infty(\partial\Omega\times(0,T))$, and is the trace of a function of $H^1(Q_T)$ (also denoted by $\bar{u}$); let $U\in\mathbb{R}$ be such that $-U\le u_0\le U$ a.e. in Ω and $-U\le\bar{u}\le U$a.e. in Ω ;

(H3) φ is a nondecreasing Lipschitz–continuous function,

(H4) f is a Lipschitz continuous function,

(H5) $\mathbf{q}\in C^1(\bar{\Omega}\times[0,T])$,

(H6) $\mathrm{div}(\mathbf{q}(x,t)) = 0$ for all $(x,t)\in\mathbb{R}^d\times(0,T)$, where

$$\mathrm{div}(\mathbf{q}(x,t)) = \sum_{i=1}^{d}\frac{\partial\mathbf{q}}{\partial x_i}(x,t), \qquad \text{and}$$

$$\mathbf{q}(x,t).\mathbf{n}(x) = 0, \quad \text{for a.e. } (x,t)\in\partial\Omega\times(0,T), \tag{2.4}$$

(where $\mathbf{n}(x)$ denotes, for a.e. $x\in\partial\Omega$, the normal to $\partial\Omega$ at point x, outward to Ω).

Because of the presence of a nonlinear convection term, the expected solution of Problem (2.1)-(2.3) is an entropy weak solution in the following sense which was introduced by several authors [5], [16].

Definition 2.1 (Entropy weak solution)**.** Under hypotheses H, a function u is said to be an entropy weak solution to Problem (2.1)-(2.3) if it verifies, for all $T>0$:

$$u\in L^\infty(\Omega\times(0,T)), \tag{2.5}$$

$$\varphi(u)-\varphi(\bar{u})\in L^2(0,T;H_0^1(\Omega)), \tag{2.6}$$

and

$$\int_{Q_T}\Big[\eta(u(x,t))\psi_t(x,t) + \Phi(u(x,t))\mathbf{q}(x,t)\cdot\nabla\psi(x,t) \tag{2.7}$$

$$- \nabla\theta(u)(x,t)\cdot\nabla\psi(x,t)\Big]dxdt + \int_\Omega \eta(u_0(x))\psi(x,0)dx \geq 0,$$
$$\forall\ \psi \in \mathcal{C},\ \forall \eta \in C^2(\mathbb{R},\mathbb{R}), \eta'' \geq 0,\ \Phi' = \eta'(\cdot)f'(\cdot),\ \theta' = \eta'(\cdot)\varphi'(\cdot),$$

where $\mathcal{C} = \{\psi \in C_c^\infty(\mathbb{R}^d \times \mathbb{R}),\ \psi \geq 0 \text{ and } \psi = 0 \text{ on } (\partial\Omega\times(0,T))\cup\Omega\times(\{T\})\}$.

Remark 2.1. Thanks to condition (2.4), there is no need to take the boundary conditions into account in the entropy inequality (2.7). A non homogeneous boundary condition stands without condition (2.4) in [16], using the trace of the weak solution u on the boundary, thus following the classical Bardos-Leroux-Nédélec formulation [1]. Carrillo gives a weak entropy formulation (see [5]) in the case of a homogeneous Dirichlet boundary condition on $\partial\Omega$ without condition (2.4).

In the present work, we prove an estimate of order $\varepsilon^{1/5}$ between the entropy weak solution u of (2.1) and the entropy weak solution u_ε of the following regularized problem:

$$\begin{aligned} &(u_\varepsilon)_t(x,t) + \operatorname{div}\big(\mathbf{q}.f(u)\big)(x,t) - \Delta\big(\varphi(u) + \varepsilon u\big)(x,t) = 0,\\ &\text{for a.e. } (x,t) \in \Omega\times(0,T). \end{aligned} \tag{2.8}$$

In the case $\Omega = \mathbb{R}^d$, the existence of the entropy weak solution is proven in [3] by using a regularization of the problem in the "general kinetic BGK" framework to yield estimates on translates of the approximate solutions. In [7], some explicit estimates for the continuous dependence with respect to the data of the solutions in the semi-group sense as introduced in [2] are given: these estimates yield an estimate between u and u_ε of order $\varepsilon^{1/2}$ in the case of the problem we consider here (our estimate is of lower order because of the boundary conditions).

In the case of bounded domains, the existence of the entropy weak solution is proved in [5] and [16]. In [16], the proof of existence uses strong BV estimates in order to derive estimates in time and space for the solution of the regularized problem (2.8). In [5], the existence of a weak solution is proved using semigroup theory (see [2]), and the uniqueness of the entropy weak solution is proved using techniques which have been introduced by Krushkov [15] and extended by Carrillo. It is now well known that Krushkov entropies are a good way to obtain an error estimate on nonlinear scalar hyperbolic problems, see e.g. [12, 6, 18]. It is however not so easy, to extend the use of the Krushkov entropies to the hyperbolic-parabolic case because of the diffusion term. This is a major breakthrough in Carillo's work [5]. Following this work, we shall prove here the following theorem :

Theorem 2.1 (Error estimate)**.** *Under hypotheses H, for all $\varepsilon > 0$, let $u_\varepsilon \in L^2(0,T;H^1(\Omega))$ be the unique weak solution of the problem* (2.8) *with initial condition* (2.2) *and boundary condition* (2.3)*. Let $u \in L^\infty(Q_T)$ be the unique entropy weak solution of Problem* (2.1)-(2.3) *in the sense of Definition* 2.1*. Then there exists $C > 0$, which depends only on Ω, T, u_0, $\bar{u}$, $\mathbf{q}$, f and φ such that, for all $\varepsilon > 0$,*

$$\|u_\varepsilon - u\|_{L^1(Q_T)} \leq C\varepsilon^{1/5}. \tag{2.9}$$

The proof of this estimate follows the same steps than those of the proof of the error estimate of [12] for the finite volume approximations of nonlinear hyperbolic conservation laws (in which case $\varphi = 0$).

The first step is to prove the following lemma :

Lemma 2.1 (Measure estimate)**.** *Under hypotheses H, let $\varepsilon > 0$ and let $u_\varepsilon \in L^2(0,T;H^1(\Omega))$ be the unique weak solution of the problem*

$$\begin{aligned} &(u_\varepsilon)_t(x,t) + div\big(\mathbf{q}.f(u_\varepsilon)\big)(x,t) - \Delta\big(\varphi(u_\varepsilon) + \varepsilon u_\varepsilon\big)(x,t) = 0,\\ &\textit{for a.e. } (x,t) \in \Omega \times (0,T), \end{aligned} \tag{2.10}$$

with initial condition (2.2) *and boundary condition* (2.3)*. Let m_ε be the measure of density $\varepsilon|\nabla u_\varepsilon|$. Let $u \in L^\infty(Q_T)$ be the unique entropy weak solution of Problem* (2.1)-(2.3) *in the sense of Definition* 2.1*. Then there exist $C_1 > 0$ and $C_2 > 0$ which only depend on Ω, T, u_0, $\bar{u}$, f, φ and $\mathbf{q}$, such that, for all $a > 0$,*

$$\begin{aligned} &\int_{Q_T} \Big[|u_\varepsilon(x,t) - u(x,t)|\psi_t(x,t) + (f(u_\varepsilon(x,t)\top u(x,t)) \\ &- f(u_\varepsilon(x,t)\bot u(x,t)))\mathbf{q}(x,t)\cdot\nabla\psi(x,t) \\ &- \nabla|\varphi(u_\varepsilon)(x,t) - \varphi(u)(x,t)|\cdot\nabla\psi(x,t)\Big]dxdt \\ &\geq -C_1 a(\|\nabla\psi\|_\infty + \|\psi_t\|_\infty + \|\psi\|_\infty + \|\psi(\cdot,0)\|_\infty + \|\Delta\psi\|_2) \\ &- C_2 m(Q_T)(\frac{\|\psi\|_\infty}{a} + \|\nabla\psi\|_\infty), \end{aligned} \tag{2.11}$$

for all functions $\psi \in C^1(\overline{Q_T})$ such that $\psi \geq 0$, $\Delta\psi \in L^2(Q_T)$, $\psi(\cdot,T) = 0$ and $\psi(x,t) = 0$ for all $(x,t) \in Q_T$ with $d(x,\partial\Omega) \leq a$.

Remark 2.2. The measure estimate (2.11) of Lemma 2.1 is important for its own sake. Indeed, when transposed to the discrete setting of the numerical scheme of [12], it may yield some error indicators which are useful for automatic refinement procedures, see [14].

The proof of Lemma 2.1 is given in Section 3. The second step in the proof of Theorem 2.1 consists in making an adequate choice for the function ψ in (2.11) of Lemma 2.1. This will be done in Section 4.

3. Proof of lemma 2.1

Throughout the paper, we shall denote by C_i various real positive values which only depend on Ω, T, u_0, $\bar{u}$, f, φ and $\mathbf{q}$.

Let us assume that hypotheses (H) hold. Let $\varepsilon > 0$ and let u (resp. u_ε) be the entropy weak solution to (2.1), (2.2), (2.3) (resp. (2.8), (2.2), (2.3)). Let us define the function ζ to be a primitive of $\sqrt{\varphi'}$. With some slight adaptations of the results of [5], the following estimates on u and u_ε hold :

$$-U \leq u(x,t) \leq U, \text{ for a.e. } (x,t) \in Q_T, \tag{3.1}$$

$$-U \leq u_\varepsilon(x,t) \leq U, \text{ for a.e. } (x,t) \in Q_T, \tag{3.2}$$

$$\int_{Q_T} (\nabla\zeta(u(x,t)))^2 dxdt \leq C_3, \tag{3.3}$$

$$\int_{Q_T} (\nabla\zeta(u_\varepsilon(x,t)))^2 dxdt \leq C_3, \tag{3.4}$$

$$\int_{\Omega\times(0,T-s)} (\zeta(u(x,t+s)) - \zeta(u(x,t)))^2 dxdt \leq \ s\ C_4, \quad \forall s \in (0,T), \tag{3.5}$$

$$\int_{\Omega\times(0,T-s)} (\zeta(u_\varepsilon(x,t+s)) - \zeta(u_\varepsilon(x,t)))^2 dxdt \leq \ s\ C_4, \quad \forall s \in (0,T), \tag{3.6}$$

$$\|u\|_{BV(Q_T)} \leq C_5. \tag{3.7}$$

$$\int_{Q_T} |\nabla u_\varepsilon(x,t)| dxdt \leq C_5, \tag{3.8}$$

$$\|u_\varepsilon\|_{BV(Q_T)} \leq C_5. \tag{3.9}$$

Let us then multiply (2.8) by $\psi(x,t)\eta'(u_\varepsilon(x,t))$ and integrate on Q_T; we obtain

$$\begin{aligned} &\int_{Q_T} \Big[\eta(u_\varepsilon(x,t))\psi_t(x,t) + \Phi(u_\varepsilon(x,t))\mathbf{q}(x,t)\cdot\nabla\psi(x,t) \\ &- \nabla\theta(u_\varepsilon)(x,t)\cdot\nabla\psi(x,t) - \eta''(u(x,t))(\nabla\zeta(u)(x,t))^2\psi(x,t) \\ &+ \varepsilon\eta(u_\varepsilon(x,t))\Delta\psi(x,t)\Big] dxdt + \int_\Omega \eta(u_0(x))\psi(x,0)dx \geq 0, \\ &\forall\ \psi \in \mathcal{C}, \forall \eta \in C^2(\mathbb{R},\mathbb{R}), \eta'' \geq 0,\ \Phi' = \eta'(\cdot)f'(\cdot),\ \theta' = \eta'(\cdot)\varphi'(\cdot). \end{aligned} \tag{3.10}$$

We shall use two inequalities which are both consequences of (3.10) in order to obtain the measure estimate of Lemma 1.

The first inequality is obtained by taking, in (3.10), entropies μ such that $\mu' = \eta' \circ \varphi$, where η is itself a C^2 convex function. This yields

$$\int_{Q_T} \Big[\mu(u_\varepsilon(x,t))\psi_t(x,t) + \nu(u_\varepsilon(x,t))\mathbf{q}(x,t)\cdot\nabla\psi(x,t) \tag{3.11}$$
$$- \nabla\eta(\varphi(u_\varepsilon)(x,t))\cdot\nabla\psi(x,t) - \eta''(\varphi(u_\varepsilon)(x,t))(\nabla\varphi(u_\varepsilon))^2(x,t)\psi(x,t)\Big]dxdt$$
$$+ \int_\Omega \mu(u_0(x))\psi(x,0)dx \geq - \sup_{s\in[\varphi(-U),\varphi(U)]} |\eta'(s)| \int_{Q_T} |\nabla\psi(x,t)|dm_\varepsilon(x,t),$$
$$\forall\ \psi\in\mathcal{C},\ \forall\eta\in C^2(\mathbb{R},\mathbb{R}), \eta''\geq 0,\ \mu' = \eta'(\varphi(\cdot)),\ \nu' = \eta'(\varphi(\cdot))f'(\cdot),$$

where m_ε is the measure of density $\varepsilon|\nabla u_\varepsilon|$. This inequality will be used when u_ε "acts parabolic", that is when $\varphi(u_\varepsilon)$ is not constant, so that the diffusive term does not vanish.

The second inequality is obtained by taking Kruskov entropies in (3.10) (in order to so, one should notice that the term $\eta''(u(x,t))\ (\nabla\zeta(u)(x,t))^2\psi(x,t)$ is non negative and can therefore be dropped out of the inequality). This yields the following inequality, which will be used when u_ε "acts hyperbolic", that is when $\varphi(u_\varepsilon)$ is constant so that the diffusive term vanishes

$$\int_{Q_T} \Big[|u_\varepsilon(x,t) - \kappa|\psi_t(x,t) + (f(u_\varepsilon(x,t)\top\kappa) \tag{3.12}$$
$$- f(u_\varepsilon(x,t)\bot\kappa))\mathbf{q}(x,t)\cdot\nabla\psi(x,t) - \nabla|\varphi(u_\varepsilon)(x,t) - \varphi(\kappa)|\cdot\nabla\psi(x,t)\Big]dxdt$$
$$+ \int_\Omega |u_0(x) - \kappa|\psi(x,0)dx \geq - \int_{Q_T} |\nabla\psi(x,t)|dm_\varepsilon(x,t), \quad \forall\ \psi\in\mathcal{C},\ \forall\kappa\in\mathbb{R}.$$

From the results of [5] (see also [13] in the discrete setting of numerical schemes), one gets the convergence in $L^1(Q_T)$ of u_ε to u as ε tends to 0. Let us now state the limit problems of (3.10), (3.11) and (3.12) as ε tends to 0. Thanks to the estimate (3.4), one may show that:

$$\liminf_{\varepsilon\longrightarrow 0} \int_{Q_T} \psi(x,t)\Big(\nabla(\zeta(u_\varepsilon)(x,t)\Big)^2 dxdt \tag{3.13}$$
$$\geq \int_{Q_T} \psi(x,t)\Big(\nabla(\zeta(u))(x,t)\Big)^2 dxdt, \quad \forall\psi\in L^\infty(Q_T),\ \psi\geq 0,$$

Hence, passing to the limit as $\varepsilon \longrightarrow 0$ in (3.10) yields:

$$\int_{Q_T} \Big[\eta(u(x,t))\psi_t(x,t) + \Phi(u(x,t))\ \mathbf{q}(x,t)\cdot\nabla\psi(x,t) \tag{3.14}$$
$$- \nabla\theta(u)(x,t)\cdot\nabla\psi(x,t) - \eta''(u(x,t))(\nabla\zeta(u)(x,t))^2\psi(x,t)\Big]dxdt$$
$$+ \int_\Omega \eta(u_0(x))\psi(x,0)dx \geq 0,$$
$$\forall\ \psi\in\mathcal{C},\ \forall\eta\in C^2(\mathbb{R},\mathbb{R}), \eta''\geq 0,\ \Phi' = \eta'(\cdot)f'(\cdot),\ \theta' = \eta'(\cdot)\varphi'(\cdot).$$

Similarly, passing to the limit in inequalities (3.11) and (3.12) yields:

$$\int_{Q_T} \Big[\mu(u(x,t))\psi_t(x,t) + \nu(u(x,t))\mathbf{q}(x,t)\cdot\nabla\psi(x,t) \tag{3.15}$$
$$- \nabla\eta(\varphi(u)(x,t))\cdot\nabla\psi(x,t) - \eta''(\varphi(u)(x,t))(\nabla\varphi(u))^2(x,t)\psi(x,t)\Big]dxdt$$
$$+ \int_\Omega \mu(u_0(x))\psi(x,0)dx \geq 0,$$
$$\forall\ \psi\in\mathcal{C},\ \forall\eta\in C^2(\mathbb{R},\mathbb{R}), \eta''\geq 0,\ \mu' = \eta'(\varphi(\cdot)),\ \nu' = \eta'(\varphi(\cdot))f'(\cdot),$$

and

$$\int_{Q_T} \Big[|u(x,t)-\kappa|\psi_t(x,t) + (f(u(x,t)\top\kappa) \tag{3.16}$$
$$- f(u(x,t)\bot\kappa))\mathbf{q}(x,t)\cdot\nabla\psi(x,t) - \nabla|\varphi(u)(x,t)-\varphi(\kappa)|\cdot\nabla\psi(x,t)\Big]dxdt$$
$$+ \int_\Omega |u_0(x)-\kappa|\psi(x,0)dx \geq 0, \forall\ \psi\in\mathcal{C},\ \forall\kappa\in\mathbb{R}.$$

Let us define, for all $\delta > 0$, a regularization $S_\delta \in C^1(\mathbb{R},\mathbb{R})$ of the sign function given by:

$$\begin{array}{ll} S_\delta(a) = -1, & \forall a\in(-\infty,-\delta], \\ S_\delta(a) = \frac{3\delta^2 a - a^3}{2\ \delta^3}, & \forall a\in[-\delta,\delta], \\ S_\delta(a) = 1, & \forall a\in[\delta,+\infty). \end{array} \tag{3.17}$$

On the set $\mathbb{R}_\varphi$ defined by

$$\mathbb{R}_\varphi = \{a\in\mathbb{R}, \forall b\in\mathbb{R}\setminus\{a\},\ \varphi(b)\neq\varphi(a)\},$$

the function φ is "genuinely non constant"; the set $\varphi(\mathbb{R}\setminus\mathbb{R}_\varphi)$ is countable, since for all $s\in\varphi(\mathbb{R}\setminus\mathbb{R}_\varphi)$, there exists $(a,b)\in\mathbb{R}^2$ with $a<b$ and $\varphi((a,b)) = \{s\}$, and therefore there exists at least one $r\in\mathbb{Q}$ with $r\in(a,b)$ verifying $\varphi(r) = s$.

Let $\kappa \in \mathbb{R}_\varphi$ and let $\delta > 0$. Let us take in (3.11) the entropy function defined by

$$\mu_{\delta,\kappa}(a) = \int_\kappa^a \eta'_{\delta,\kappa}(\varphi(s))ds$$

for $a \in \mathbb{R}$, where η is a regularization of a Kruskov entropy:

$$\eta_{\delta,\kappa}(a) = \int_{\varphi(\kappa)}^a S_\delta(s - \varphi(\kappa))ds \quad \text{for } a \in \mathbb{R}.$$

Let ν be the flux function associated to $\mu_{\delta,\kappa}$: for $a \in \mathbb{R}$,

$$\nu_{\delta,\kappa}(a) = \int_\kappa^a \eta'_{\delta,\kappa}(\varphi(s))f'(s)ds.$$

With this choice of entropy-flux pair, Inequality (3.11) may be written:

$$\begin{aligned}
&\int_{Q_T} \Big[|u_\varepsilon(x,t) - \kappa|\psi_t(x,t) + (f(u_\varepsilon(x,t)\top\kappa) \qquad (3.18)\\
&- f(u_\varepsilon(x,t)\perp\kappa))\mathbf{q}(x,t)\cdot\nabla\psi(x,t) - S_\delta(\varphi(u_\varepsilon)(x,t)\\
&- \varphi(\kappa))\nabla\varphi(u_\varepsilon)(x,t)\cdot\nabla\psi(x,t)\Big]dxdt\\
&- \int_{Q_T}\Big[S'_\delta(\varphi(u_\varepsilon)(x,t) - \varphi(\kappa))(\nabla\varphi(u_\varepsilon))^2(x,t)\psi(x,t)\Big]dxdt\\
&+ \int_\Omega |u_0(x) - \kappa|\psi(x,0)dx \geq A(\delta, u_\varepsilon, \kappa, \psi) - \int_{Q_T}|\nabla\psi(x,t)|dm_\varepsilon(x,t),\ \forall\psi\in\mathcal{C},
\end{aligned}$$

where $A(\delta, u_\varepsilon, \kappa, \psi)$ is defined by

$$\begin{aligned}
&A(\delta, u_\varepsilon, \kappa, \psi) = \int_{Q_T}\Big[\Big(|u_\varepsilon(x,t) - \kappa| - \mu_{\delta,\kappa}(u_\varepsilon(x,t))\Big)\psi_t(x,t) \qquad (3.19)\\
&+ \Big((f(u_\varepsilon(x,t)\top\kappa) - f(u_\varepsilon(x,t)\perp\kappa)) - \nu_{\delta,\kappa}(u_\varepsilon(x,t))\Big)\mathbf{q}(x,t)\cdot\nabla\psi(x,t)\Big]dxdt\\
&+ \int_\Omega\Big(|u_0(x) - \kappa| - \mu_{\delta,\kappa}(u_0(x))\Big)\psi(x,0)dx.
\end{aligned}$$

Let us now make δ tend to 0 in inequality (3.18). Thanks to the dominated convergence theorem, one gets that for all $a \in \mathbb{R}$, $\lim_{\delta\longrightarrow 0}\eta_{\delta,\kappa}(a) = |a - \varphi(\kappa)|$, $\lim_{\delta\longrightarrow 0}\mu_{\delta,\kappa}(a) = |a - \kappa|$ and $\lim_{\delta\longrightarrow 0}\nu_{\delta,\kappa}(a) = f(a\top\kappa) - f(a\perp\kappa)$. Hence the passage to the limit in (3.19) as δ tends to 0 yields:

$$\lim_{\delta\longrightarrow 0} A(\delta, u, \kappa, \psi) = 0. \qquad (3.20)$$

One may also write (3.12), for all $\kappa \in \mathbb{R}$, as:

$$\int_{Q_T}\Big[|u_\varepsilon(x,t)-\kappa|\psi_t(x,t)+(f(u_\varepsilon(x,t)\top\kappa)-f(u_\varepsilon(x,t)\bot\kappa))\mathbf{q}(x,t)\cdot\nabla\psi(x,t)$$
$$-S_\delta(\varphi(u_\varepsilon)(x,t)-\varphi(\kappa))\nabla\varphi(u_\varepsilon)(x,t)\cdot\nabla\psi(x,t)\Big]dxdt \quad (3.21)$$
$$+\int_\Omega|u_0(x)-\kappa|\psi(x,0)dx \geq B(\delta,u_\varepsilon,\kappa,\psi)-\int_{Q_T}|\nabla\psi(x,t)|dm_\varepsilon(x,t),$$

$\forall\ \psi \in \mathcal{C}$, where for any u_ε, any $\psi \in \mathcal{C}$, $\kappa \in \mathbb{R}$ and $\delta > 0$, $B(\delta,u_\varepsilon,\kappa,\psi)$ is defined by

$$B(\delta,u_\varepsilon,\kappa,\psi)= \quad (3.22)$$
$$\int_{Q_T}\Big[\nabla\Big(|\varphi(u_\varepsilon)(x,t)-\varphi(\kappa)|-\eta_{\delta,\kappa}(\varphi(u_\varepsilon)(x,t))\Big)\cdot\nabla\psi(x,t)\Big]dxdt.$$

Remarking that for all $\psi \in \mathcal{C}$, one has

$$B(\delta,u_\varepsilon,\kappa,\psi)= \quad (3.23)$$
$$-\int_{Q_T}\Big[\Big(|\varphi(u_\varepsilon)(x,t)-\varphi(\kappa)|-\eta_{\delta,\kappa}(\varphi(u_\varepsilon)(x,t))\Big)\Delta\psi(x,t)\Big]dxdt,$$

yields that

$$\lim_{\delta\longrightarrow 0} B(\delta,u_\varepsilon,\kappa,\psi)=0,\ \text{for all } \psi\in\mathcal{C},\delta>0 \text{ and } \kappa\in\mathbb{R}. \quad (3.24)$$

Let us now define the sets $E_{u_\varepsilon} = \{(x,t) \in Q_T,\ u_\varepsilon(x,t) \in \mathbb{R}_\varphi\}$ and $E_u = \{(x,t) \in Q_T,\ u(x,t) \in \mathbb{R}_\varphi\}$ where u_ε and u have a "genuinely parabolic" contribution, that is where $\varphi(u_\varepsilon)$ and $\varphi(u)$ are non constant. Let $\xi \in C_c^\infty(\mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R})$ such that, for all $(x,t) \in \Omega \times [0,T)$, $\xi(x,t,\cdot,\cdot) \in \mathcal{C}$ and for all $(y,s) \in \Omega \times [0,T)$, $\xi(\cdot,\cdot,y,s) \in \mathcal{C}$. Let us now use Kruskov's technique [15] of "dedoubling the variables" and take $\kappa = u(y,s)$ in (3.18), for $(y,s) \in E_u$ (where u acts "parabolic"), and $\psi = \xi(\cdot,\cdot,y,s)$. Integrating the result over E_u yields:

$$\int_{E_u}\int_{Q_T}\Big[|u_\varepsilon(x,t)-u(y,s)|\xi_t(x,t,y,s)+(f(u_\varepsilon(x,t)\top u(y,s)) \quad (3.25)$$
$$-f(u_\varepsilon(x,t)\bot u(y,s)))\mathbf{q}(x,t)\cdot\nabla_x\xi(x,t,y,s)-S_\delta(\varphi(u_\varepsilon)(x,t)-\varphi(u)(y,s))$$
$$\times\nabla\varphi(u_\varepsilon)(x,t)\cdot\nabla_x\xi(x,t,y,s)-S'_\delta(\varphi(u_\varepsilon)(x,t)-\varphi(u)(y,s))$$
$$\times(\nabla\varphi(u_\varepsilon))^2(x,t)\xi(x,t,y,s)\Big]dx\,dt\,dy\,ds$$

$$+\int_{E_u}\int_\Omega |u_0(x)-u(y,s)|\xi(x,0,y,s)dxdyds$$
$$\geq \int_{E_u} A(\delta,u_\varepsilon,u(y,s),\xi(\cdot,\cdot,y,s))dyds - \int_{E_u}\int_{Q_T}|\nabla_x\xi(x,t,y,s)|dm_\varepsilon(x,t)dyds.$$

Let us now take $\kappa = u(y,s)$ in (3.21), for $(y,s)\in Q_T\setminus E_u$ (where u "acts hyperbolic", and $\psi = \xi(\cdot,\cdot,y,s)$ and integrate over $Q_T\setminus E_u$; this yields

$$\int_{Q_T\setminus E_u}\int_{Q_T}\Big[|u_\varepsilon(x,t)-u(y,s)|\xi_t(x,t,y,s)+\Big(f(u_\varepsilon(x,t)\top u(y,s)) \tag{3.26}$$
$$-f(u_\varepsilon(x,t)\perp u(y,s))\Big)\mathbf{q}(x,t)\cdot\nabla_x\xi(x,t,y,s) - S_\delta(\varphi(u_\varepsilon)(x,t)-\varphi(u)(y,s))$$
$$\times\nabla\varphi(u_\varepsilon)(x,t)\cdot\nabla_x\xi(x,t,y,s)\Big]dxdt\,dyds$$
$$+\int_{Q_T\setminus E_u}\int_\Omega |u_0(x)-u(y,s)|\xi(x,0,y,s)dxdyds$$
$$\geq \int_{Q_T\setminus E_u} B(\delta,u_\varepsilon,u(y,s),\xi(\cdot,\cdot,y,s))dyds$$
$$-\int_{Q_T\setminus E_u}\int_{Q_T}|\nabla_x\xi(x,t,y,s)|dm_\varepsilon(x,t)dyds.$$

Adding (3.25) and (3.26) gives

$$\int_{Q_T}\int_{Q_T}\Big[|u_\varepsilon(x,t)-u(y,s)|\xi_t(x,t,y,s)+(f(u_\varepsilon(x,t)\top u(y,s)) \tag{3.27}$$
$$-f(u_\varepsilon(x,t)\perp u(y,s)))\mathbf{q}(x,t)\cdot\nabla_x\xi(x,t,y,s) - S_\delta(\varphi(u_\varepsilon)(x,t)-\varphi(u)(y,s))$$
$$\times\nabla\varphi(u_\varepsilon)(x,t)\cdot\nabla_x\xi(x,t,y,s)\Big]dxdt\,dyds$$
$$-\int_{E_u}\int_{Q_T}\Big[S'_\delta(\varphi(u_\varepsilon)(x,t)-\varphi(u)(y,s))$$
$$\times(\nabla\varphi(u_\varepsilon))^2(x,t)\xi(x,t,y,s)\Big]dxdt\,dyds$$
$$+\int_{Q_T}\int_\Omega |u_0(x)-u(y,s)|\xi(x,0,y,s)dxdyds$$
$$\geq \int_{E_u} A(\delta,u_\varepsilon,u(y,s),\xi(\cdot,\cdot,y,s))dyds + \int_{Q_T\setminus E_u} B(\delta,u,u(y,s),\xi(\cdot,\cdot,y,s))dyds$$
$$-\int_{Q_T}\int_{Q_T}|\nabla_x\xi(x,t,y,s)|dm_\varepsilon(x,t)dyds.$$

One now exchanges the roles of u_ε and u, and add the resulting equations (the only difference being in the right hand sides). This gives

$$T_1 + T_2 + T_3(\delta) + T_4(\delta) + T_5(\delta) \geq T_6(\delta) - T_7, \tag{3.28}$$

where

$$T_1 = \int_{Q_T}\int_{Q_T}\Big[|u_\varepsilon(x,t) - u(y,s)|(\xi_t(x,t,y,s) + \xi_s(x,t,y,s)) \tag{3.29}$$
$$+ (f(u_\varepsilon(x,t)\top u(y,s)) - f(u_\varepsilon(x,t)\bot u(y,s)))$$
$$\times \Big(\mathbf{q}(x,t)\cdot\nabla_x\xi(x,t,y,s) + \mathbf{q}(y,s)\cdot\nabla_y\xi(x,t,y,s)\Big)\Big]dxdtdyds,$$

$$T_2 = \int_{Q_T}\int_\Omega |u_0(x) - u(y,s)|\xi(x,0,y,s)dxdyds \tag{3.30}$$
$$+ \int_{Q_T}\int_\Omega |u_0(y) - u_\varepsilon(x,t)|\xi(x,t,y,0)dydxdt,$$

$$T_3(\delta) = -\int_{Q_T}\int_{Q_T}\Big[S_\delta(\varphi(u_\varepsilon)(x,t) - \varphi(u)(y,s)) \tag{3.31}$$
$$\times \nabla\varphi(u_\varepsilon)(x,t)\cdot\Big(\nabla_x\xi(x,t,y,s) + \nabla_y\xi(x,t,y,s)\Big)$$
$$+ S_\delta(\varphi(u)(y,s) - \varphi(u_\varepsilon)(x,t))$$
$$\times \nabla\varphi(u)(y,s)\cdot\Big(\nabla_x\xi(x,t,y,s) + \nabla_y\xi(x,t,y,s)\Big)\Big]dxdtdyds,$$

$$T_4(\delta) = \int_{Q_T}\int_{Q_T}\Big[S_\delta(\varphi(u_\varepsilon)(x,t) - \varphi(u)(y,s)) \tag{3.32}$$
$$\times \nabla\varphi(u_\varepsilon)(x,t)\cdot\nabla_y\xi(x,t,y,s) + S_\delta(\varphi(u)(y,s) - \varphi(u_\varepsilon)(x,t))$$
$$\times \nabla\varphi(u)(y,s)\cdot\nabla_x\xi(x,t,y,s)\Big]dxdtdyds,$$

$$T_5(\delta) = -\int_{E_u}\int_{Q_T}\Big[S'_\delta(\varphi(u_\varepsilon)(x,t) - \varphi(u)(y,s)) \tag{3.33}$$
$$\times (\nabla\varphi(u_\varepsilon))^2(x,t)\xi(x,t,y,s)\Big]dxdtdyds$$
$$-\int_{Q_T}\int_{E_{u_\varepsilon}}\Big[S'_\delta(\varphi(u_\varepsilon)(x,t) - \varphi(u)(y,s))(\nabla\varphi(u))^2(y,s)\xi(x,t,y,s)\Big]dxdtdyds,$$

$$T_6(\delta) = \int_{E_u} A(\delta, u_\varepsilon, u(y,s), \xi(\cdot,\cdot,y,s))dyds \tag{3.34}$$
$$+ \int_{Q_T\setminus E_u} B(\delta, u, u(y,s), \xi(\cdot,\cdot,y,s))dyds$$

$$+\int_{E_{u_\varepsilon}} A(\delta, u_\varepsilon, u(x,t), \xi(x,t,\cdot,\cdot))dxdt + \int_{Q_T\setminus E_{u_\varepsilon}} B(\delta, u, u(x,t), \xi(x,t,\cdot,\cdot))dxdt,$$

$$T_7 = \int_{Q_T}\int_{Q_T} |\nabla_x \xi(x,t,y,s)| dm_\varepsilon(x,t)dyds. \quad (3.35)$$

An integration by parts in (3.32) together with the fact that ξ vanishes on $\partial\Omega \times (0,T) \times \Omega \times (0,T)$ and on $\Omega \times (0,T) \times \partial\Omega \times (0,T)$ yields that

$$T_4(\delta) = \int_{Q_T}\int_{Q_T} \Big[S'_\delta(\varphi(u_\varepsilon)(x,t) - \varphi(u)(y,s)) \quad (3.36)$$
$$\times \xi(x,t,y,s)\nabla\varphi(u_\varepsilon)(x,t)\cdot\nabla\varphi(u)(y,s) + S'_\delta(\varphi(u)(y,s) - \varphi(u_\varepsilon)(x,t))$$
$$\times \xi(x,t,y,s)\nabla\varphi(u)(y,s)\cdot\nabla\varphi(u_\varepsilon)(x,t)\Big] dxdt\,dyds.$$

Introducing $E_s = \{(x,t) \in Q_T, \varphi(u_\varepsilon)(x,t) = s\}$ for all $s \in \mathbb{R}$, one has $\nabla\varphi(u_\varepsilon) = 0$ a.e. on E_s (see [4] for instance). Since $Q_T \setminus E_{u_\varepsilon} = \cup_{s\in\varphi(\mathbb{R}\setminus\mathbb{R}_\varphi)} E_s$, and since $\varphi(\mathbb{R}\setminus\mathbb{R}_\varphi)$ is countable, the following equations hold:

$$\nabla\varphi(u_\varepsilon) = 0, \text{ a.e. on } Q_T \setminus E_{u_\varepsilon} \quad (3.37)$$
$$\nabla\varphi(u) = 0, \text{ a.e. on } Q_T \setminus E_u. \quad (3.38)$$

Hence the terms T_4 and T_5 may be written:

$$T_4(\delta) = \int_{E_{u_\varepsilon}\times E_u} \Big[S'_\delta(\varphi(u_\varepsilon)(x,t) - \varphi(u)(y,s)) \quad (3.39)$$
$$\times \xi(x,t,y,s)\nabla\varphi(u_\varepsilon)(x,t)\cdot\nabla\varphi(u)(y,s) + S'_\delta(\varphi(u)(y,s) - \varphi(u_\varepsilon)(x,t))$$
$$\times \xi(x,t,y,s)\nabla\varphi(u)(y,s)\cdot\nabla\varphi(u_\varepsilon)(x,t)\Big] dxdt\,dyds,$$

$$T_5(\delta) = -\int_{E_{u_\varepsilon}\times E_u} \Big[S'_\delta(\varphi(u_\varepsilon)(x,t) - \varphi(u)(y,s)) \quad (3.40)$$
$$\times (\nabla\varphi(u_\varepsilon))^2(x,t)\xi(x,t,y,s) + S'_\delta(\varphi(u_\varepsilon)(x,t) - \varphi(u)(y,s))$$
$$\times (\nabla\varphi(u))^2(y,s)\xi(x,t,y,s)\Big] dxdt\,dyds.$$

Therefore,

$$T_4(\delta) + T_5(\delta) = -\int_{E_u}\int_{E_{u_\varepsilon}} \Big[S'_\delta(\varphi(u_\varepsilon)(x,t) - \varphi(u)(y,s))\xi(x,t,y,s) \quad (3.41)$$
$$\times \Big(\nabla\varphi(u_\varepsilon)(x,t) - \nabla\varphi(u)(y,s)\Big)^2\Big] dxdt\,dyds \leq 0, \quad \forall\delta > 0.$$

We may thus get rid of $T_4 + T_5$ in (3.28) and obtain :

$$T_1 + T_2 + T_3(\delta) \geq T_6(\delta) - T_7,\ \forall \delta > 0. \tag{3.42}$$

One can now let δ tend to 0 in (3.42). This gives

$$\begin{aligned}
&\int_{Q_T}\int_{Q_T}\Big[|u_\varepsilon(x,t) - u(y,s)|(\xi_t(x,t,y,s) + \xi_s(x,t,y,s)) \\
&+ (f(u_\varepsilon(x,t)\top u(y,s)) - f(u_\varepsilon(x,t)\bot u(y,s))) \\
&\times \Big(\mathbf{q}(x,t)\cdot\nabla_x\xi(x,t,y,s) + \mathbf{q}(y,s)\cdot\nabla_y\xi(x,t,y,s)\Big) \\
&- \Big(\nabla_x|\varphi(u_\varepsilon)(x,t) - \varphi(u)(y,s)| + \nabla_y|\varphi(u_\varepsilon)(x,t) - \varphi(u)(y,s)|\Big) \\
&\cdot (\nabla_x\xi(x,t,y,s) + \nabla_y\xi(x,t,y,s))\Big]dxdt\,dyds \\
&+ \int_{Q_T}\int_\Omega |u_0(x) - u(y,s)|\xi(x,0,y,s)dxdyds \\
&+ \int_{Q_T}\int_\Omega |u_0(y) - u_\varepsilon(x,t)|\xi(x,t,y,0)dydxdt \\
&\geq -\int_{Q_T}\int_{Q_T}|\nabla_x\xi(x,t,y,s)|dm_\varepsilon(x,t)dyds.
\end{aligned} \tag{3.43}$$

Let us now take in (3.16) for $x \in \Omega$, $\kappa = u_0(x)$ and $\psi(y,s) = \int_s^T \xi(x,0,y,\tau)d\tau$. Integrating the result on Ω leads to

$$\begin{aligned}
&\int_\Omega\int_{Q_T}\Big[-|u(y,s) - u_0(x)|\ \xi(x,0,y,s) + (f(u(y,s)\top u_0(x)) \\
&- f(u(y,s)\bot u_0(x)))\mathbf{q}(y,s)\cdot\nabla_y\int_s^T \xi(x,0,y,\tau)d\tau \\
&- \nabla_y|\varphi(u)(y,s) - \varphi(u_0(x))|\cdot\int_s^T \nabla_y\xi(x,0,y,\tau)d\tau\Big]dydsdx \\
&+ \int_\Omega\int_\Omega |u_0(x) - u_0(y)|\int_0^T \xi(x,0,y,\tau)d\tau dxdy \geq 0.
\end{aligned} \tag{3.44}$$

Adding (3.43) and (3.44) gives

$$\begin{aligned}
&\int_{Q_T}\int_{Q_T}\Big[|u_\varepsilon(x,t) - u(y,s)|(\xi_t(x,t,y,s) + \xi_s(x,t,y,s)) \\
&+ (f(u_\varepsilon(x,t)\top u(y,s)) - f(u_\varepsilon(x,t)\bot u(y,s))) \\
&\times \Big(\mathbf{q}(x,t)\cdot\nabla_x\xi(x,t,y,s) + \mathbf{q}(y,s)\cdot\nabla_y\xi(x,t,y,s)\Big)
\end{aligned} \tag{3.45}$$

$$
\begin{aligned}
&- \Big(\nabla_x|\varphi(u_\varepsilon)(x,t) - \varphi(u)(y,s)| + \nabla_y|\varphi(u_\varepsilon)(x,t) - \varphi(u)(y,s)|\Big)\\
&\cdot (\nabla_x\xi(x,t,y,s) + \nabla_y\xi(x,t,y,s))\Big]dxdt\,dyds + \int_\Omega\int_{Q_T}\Big[(f(u(y,s)\top u_0(x))\\
&- f(u(y,s)\bot u_0(x)))\mathbf{q}(y,s)\cdot\nabla_y\int_s^T\xi(x,0,y,\tau)d\tau\\
&- \nabla_y|\varphi(u)(y,s) - \varphi(u_0(x))|\cdot\int_s^T\nabla_y\xi(x,0,y,\tau)d\tau\Big]dydsdx\\
&+ \int_\Omega\int_\Omega|u_0(x)-u_0(y)|\int_0^T\xi(x,0,y,\tau)d\tau dxdy\\
&+ \int_{Q_T}\int_\Omega|u_0(y)-u_\varepsilon(x,t)|\xi(x,t,y,0)dydxdt\\
&\geq -\int_{Q_T}\int_{Q_T}|\nabla_x\xi(x,t,y,s)|dm_\varepsilon(x,t)dyds.
\end{aligned}
$$

Some mollifiers in $\mathbb{R}$ and $\mathbb{R}^d$ are now used. Let $\rho \in C_c^\infty(\mathbb{R}^d,\mathbb{R}_+)$ and $\bar\rho \in C_c^\infty(\mathbb{R},\mathbb{R}_+)$ be such that

$$\{x\in\mathbb{R}^d;\rho(x)\neq 0\}\subset\{x\in\mathbb{R}^d;|x|\leq 1\},\ \{x\in\mathbb{R};\bar\rho(x)\neq 0\}\subset[-1,0] \tag{3.46}$$

and

$$\int_{\mathbb{R}^d}\rho(x)dx = 1,\ \int_{\mathbb{R}}\bar\rho(x)dx = 1. \tag{3.47}$$

For some positive real values a and b which will be chosen later, let us define $\rho_a = \frac{1}{a^d}\rho(\frac{x}{a})$ for all $x\in\mathbb{R}^d$ and $\bar\rho_b = \frac{1}{b}\bar\rho(\frac{x}{b})$ for all $x\in\mathbb{R}$. In the remainder of the paper, we shall denote by $\Omega_a = \{x\in\Omega, d(x,\partial\Omega) < a\}$.

One sets $\xi(x,t,y,s) = \psi(x,t)\rho_a(x-y)\bar\rho_b(t-s)$, where $\psi\in\mathcal{C}$ is such that $\psi(x,t) = 0$ for all $(x,t)\in(\Omega_a\times[0,T))\cup(\Omega\times(T-b,T))$. Thus for all $(x,t)\in\Omega\times[0,T)$, one has $\xi(x,t,\cdot,\cdot)\in\mathcal{C}$ and for all $(y,s)\in\Omega\times[0,T)$, one has $\xi(\cdot,\cdot,y,s)\in\mathcal{C}$. Note that $\xi(\cdot,\cdot,\cdot,0) = 0$. One gets, from (3.45),

$$E_1 + E_2 + E_3 + E_4 \geq -E_5, \tag{3.48}$$

with

$$E_1 = \int_{Q_T}\int_{Q_T}\rho_a(x-y)\bar\rho_b(t-s)|u_\varepsilon(x,t)-u(y,s)|\psi_t(x,t)dxdtdyds \tag{3.49}$$

$$
\begin{aligned}
E_2 = \int_{Q_T}\int_{Q_T}\Big[\Big(f(u_\varepsilon(x,t)\top u(y,s)) - f(u_\varepsilon(x,t)\bot u(y,s))\Big) \qquad (3.50)\\
\times\Big(\rho_a(x-y)\bar\rho_b(t-s)\mathbf{q}(x,t)\cdot\nabla\psi(x,t)
\end{aligned}
$$

$$+ \psi(x,t)\bar{\rho}_b(t-s)(\mathbf{q}(x,t)-\mathbf{q}(y,s))\cdot\nabla\rho_a(x-y)\Big)\Big]dxdt\,dyds$$

$$E_3 = -\int_{Q_T}\int_{Q_T}\Big[\rho_a(x-y)\bar{\rho}_b(t-s)\Big(\nabla_x|\varphi(u_\varepsilon)(x,t)-\varphi(u)(y,s)| \tag{3.51}$$

$$+ \nabla_y|\varphi(u_\varepsilon)(x,t)-\varphi(u)(y,s)|\Big)\cdot\nabla\psi(x,t)\Big]dxdt\,dyds$$

$$E_4 = \int_\Omega\int_{Q_T}\Big[(f(u(y,s)\top u_0(x)) - f(u(y,s)\bot u_0(x))) \tag{3.52}$$

$$\times\,\mathbf{q}(y,s)\cdot\psi(x,0)\nabla\rho_a(x-y)\int_s^T\bar{\rho}_b(-\tau)d\tau$$

$$+ \nabla_y|\varphi(u)(y,s)-\varphi(u_0(x))|\cdot\psi(x,0)\nabla\rho_a(x-y)\int_s^T\bar{\rho}_b(-\tau)d\tau\Big]dydsdx$$

$$+ \int_\Omega\int_\Omega|u_0(x)-u_0(y)|\psi(x,0)\rho_a(x-y)dxdy$$

$$E_5 = \int_{Q_T}\int_{Q_T}|\nabla_x\rho_a(x-y)\psi(x,t)|\bar{\rho}_b(t-s)dm_\varepsilon(x,t)dyds. \tag{3.53}$$

One sets

$$D_1 = \int_{Q_T}|u_\varepsilon(x,t)-u(x,t)|\ \psi_t(x,t)dxdt \tag{3.54}$$

$$D_2 = \int_{Q_T}\Big[\Big(f(u_\varepsilon(x,t)\top u(x,t)) - f(u_\varepsilon(x,t)\bot u(x,t))\Big) \tag{3.55}$$

$$\times\,\mathbf{q}(x,t)\cdot\nabla\psi(x,t)\Big]dxdt$$

$$D_3 = -\int_{Q_T}\Big[\nabla|\varphi(u_\varepsilon)(x,t)-\varphi(u)(x,t)|\cdot\nabla\psi(x,t)\Big]dxdtdyds. \tag{3.56}$$

Let

$$V_a = \sup_{y\in B(0,a)}\|u(\cdot,\cdot)-u(\cdot+y,\cdot)\|_{L^1(\Omega\setminus\Omega_a\times(0,T))},$$
$$\bar{V}_b = \sup_{\tau\in(0,b)}\|u(\cdot,\cdot)-u(\cdot,\cdot+\tau)\|_{L^1(\Omega\times(0,T-b))}.$$

Then

$$|E_1 - D_1| \le C_1(V_a+\bar{V}_b)\|\psi_t\|_\infty, \tag{3.57}$$

and

$$|E_2 - D_2| \le C_2(V_a+\bar{V}_b)(\|\nabla\psi\|_\infty + \|\psi\|_\infty), \tag{3.58}$$

where one denotes by $\|\cdot\|_\infty$ both the $L^\infty(Q_T)$ or $L^\infty(\Omega)$ norms, thanks to the fact that f is Lipschitz continuous. This was proved in [11]) for $f \in C^1$

but in fact the proof holds with no modification in the case of a Lipschitz continuous f. Integrations by parts in (3.51) lead to

$$E_3 = \int_{Q_T}\int_{Q_T}\Big[\rho_a(x-y)\bar\rho_b(t-s) \qquad (3.59)$$
$$\times |\varphi(u_\varepsilon)(x,t) - \varphi(u)(y,s)|\Delta\psi(x,t)\Big] dxdt\, dyds.$$

Let $W_a = \sup\{\|\varphi(u(\cdot,\cdot)) - \varphi(u(\cdot+y,\cdot))\|_{L^2(\Omega\setminus\Omega_a\times(0,T))}, y \in B(0,a)\}$ and $\bar W_b = \sup\{\|\varphi(u(\cdot,\cdot)) - \varphi(u(\cdot,\cdot+\tau))\|_{L^2(\Omega\times(0,T-b))}, \tau\in(0,b)\}$. Then, by the Cauchy-Schwarz inequality,

$$|E_3 - D_3| \le C_3(W_a + \bar W_b)\|\Delta\psi\|_2, \qquad (3.60)$$

where one denotes by $\|\cdot\|_2$ the $L^2(Q_T)$ norm. Let

$$V_{0,a} = \sup\{\|u_0(\cdot) - u_0(\cdot+y)\|_{L^1(\Omega\setminus\Omega_a\times(0,T))}, y\in B(0,a)\}.$$

Using the fact that $\int_{\mathbb{R}^d}|\nabla\rho_a(x)|dx \le \frac{C_4}{a}$, and that the length of the time support of $\bar\rho_b$ is b, one gets

$$|E_4| \le C_{10}(V_{0,a} + \frac{b}{a})\|\psi(\cdot,0)\|_\infty. \qquad (3.61)$$

Denoting by $M_\varepsilon = m_\varepsilon(Q_T)$,

$$|E_5| \le C_{11}M_\varepsilon(\frac{\|\psi\|_\infty}{a} + \|\nabla\psi\|_\infty). \qquad (3.62)$$

Since $u_0\in BV(\Omega)$, there exists C_{12} in $\mathbb{R}$ such that : $V_{0,a}\le C_{12}a$. Using the BV estimate (3.7), one gets $V_a\le C_{13}a$ and $\bar V_b \le C_{14}b$. The translates estimates (3.5) give $W_a\le C_{15}a$ and $\bar W_b\le C_{16}\sqrt{b}$. One concludes, using (3.57)-(3.62), that

$$D1 + D2 + D3 \ge -C_{17}a\,(\|\nabla\psi\|_\infty + \|\psi_t\|_\infty + \|\psi\|_\infty + \|\psi(\cdot,0)\|_\infty) \qquad (3.63)$$
$$- C_{18}(a+\sqrt{b})\ \|\Delta\psi\|_2 - \frac{b}{a}\ \|\psi(\cdot,0)\|_\infty - C_{19}\ b\ (\|\nabla\psi\|_\infty + \|\psi_t\|_\infty + \|\psi\|_\infty)$$
$$- C_{20}M_\varepsilon(\frac{\|\psi\|_\infty}{a} + \|\nabla\psi\|_\infty).$$

Let us now let b tend to 0 in (3.63). This gives

$$\int_{Q_T}\Big[|u_\varepsilon(x,t) - u(x,t)|\ \psi_t(x,t) + (f(u_\varepsilon(x,t)\top u(x,t)) \qquad (3.64)$$
$$- f(u_\varepsilon(x,t)\perp u(x,t)))\mathbf{q}(x,t)\cdot\nabla\psi(x,t)$$
$$- \nabla|\varphi(u_\varepsilon)(x,t) - \varphi(u)(x,t)|\cdot\nabla\psi(x,t)\Big] dxdt$$

$$\geq -C_{21}a(\|\nabla\psi\|_\infty + \|\psi_t\|_\infty + \|\psi\|_\infty + \|\psi(\cdot,0)\|_\infty + \|\Delta\psi\|_2)$$
$$- C_{20}M_\varepsilon(\frac{\|\psi\|_\infty}{a} + \|\nabla\psi\|_\infty).$$

One remarks that (3.64) applies for all functions $\psi \in C^1(\overline{Q_T})$ such that $\psi \geq 0$, $\Delta\psi \in L^2(Q_T)$, $\psi(\cdot,T) = 0$ and $\psi(x,t) = 0$ for all $x \in \Omega_a \times (0,T)$. This concludes the proof of Lemma 2.1.

4. Conclusion of the proof of Theorem 2.1

Let $\delta > 0$ and $a > 0$ with $1 \geq \delta \geq a$ and $\delta + 2a \leq r_\Omega$ where r_Ω is such that one has $d(\cdot,\partial\Omega) \in C^2(\Omega_{r_\Omega})$. The two values δ and a will be chosen at the very end of the proof.

Let the function $g \in C^1(\mathbb{R}_+,\mathbb{R}_+)$ be defined by $g''(s) = 0$ for $s \in (0,a)$, $g''(s) = \frac{1}{a}$ for $s \in (a,2a)$, $g''(s) = -\frac{1}{\delta}$ for $s \in (2a,2a+\delta)$, $g''(s) = 0$ for $s \in (2a+\delta,\infty)$, $g'(s) = \int_{(0,s)} g''(t)dt$, $g(s) = \int_{(0,s)} g'(t)dt$. One can easily verify the following properties : $g' \geq 0$, $\sup_{s\in\mathbb{R}_+} g'(s) = 1$, the support of g' is included in $[a,\delta+2a]$, $g(s) = 0$ for $s \in [0,a]$, g is nondecreasing and $g(s) = \frac{\delta+a}{2}$ for $s \in [\delta+2a,\infty)$.

We shall take for ψ, in Lemma 2.1, the function defined by : $\psi(x,t) = (T-t)g(d(x,\partial\Omega))$, for all $(x,t) \in Q_T$. This function satisfies the conditions $\psi \in C^1(\overline{Q_T})$, $\psi \geq 0$, $\Delta\psi \in L^2(Q_T)$, $\psi(\cdot,T) = 0$ and $\psi(x,t) = 0$ for all $x \in \Omega_a$ and $t \in (0,T)$. Thanks to the property $|\nabla d(\cdot,\partial\Omega)| = 1$, one can easily check that $|\nabla\psi(x,t)| = (T-t)g'(d(x,\partial\Omega))$, $\Delta\psi(x,t) = (T-t)[g''(d(x,\partial\Omega)) + g'(d(x,\partial\Omega))\Delta d(x,\partial\Omega)]$, and therefore, $\psi_t \leq 0$, $\|\psi\|_\infty = \|\psi(\cdot,0)\|_\infty = T\delta + \frac{a}{2}$, $\|\nabla\psi\|_\infty = T$, and $\|\Delta\psi\|_2 \leq C_5\frac{1}{\sqrt{a}}$, using $a \leq \delta \leq 1$. Therefore,

$$-\delta + \frac{a}{2}\int_{Q_T} |u_\varepsilon(x,t) - u(x,t)|dxdt \tag{4.1}$$
$$\geq \int_{Q_T} |u_\varepsilon(x,t) - u(x,t)|\psi_t(x,t)dxdt - C_{23}\delta^2,$$

and, using condition (2.4)

$$\Big|\int_{Q_T} \Big(f(u_\varepsilon(x,t)\top u(x,t)) - f(u_\varepsilon(x,t)\bot u(x,t))\Big) \tag{4.2}$$
$$\times\, \mathbf{q}(x,t)\cdot\nabla\psi(x,t)dxdt\Big| \leq C_{24}\delta^2.$$

Thanks to Hardy's inequality, which writes [4]

$$\int_\Omega \frac{w(x)^2}{d(x,\partial\Omega)^2}dx \le C_{25}\int_\Omega (\nabla w(x))^2 dx,\ \forall w \in H_0^1(\Omega),$$

by the Cauchy–Schwarz inequality and thanks to the $L^2(0,T;H^1(\Omega))$ estimates (3.3) one has

$$\int_{\Omega_{2a}\times(0,T)} |\varphi(u_\varepsilon)(x,t)-\varphi(u)(x,t)|(T-t)dxdt \le C_{26}a^{3/2},$$

which also gives

$$\int_{\Omega_{2a}\times(0,T)} |\varphi(u_\varepsilon)(x,t)-\varphi(u)(x,t)|g''(d(x,\partial\Omega))(T-t)dxdt \le C_{26}\sqrt{a}.$$

There exists C_{27}, such that, for $\delta \le C_{27}$, one has

$$\begin{aligned}&\Big|\int_{\Omega_{\delta+2a}\times(0,T)} \Big[|\varphi(u_\varepsilon)(x,t)-\varphi(u)(x,t)| \\ &\times g'(d(x,\partial\Omega))\Delta d(x,\partial\Omega)(T-t)\Big]dxdt\Big| \\ &\le \frac{1}{2}\int_{\Omega_{\delta+2a}\times(0,T)} |\varphi(u_\varepsilon)(x,t)-\varphi(u)(x,t)|\frac{T-t}{\delta}dxdt.\end{aligned} \tag{4.3}$$

Therefore, using the expression of $\Delta\psi$ and the properties of g, one gets

$$\begin{aligned}0 &\ge -\frac{1}{2}\int_{\Omega_{\delta+2a}\times(0,T)} |\varphi(u_\varepsilon)(x,t)-\varphi(u)(x,t)|\frac{T-t}{\delta}dxdt \\ &\ge \int_{Q_T} |\varphi(u_\varepsilon)(x,t)-\varphi(u)(x,t)|\Delta\psi(x,t)dxdt - 2C_{26}\sqrt{a}.\end{aligned} \tag{4.4}$$

On the other hand, one has

$$\begin{aligned}&-a\big(\|\nabla\psi\|_\infty+\|\psi_t\|_\infty+\|\psi\|_\infty+\|\psi(\cdot,0)\|_\infty+\|\Delta\psi\|_2\big) - M_\varepsilon(\frac{\|\psi\|_\infty}{a}+\|\nabla\psi\|_\infty) \\ &\ge -C_{28}(\sqrt{a}+M_\varepsilon(\frac{\delta}{a}+1)).\end{aligned} \tag{4.5}$$

Using (4.1), (4.2), (4.4), (4.5), Lemma 2.1 and dividing by δ, one gets

$$\int_{Q_T} |u_\varepsilon(x,t)-u(x,t)|dxdt \le C_{29}(\delta + \frac{\sqrt{a}}{\delta}+\frac{M_\varepsilon}{a}). \tag{4.6}$$

One can now take in (4.6), $a = M_\varepsilon^{4/5}$ and $\delta = M_\varepsilon^{1/5}$, for $M_\varepsilon \le 1$. Using the BV estimate (3.9) for u_ε, which reads $M_\varepsilon \le C_{30}\varepsilon$, one gets

$$\|u_\varepsilon - u\|_{L^1(Q_T)} \le C_{31}\varepsilon^{1/5}. \tag{4.7}$$

Condition $a \leq \delta \leq C_{27}$ is then satisfied for ε small enough. Since $\|u_\varepsilon - u\|_{L^1(Q_T)} \leq C_{32}$, the error estimate is then proved for all $\varepsilon > 0$.

References

[1] C. Bardos, A.Y. Leroux, and J.C. Nédélec, *First order quasilinear equations with boundary conditions*, Commun. Partial Differ. Eq., 4 (1979), 1017-1034.

[2] Ph. Bénilan, *Equations d'évolution dans un espace de Banach quelconque et applications*, Thèse d'état, Université d'Orsay (1972).

[3] F. Bouchut, F. Guarguaglini, and R. Natalini, *Diffusive BGK approximations for nonlinear multidimensional parabolic equations*, Indiana Univ. Math. J., 49 (2000).

[4] H. Brezis "Analyse Fonctionnelle: Théorie et Applications", Masson, Paris, (1983).

[5] J. Carrillo *Entropy solutions for nonlinear degenerate problems,* Arch. Rat. Mech. Anal., 147 (1999), 269-361.

[6] B.Cockburn, F. Coquel, and P. LeFloch, *An error estimate for finite volume methods for multidimensional conservation laws,* Math. Comput., 63 (1994), 207, 77-103.

[7] B. Cockburn and G. Gripenberg, *Continuous dependence on the nonlinearities of solutions of degenerate parabolic equations,* J. Differential Equations, 151 (1999), 231-251.

[8] P. Dangla, O. Coussy, and R. Eymard, *A vanishing diffusion process in unsaturated soils,* Int. J. Non-Linear Mechanics, 33 (1998), 1027-1037.

[9] R. DiPerna, *Measure-valued solutions to conservation laws*, Arch. Rat. Mech. Anal., 88 (1985), 223-270.

[10] R. Eymard, *Simulation of air-water flows in soils*, First M.I.T. Conference on Computational Fluid and Solid Mechanics, June 12 - 14 (2001).

[11] R. Eymard, T. Gallouët, and R. Herbin *Existence and Uniqueness of the Entropy Solution to a Nonlinear Hyperbolic Equation*. Chi. Ann. of Math., 16, Ser B 1 (1995), 1-14.

[12] R. Eymard, T. Gallouët, M. Ghilani, and R. Herbin *Error estimates for the approximate solutions of a nonlinear hyperbolic equation given by finite volume schemes.* IMA Journal of Numerical Analysis, 18 (1997), 563-594.

[13] R. Eymard, T. Gallouët, R. Herbin, and A. Michel, *Convergence of a finite volume scheme to the entropy weak solution of a nonlinear hyperbolic degenerate parabolic equation*, accepted for publication in Num. Math.

[14] D. Kröner and M. Ohlberger, *A posteriori error estimates for upwind finite volume schemes for nonlinear conservation laws in multi dimensions.* Math. Comput., 69 (2000), 25-39.

[15] S.N. Krushkov, *First Order quasilinear equations with several space variables*, Math. USSR. Sb., 10 (1970), 217-243.

[16] A. Lagha-Benabdallah and F. Smadhi, *Existence de solutions faibles pour un problème aux limites associé à une équation parabolique dégénérée*, Maghreb Math. Rev., 2 (1993), 201-222.

[17] B.A. Schrefler and D. Gawin, *The effective stress principle: incremental or finite form?*, Int. Jour. for Num. or Ana. Methods in Geomechanics, 20 (1996), 785-814.

[18] J.P. Vila *Convergence and error estimate in finite volume schemes for general multidimensional conservation laws, I. explicit monotone schemes*, Model. Math. Anal. Numer., 28 (1994), 267-285.