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APPROXIMATION BY THE FINITE VOLUME METHOD OF AN ELLIPTIC-PARABOLIC EQUATION ARISING IN ENVIRONMENTAL STUDIES

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We prove the convergence of a finite volume scheme for the Richards equation $\beta(p)_t - \mathrm{div}\big(\Lambda(\beta(p))(\nabla p - \rho \mathbf{g})\big) = 0$ together with a Dirichlet boundary condition and an initial condition in a bounded domain $\Omega \times (0,T)$. We consider the hydraulic charge $u = \frac{p}{\rho g} - z$ as the main unknown function so that no upwinding is necessary. The convergence proof is based on the strong convergence in L^2 of the water saturation $\beta(p)$, which one obtains by estimating differences of space and time translates and applying Kolmogorov's theorem. This implies the convergence in L^2 of the approximate water mobility towards $\Lambda(\beta(p))$ as the time and mesh steps tend to 0, which in turn implies the convergence of the approximate pressure to a weak solution p of the continuous problem.

Keywords: Richards equation, Finite volume scheme, Kolmogorov's theorem.

1. The Richards equation

When modelling the two phase flow of air and water in a soil, it is often realistic to assume that the air is at the atmospheric pressure; this assumption yields a simplified model (see Richards[24] or Bear[3] p. 487) which is often used by hydrologists and known as the Richards equation, namely

$$\beta(p)_t(x,t) - \mathrm{div}\Big(\Lambda\big(\beta(p)\big)\,\big(\nabla p - \rho \mathbf{g}\big)\Big)(x,t) = 0, \qquad (x,t) \in \Omega \times \mathbb{R}^+, \qquad (1.1)$$

where p is the water pressure and Ω is a bounded open domain of $\mathbb{R}^d (d = 1,2,3)$.

The function β is the water storage capacity, the function Λ is the water phase mobility, the real value $\rho > 0$ is the water density and $\mathbf{g}$ is the gravity acceleration. We prescribe a Dirichlet boundary condition, namely

$$p(x,t) = \bar{p}(x,t), \qquad (x,t) \in \partial\Omega \times \mathbb{R}^+. \tag{1.2}$$

We also prescribe an initial condition for the water pressure

$$p(x,0) = p_0(x), \qquad x \in \Omega. \tag{1.3}$$

Approximation methods of the equations (1.1)-(1.3) are frequently made by considering (see e.g. Alt *et al* [1], Yin[25], Eymard *et al* [10]) the term div $(\Lambda(\beta(p))(\nabla p - \rho\mathbf{g}))$ as the sum of a diffusion term div $(\Lambda(\beta(p))\nabla p)$ and a convective term div $(\Lambda(\beta(p))\rho\mathbf{g})$. The diffusion term is rewritten as $\Delta H(p)$, with $H' = \Lambda o \beta$, and a change of unknowns is made, using the reciprocal function of H (this change of functions is called the Kirchoff transform). A drawback of this method is that one should use an upstream weighting scheme in the discretization of the convective term (Forsyth *et al* [13] or Eymard *et al* [10]) in order to preserve the stability of the scheme, therefore degrading the order of convergence. In this paper, we consider another approach in the modelling of water flow through soils; it consists in introducing the hydraulic charge u, defined by

$$u(x,t) = \frac{p(x,t)}{\rho\, g} - z(x) \text{ for all } (x,t) \in \Omega \times (0,T), \text{ for all } T > 0, \tag{1.4}$$

where g denotes the modulus of $\mathbf{g}$ and the function $z(x)$ is the projection of point x on the vertical axis, oriented by $\mathbf{g}/g$.
With a simple adimensionalisation, one may assume that $\rho\, g = 1$ and one may then rewrite equation (1.1) as follows:

$$\beta(u+z)_t(x,t) - \text{div}\Big(\Lambda\big(\beta(u+z)\big)\,\nabla u\Big)(x,t) = 0, \qquad (x,t) \in \Omega \times \mathbb{R}^+. \tag{1.5}$$

In (1.5), only a diffusion term appears so that the numerical approximation will not involve any upwinding. A natural space for studying solutions of equation (1.5) is the space $L^2(0,T;H^1(\Omega))$, and therefore the trace of the function u on $\partial\Omega$ is well defined. In order to simplify the notation, we shall denote a function u defined on Ω and its trace defined on $\partial\Omega$ (whenever it exists) by the same symbol; hence the boundary condition may be written as:

$$u(x,t) = \bar{u}(x,t), \text{ for all } x \in \partial\Omega \text{ and a.e. } t > 0. \tag{1.6}$$

In (1.6), the function $\bar{u} = \bar{p} - z$ is defined on the boundary $\partial\Omega \times (0,T)$ as the trace of a function $\bar{u}$, which belongs to $H^1(\Omega \times (0,T))$. We prescribe an initial condition on u

$$u(x,0) = u_0(x), \; x \in \Omega, \tag{1.7}$$

where $u_0(x) = p_0(x) - z(x)$. We denote by (P) the problem given by (1.5), (1.6) and (1.7). We shall make the following assumptions.

Hypotheses H

(H1) the domain $\Omega \subset \mathbb{R}^d$ *is polygonal in a general sense (that is polygonal if* $d = 2$ *and polyhedral if* $d = 3$*),*

(H2) $u_0 \in L^2(\Omega)$*,*

(H3) Λ *is a Lipschitz-continuous function defined on* $\mathbb{R}$ *with Lipschitz constant* $L > 0$*, such that there exists* $\Lambda_m > 0$ *with* $\Lambda \geq \Lambda_m$*,*

(H4) β *is a nondecreasing Lipschitz continuous function, with Lipschitz constant* $L_\beta > 0$*, such that there exist* $\beta_I \in \mathbb{R}$ *and* $\beta_S \in \mathbb{R}$ *with* $\beta_I \leq \beta \leq \beta_S$*,*

(H5) $\bar{u} \in H^1(\Omega \times (0,T))$*, for all* $T > 0$*.*

Definition 1.1 Under Hypotheses H, a function u is said to be a weak solution of Problem (P) if

$$\left.\begin{array}{ll} (i) & u - \bar{u} \in L^2(0,T;H_0^1(\Omega)); \\ (ii) & \displaystyle\int_0^T \int_\Omega \Big\{\beta\big(u(x,t) + z(x)\big) - \beta\big(u_0(x) + z(x)\big)\Big\}\, \psi_t(x,t)\, dx\, dt = \\ & \displaystyle\int_0^T \int_\Omega \Big\{\Lambda\Big(\beta\big(u(x,t) + z(x)\big)\Big)\, \nabla u(x,t)\ \nabla\psi(x,t)\Big\}\, dx\, dt, \\ & \text{for all } \psi \in L^2(0,T;H_0^1(\Omega)) \\ & \text{such that } \psi_t \in L^\infty(\Omega \times (0,T)),\ \psi(.,T) = 0. \end{array}\right\} \quad (1.8)$$

It follows from Otto[23] that if $\bar{u}$ only depends on the space variable, Problem (P) has at most one weak solution in the sense of the above definition.

The discretization of the Richards equation was performed by means of the finite difference method by Hornung[18] and by means of the finite element method by Knabner[20]. Kelanemer[19] and Chounet *et al* [5] implemented a mixed finite element method and Folkovic *et al* [12] applied a finite volume scheme with a mesh which is constructed as dual to a finite element mesh, and which is adapted according to an error indicator.

Finite volume schemes have first been developed by engineers in order to study complex coupled physical phenomena where the conservation of extensive quantities (such as masses, energy, impulsion...) must be carefully respected by the approximate solution. Another advantage of such schemes is that a large variety of meshes can be used. The basic idea is the following : one integrates the partial differential equations in each control volume and then approximates the fluxes across the volume boundaries. The finite volume method is one of the most popular method in computational hydrology. Therefore it is interesting from a mathematical point of view to present convergence proofs for this method.

In section 2, we introduce the finite volume scheme and define the approximate problem. Then we prove in section 3 the existence and uniqueness of the discrete solution $u_{\mathcal{T},k}$ to this problem. The existence and uniqueness proof is based on the fact that the semigroup corresponding to $\beta(u(.,t)+z(.))$ satisfies a contraction property.
In section 4, we derive a priori estimates. We prove an estimate on $u_{\mathcal{T},k}$ in a discrete $L^2(0,T;H^1(\Omega))$ norm. We then deduce estimates on differences of space translates of $u_{\mathcal{T},k}$ and on differences of space and time translates of $\beta(u_{\mathcal{T},k}+z)$, which imply that the sequence $\{\beta(u_{\mathcal{T},k}+z)\}$ is relatively compact in $L^2(\Omega\times(0,T))$. Basic ingredients that we use to obtain these estimates are discrete forms of the Poincaré inequality and of trace theorems that we recall at the end of section 2.
From these estimates, we deduce in section 5, for any sequence of approximate solutions, the existence of a subsequence which converges to a function $u \in L^2(0,T;H^1(\Omega))$ weakly in $L^2(\Omega\times(0,T))$ and such that $\{\beta(u_{\mathcal{T},k}+z)\}$ converges to a function $\bar{\beta}$ strongly in $L^2(\Omega\times(0,T))$ as the mesh and time steps tend to 0. Finally, we prove that $\bar{\beta}=\beta(u+z)$, where u is a weak solution of Problem (P). In the case that $\bar{u}=\bar{u}(x)$, the limit u is the unique weak solution of Problem (P) and the whole family of approximate solutions converges to u.
For references on the convergence of the finite volume method for elliptic equations we refer to e.g. Heinrich[15], Herbin[16], Lazarov *et al* [21], Mishev[22], Eymard *et al* [8], Eymard *et al* [6], Eymard *et al* [7], Gallouët *et al* [14]; for linear or nonlinear parabolic equations we refer to Baughman *et al* [2], Herbin[17], Eymard *et al* [8], Eymard *et al* [9] and Eymard *et al* [10]; see also Feistauer *et al* [11] for the convergence of a coupled finite-volume finite-element scheme for a semilinear parabolic equation.

2. The numerical scheme

In this section, we construct approximate solutions of Problem (P). To this purpose, we introduce a notion of admissible finite volume mesh (see also Eymard *et al* [8]or Eymard *et al* [6]).

Definition 2.2 (Admissible meshes) Let Ω be an open bounded polygonal subset of $\mathbb{R}^d$. An admissible finite volume mesh of Ω, denoted by $\mathcal{T}$, is given by a family of "control volumes", which are open polygonal convex subsets of Ω (with positive measure), a family of subsets of $\overline{\Omega}$ contained in hyperplanes of $\mathbb{R}^d$ denoted by $\mathcal{E}$ (these are the edges of the control volumes), with strictly positive length, and a family of points of Ω satisfying the following properties (in fact we also denote by $\mathcal{T}$ the family of control volumes):

(i) The closure of the union of all the control volumes is $\overline{\Omega}$;

(ii) For any $K \in \mathcal{T}$, there exists a subset $\mathcal{E}_K$ of $\mathcal{E}$ such that $\partial K = \bar{K} \backslash K = \cup_{\sigma \in \mathcal{E}_K} \overline{\sigma}$. Furthermore, $\mathcal{E} = \cup_{K \in \mathcal{T}} \mathcal{E}_K$.

(iii) For any $(K, L) \in \mathcal{T}^2$ with $K \neq L$, either the length of $\bar{K} \cap \bar{L}$ is 0 or $\bar{K} \cap \bar{L} = \overline{\sigma}$ for some $\sigma \in \mathcal{E}$, which will then be denoted by $K|L$.

(iv) The family of points $(x_K)_{K \in \mathcal{T}}$ is such that $x_K \in K$ (for all $K \in \mathcal{T}$) and, if $\sigma = K|L$, it is assumed that the straight line (x_K, x_L) is orthogonal to σ.

In the sequel, the following notations are used. The mesh size is defined by size$(\mathcal{T}) = \sup\{\delta(\mathrm{K}),\, K \in \mathcal{T}\}$ where $\delta(K)$ denotes the diameter of the control volume K. For any $K \in \mathcal{T}$ and $\sigma \in \mathcal{E}$, $m(K)$ is the area of K and $m(\sigma)$ the length of σ. The set of interior (resp. boundary) edges is denoted by $\mathcal{E}_{int}$ (resp. $\mathcal{E}_{ext}$), that is $\mathcal{E}_{int} = \{\sigma \in \mathcal{E};\ \sigma \not\subset \partial\Omega\}$ (resp. $\mathcal{E}_{ext} = \{\sigma \in \mathcal{E};\ \sigma \subset \partial\Omega\}$). For all $K \in \mathcal{T}$ and $\sigma \in \mathcal{E}_K$, we denote by $d_{K,\sigma}$ the Euclidean distance between x_K and σ; the "transmissibility" through σ is defined by $\tau_{K,\sigma} = \dfrac{m(\sigma)}{d_{K,\sigma}}$. We finally set $z_K = z(x_K)$, for all $K \in \mathcal{T}$.

Let $k \in (0, T)$ be the time step and $((K)_{K \in \mathcal{T}},\ (\sigma)_{\sigma \in \mathcal{E}},\ (x_K)_{K \in \mathcal{T}})$ be an admissible mesh. A semi-implicit finite volume scheme may be defined by the following equations:

$$\left.\begin{aligned} & u_K^0 = \frac{1}{m(K)} \int_K u_0(x)dx, \\ & w_K^0 = \beta(u_K^0 + z_K), \\ & \text{for all } K \in \mathcal{T}. \end{aligned}\right\} \tag{2.9}$$

$$\left.\begin{aligned} & m(K)\frac{w_K^{n+1} - w_K^n}{k} - \sum_{\sigma \in \mathcal{E}_K} \tau_{K,\sigma} \Lambda(w_K^n)(u_\sigma^{n+1} - u_K^{n+1}) = 0, \\ & w_K^{n+1} = \beta(u_K^{n+1} + z_K), \\ & \text{for all } K \in \mathcal{T} \text{ and } n \in \mathbb{N}. \end{aligned}\right\} \tag{2.10}$$

where the values $(u_\sigma^{n+1})_{\sigma \in \mathcal{E}}$ are obtained from the Dirichlet boundary conditions on the boundary edges and from the conservation of fluxes on the internal edges:

$$\left.\begin{aligned} & u_\sigma^{n+1} = \frac{1}{k\ m(\sigma)} \int_{nk}^{(n+1)k} \int_\sigma \bar{u}(x,t) d\gamma(x) dt, \\ & \text{for all } \sigma \in \mathcal{E}_{ext}, \\ & \tau_{K,\sigma}\, \Lambda(w_K^n)\, (u_\sigma^{n+1} - u_K^{n+1}) + \tau_{L,\sigma}\, \Lambda(w_L^n)\, (u_\sigma^{n+1} - u_L^{n+1}) = 0, \\ & \text{for all } \sigma \in \mathcal{E}_{int} \text{ with } \sigma = K|L \text{ and } n \in \mathbb{N}, \end{aligned}\right\} \tag{2.11}$$

where we denote by $d\gamma$ the $d-1$-dimensional Lebesgue measure on edges.

Remark 2.1 *Note that scheme (2.9)-(2.11) is in fact a cell-centered finite volume scheme using a harmonic average of mobilities for all internal edges. Indeed, for an internal edge* $\sigma = K|L$, *(2.11) leads to* $u_\sigma^{n+1} = \dfrac{\tau_{K,\sigma}\Lambda(w_K^n)u_K^{n+1} + \tau_{L,\sigma}\Lambda(w_L^n)u_L^{n+1}}{\tau_{K,\sigma}\Lambda(w_K^n) + \tau_{L,\sigma}\Lambda(w_L^n)}$. *Therefore the discretization scheme (2.10) can be rewritten in the more standard form*

$$m(K)\frac{w_K^{n+1} - w_K^n}{k} - \sum_{L,K|L\in\mathcal{E}_K} F_{K,L}^n = 0, \quad \textit{for all } K \in \mathcal{T}, \tag{2.12}$$

where the expression of the flux $F_{K,L}^n$ *from* K *to* L *at time step* n *is given by*

$$F_{K,L}^n = \frac{\tau_{K,\sigma}\Lambda(w_K^n)\tau_{L,\sigma}\Lambda(w_L^n)}{\tau_{K,\sigma}\Lambda(w_K^n) + \tau_{L,\sigma}\Lambda(w_L^n)}(u_L^{n+1} - u_K^{n+1}). \tag{2.13}$$

If there exists a solution to (2.9)-(2.11), one may then build an approximate solution $u_{\mathcal{T},k} : \Omega \times (0,T) \longrightarrow \mathbb{R}$ for u, by

$$u_{\mathcal{T},k}(x,t) = u_K^{n+1}, \quad \text{for all } x \in K \text{ and } t \in (nk,(n+1)k), \tag{2.14}$$

and $w_{\mathcal{T},k} : \Omega \times (0,T) \longrightarrow \mathbb{R}$ for $\beta(u+z)$ by

$$w_{\mathcal{T},k}(x,t) = w_K^{n+1} = \beta(u_K^{n+1} + z_K), \quad \text{for all } x \in K \text{ and } t \in (nk,(n+1)k). \tag{2.15}$$

We shall make an intensive use of the following remark:

Remark 2.2 (Discrete integration by parts) *Let* $(a_K)_{K\in\mathcal{T}}$, $(a_\sigma)_{\sigma\in\mathcal{E}}$ *and* $(F_{K,\sigma})_{K\in\mathcal{T},\sigma\in\mathcal{E}_K}$ *be real values such that* $F_{K,\sigma} + F_{L,\sigma} = 0$ *for all* $\sigma \in \mathcal{E}_{int}$ *with* $\sigma = K|L$ *and* $a_\sigma = 0$ *for all* $\sigma \in \mathcal{E}_{ext}$, *then*

$$\sum_{K\in\mathcal{T}} a_K \sum_{\sigma\in\mathcal{E}_K} F_{K,\sigma} = \sum_{K\in\mathcal{T}}\sum_{\sigma\in\mathcal{E}_K} (a_K - a_\sigma)F_{K,\sigma}. \tag{2.16}$$

We also recall the following discrete Poincaré inequality.

Lemma 2.1 (Discrete Poincaré inequality) Let $(a_K)_{K\in\mathcal{T}}$, $(a_\sigma)_{\sigma\in\mathcal{E}}$ be real values such that $a_\sigma = 0$ for all $\sigma \in \mathcal{E}_{ext}$. Let $\tau_{K,\sigma}$ be the "transmissibility" through σ. Then there exists a real value $\alpha > 0$ depending only on Ω such that

$$\sum_{K\in\mathcal{T}}\sum_{\sigma\in\mathcal{E}_K} \tau_{K,\sigma}(a_\sigma - a_K)^2 \geq \alpha \sum_{K\in\mathcal{T}} m(K)(a_K)^2. \tag{2.17}$$

Proof. The proof is completely similar to that of the discrete Poincaré inequality

$$\sum_{K\in\mathcal{T}}\sum_{L,K|L\in\mathcal{E}_K} \tau_{K|L}(a_L - a_K)^2 \geq \alpha \sum_{K\in\mathcal{T}} m(K)(a_K)^2,$$

(see for instance Herbin[16], Eymard *et al* [6]or Eymard *et al* [8]).□

We will also need the following result on the local averages of an H^1 function. The proof is given in Eymard *et al* [7].

Lemma 2.2 Let Ω be an open bounded subset of $\mathbb{R}^d$, $\bar{u} \in H^1(\Omega)$ (recall that we denote the trace of a function by the same notation as the function itself). Let $\mathcal{T}$ be an admissible mesh (in the sense of Definition 2.2) such that, for some $\zeta > 0$, the inequality $d_{K,\sigma} \geq \zeta\, \delta(K)$ holds for each control volume $K \in \mathcal{T}$ and for all $\sigma \in \mathcal{E}_K$, and let $M \in \mathbb{N}$ be such that $\mathrm{card}(\mathcal{E}_K) \leq M$ for all $K \in \mathcal{T}$. We define, for all $K \in \mathcal{T}$,

$$\bar{u}_K = \frac{1}{m(K)} \int_K \bar{u}(x)\, dx, \tag{2.18}$$

and for all $\sigma \in \mathcal{E}$

$$\bar{u}_\sigma = \frac{1}{m(\sigma)} \int_\sigma \bar{u}(x)\, d\gamma(x). \tag{2.19}$$

There exists $C(\zeta, M, d) \in \mathbb{R}^{+*}$, depending only on ζ, M and d, such that

$$\sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}_K} \tau_{K,\sigma}\, (\bar{u}_\sigma - \bar{u}_K)^2 \leq C(\zeta, M, d)\, \|\bar{u}\|^2_{H^1(\Omega)}. \tag{2.20}$$

Remark 2.3 (Definition of $\bar{u}_\sigma$ on internal edges.) *Note that this lemma uses $\bar{u}_\sigma$ as the average value of the trace of $\bar{u}$ on the internal edges, which is different from the definition which is given elsewhere in this article. We remark that this lemma will always be sufficient since it is possible to first eliminate the values at the edges given by the continuity of fluxes, and then apply the Cauchy-Schwarz inequality in order to introduce the averages of the traces on the internal edges.*

3. Existence and uniqueness of the solution to the scheme

Lemma 3.3 Under hypotheses (H), let $\mathcal{T}$ be an admissible mesh of Ω in the sense of definition 2.2, $k \in (0, T)$ be the time step. Then there exists a unique solution $(u_K^n)_{K\in\mathcal{T}, n\in\mathbb{N}}$ and $(u_\sigma^{n+1})_{\sigma\in\mathcal{E}, n\in\mathbb{N}}$ to equations (2.9), (2.10), (2.11).

Proof. The proof of Lemma 3.3 is based upon an induction argument and a constructive method. Let $n \geq 0$ be given. We consider the sequences $(u_K^{(l)})_{K\in\mathcal{T}, l\in\mathbb{N}}$, $(u_\sigma^{(l+1)})_{\sigma\in\mathcal{E}, l\in\mathbb{N}}$ and $(w_K^{(l)})_{K\in\mathcal{T}, l\in\mathbb{N}}$ which are defined by

$$u_K{}^{(0)} = u_K^n,\ w_K{}^{(0)} = w_K^n, \tag{3.21}$$

and for all $l \in \mathbb{N}$ by

$$\left.\begin{aligned} & m(K) \frac{w_K^{(l)} - w_K^n + L_\beta \left(u_K^{(l+1)} - u_K^{(l)}\right)}{k} && - \\ & \sum_{\sigma\in\mathcal{E}_K} \tau_{K,\sigma} \Lambda(w_K^n)(u_\sigma^{(l+1)} - u_K^{(l+1)}) && = 0, \\ & w_K^{(l+1)} = \beta(u_K^{(l+1)} + z_K), && \end{aligned}\right\} \tag{3.22}$$

where L_β is the Lipschitz constant of β and by

$$\left.\begin{array}{l} u_\sigma^{(l+1)} = u_\sigma^{n+1}, \\ \quad \text{for all } \sigma \in \mathcal{E}_{ext}, \\ \tau_{K,\sigma}\,\Lambda(w_K^n)\,(u_\sigma^{(l+1)} - u_K^{(l+1)}) + \tau_{L,\sigma}\,\Lambda(w_L^n)\,(u_\sigma^{(l+1)} - u_L^{(l+1)}) = 0, \\ \quad \text{for all } \sigma \in \mathcal{E}_{int} \text{ with } \sigma = K|L. \end{array}\right\} \tag{3.23}$$

First we show that the matrix corresponding to the linear system (3.22)-(3.23) is invertible. Therefore this system defines for a given $l \in \mathbb{N}$ a unique family of values $(u_K^{(l+1)})_{K\in\mathcal{T}}$, $(u_\sigma^{(l+1)})_{\sigma\in\mathcal{E}}$.

Let $(a_K)_{K\in\mathcal{T}}$ and $(a_\sigma)_{\sigma\in\mathcal{E}}$ be the solution of the following system of equations: for all $K \in \mathcal{T}$

$$m(K)\frac{L_\beta}{k}a_K - \sum_{\sigma\in\mathcal{E}_K} \tau_{K,\sigma}\Lambda(w_K^n)(a_\sigma - a_K) = 0, \tag{3.24}$$

$$\left.\begin{array}{l} a_\sigma = 0, \\ \quad \text{for all } \sigma \in \mathcal{E}_{ext}, \\ \tau_{K,\sigma}\Lambda(w_K^n)(a_\sigma - a_K) + \tau_{L,\sigma}\Lambda(w_L^n)(a_\sigma - a_L) = 0, \\ \quad \text{for all } \sigma \in \mathcal{E}_{int} \text{ with } \sigma = K|L. \end{array}\right\} \tag{3.25}$$

We multiply (3.24) by a_K and sum the result over $K \in \mathcal{T}$. We obtain using a discrete integration by parts (see Remark 2.2),

$$\frac{L_\beta}{k}\sum_{K\in\mathcal{T}} m(K)a_K^2 + \sum_{K\in\mathcal{T}}\sum_{\sigma\in\mathcal{E}_K} \tau_{K,\sigma}\Lambda(w_K^n)(a_\sigma - a_K)^2 = 0,$$

which implies that the families $(a_K)_{K\in\mathcal{T}}$ and $(a_\sigma)_{\sigma\in\mathcal{E}}$ are identically equal to zero, so that the kernel of the matrix of the linear system (3.22)-(3.23) is reduced to $\{0\}$.

Next we prove that the sequences $(u_K^{(l)})_{K\in\mathcal{T},l\in\mathbb{N}}$, $(u_\sigma^{(l+1)})_{\sigma\in\mathcal{E},l\in\mathbb{N}}$ and $(w_K^{(l)})_{K\in\mathcal{T},l\in\mathbb{N}}$ converge as $l \longrightarrow +\infty$ and we denote their limits by $(u_K^{n+1})_{K\in\mathcal{T}}$, $(u_\sigma^{n+1})_{\sigma\in\mathcal{E}}$ and $(w_K^{n+1})_{K\in\mathcal{T}}$ respectively. These limits satisfy equations (2.10)-(2.11) which proves the existence of a solution to the scheme.

In order to show the convergence of the scheme, let us substract (3.22) at steps l and $l-1$. This yields

$$\begin{aligned} m(K)\frac{L_\beta\left(u_K^{(l+1)}-u_K^{(l)}\right)}{k} & - \\ \sum_{\sigma\in\mathcal{E}_K}\tau_{K,\sigma}\Lambda(w_K^n)\left((u_\sigma^{(l+1)}-u_\sigma^{(l)})-(u_K^{(l+1)}-u_K^{(l)})\right) & = \\ m(K)\frac{w_K^{(l-1)}-w_K^{(l)}+L_\beta\left(u_K^{(l)}-u_K^{(l-1)}\right)}{k}. & \end{aligned} \tag{3.26}$$

We multiply (3.26) by $u_K^{(l+1)}-u_K^{(l)}$ and sum the result over $K\in\mathcal{T}$. Since $u_\sigma^{(l+1)}-u_\sigma^{(l)}=0$ for all $\sigma\in\mathcal{E}_{ext}$ and

$$\begin{aligned} \tau_{K,\sigma}\Lambda(w_K^n)((u_\sigma^{(l+1)}-u_\sigma^{(l)})-(u_K^{(l+1)}-u_K^{(l)})) & + \\ \tau_{L,\sigma}\Lambda(w_L^n)((u_\sigma^{(l+1)}-u_\sigma^{(l)})-(u_L^{(l+1)}-u_L^{(l)})) & = 0, \end{aligned}$$

for all $\sigma\in\mathcal{E}_{int}$ with $\sigma=K|L$, a discrete integration by parts (see Remark 2.2) may be used to obtain

$$\begin{aligned} \sum_{K\in\mathcal{T}} m(K)\frac{L_\beta\left(u_K^{(l+1)}-u_K^{(l)}\right)^2}{k} & + \\ \sum_{K\in\mathcal{T}}\sum_{\sigma\in\mathcal{E}_K}\tau_{K,\sigma}\Lambda(w_K^n)\left((u_\sigma^{(l+1)}-u_\sigma^{(l)})-(u_K^{(l+1)}-u_K^{(l)})\right)^2 & = \\ \sum_{K\in\mathcal{T}} m(K)\frac{w_K^{(l-1)}-w_K^{(l)}+L_\beta\left(u_K^{(l)}-u_K^{(l-1)}\right)}{k}\left(u_K^{(l+1)}-u_K^{(l)}\right). & \end{aligned} \tag{3.27}$$

Next we prove that

$$\left(w_K^{(l-1)}-w_K^{(l)}+L_\beta\left(u_K^{(l)}-u_K^{(l-1)}\right)\right)^2\le\left(L_\beta\left(u_K^{(l)}-u_K^{(l-1)}\right)\right)^2. \tag{3.28}$$

Indeed, if we develop the left-hand-side of (3.28) and simplify, we obtain the equivalent inequality

$$\left(w_K^{(l-1)}-w_K^{(l)}\right)^2-2\,L_\beta\left(w_K^{(l-1)}-w_K^{(l)}\right)\left(u_K^{(l-1)}-u_K^{(l)}\right)\le 0. \tag{3.29}$$

Since β is nondecreasing, inequality (3.29) is equivalent to the inequality

$$\left(w_K^{(l-1)}-w_K^{(l)}\right)^2\le 2\,L_\beta\left|w_K^{(l-1)}-w_K^{(l)}\right|\left|u_K^{(l-1)}-u_K^{(l)}\right|, \tag{3.30}$$

which is satisfied since β is Lipschitz with Lipschitz constant L_β. Applying the Poincaré inequality (Lemma 2.1) to the second term of the left-hand-side of (3.27) and the Cauchy-Schwarz inequality to the right-hand-side, we finally obtain with (3.28), after simplification

$$\begin{aligned}\left(\frac{L_\beta}{k}+\Lambda_m\alpha\right)\left(\sum_{K\in\mathcal{T}} m(K)\left(u_K^{(l+1)}-u_K^{(l)}\right)^2\right)^{\frac{1}{2}} &\leq \\ \frac{L_\beta}{k}\left(\sum_{K\in\mathcal{T}} m(K)\left(u_K^{(l)}-u_K^{(l-1)}\right)^2\right)^{\frac{1}{2}}.\end{aligned} \tag{3.31}$$

We deduce from (3.31) that there exists a positive value U^n such that

$$\left(\sum_{K\in\mathcal{T}} m(K)\left(u_K^{(l)}-u_K^{(l-1)}\right)^2\right)^{\frac{1}{2}} \leq U^n C^{l-1},$$

where $C=\dfrac{L_\beta}{L_\beta+k\Lambda_m\alpha}$ satisfies $0<C<1$. Therefore for all $K\in\mathcal{T}$, $(u_K^{(l)})_{l\in\mathbb{N}}$ is a Cauchy sequence (the Cauchy residual tends to zero uniformly) so that it converges. In view of (3.23), we also obtain the convergence of $(u_\sigma^{(l+1)})_{l\in\mathbb{N}}$ for all $\sigma\in\mathcal{E}$. The respective limits u_K^{n+1} and u_σ^{n+1} satisfy equations (2.10)-(2.11); this shows the existence of the solution to these equations.

We finally show the uniqueness of these solutions. Let u_K and v_K be two solutions of (2.10). We subtract equation (2.10) for v_K from equation (2.10) for u_K. It leads to

$$\begin{aligned}-\sum_{\sigma\in\mathcal{E}_K}\tau_{K,\sigma}\Lambda(w_K^n)\Big((u_\sigma-v_\sigma)-(u_K-v_K)\Big) &= \\ m(K)\frac{\beta(v_K+z_K)-\beta(u_K+z_K)}{k}.\end{aligned} \tag{3.32}$$

where

$$\left.\begin{array}{l} u_\sigma-v_\sigma=0, \\ \text{for all } \sigma\in\mathcal{E}_{ext}, \\ \tau_{K,\sigma}\Lambda(w_K^n)(u_\sigma-u_K)+\tau_{L,\sigma}\Lambda(w_L^n)(u_\sigma-u_L)=0, \\ \text{for all } \sigma\in\mathcal{E}_{int} \text{ with } \sigma=K|L, \\ \tau_{K,\sigma}\Lambda(w_K^n)(v_\sigma-v_K)+\tau_{L,\sigma}\Lambda(w_L^n)(v_\sigma-v_L)=0, \\ \text{for all } \sigma\in\mathcal{E}_{int} \text{ with } \sigma=K|L. \end{array}\right\} \tag{3.33}$$

We multiply (3.32) by (u_K-v_K) and sum the result over $K\in\mathcal{T}$. Using a similar argument to the one yealding (3.31) and applying the discrete Poincaré inequality (Lemma 2.1), we obtain

$$(\Lambda_m\alpha)\sum_{K\in\mathcal{T}} m(K)(u_K-v_K)^2\leq 0.$$

This proves the uniqueness of the solution of (2.10) and concludes the proof of Lemma 3.3.□

4. A priori estimates

4.1. *Space translates estimate*

The space translates estimate is based on the following lemma.

Lemma 4.4 ($L^2(0,T;H^1(\Omega))$ estimate) Under hypotheses (H), let $T>0$ and $\mathcal{T}$ be an admissible mesh of Ω in the sense of Definition 2.2, $k \in (0,T)$ be the time step. Let $\zeta > 0$ be such that

$$\zeta \le \inf_{\substack{K\in\mathcal{T},\\ \sigma\in\mathcal{E}_K}} \frac{d_{K,\sigma}}{\delta(K)}, \tag{4.34}$$

and $M > 0$ such that

$$M \ge \max_{K\in\mathcal{T}} \operatorname{card}(\mathcal{E}_{\mathrm{K}}). \tag{4.35}$$

Let $(u_K^n)_{K\in\mathcal{T},n\in\mathbb{N}}$ and $(u_\sigma^{n+1})_{\sigma\in\mathcal{E},n\in\mathbb{N}}$ be the unique solution of equations (2.9), (2.10) and (2.11). Let $(\bar u_K^{n+1})_{K\in\mathcal{T},n\in\mathbb{N}}$ and $(\bar u_\sigma^{n+1})_{\sigma\in\mathcal{E},n\in\mathbb{N}}$ be given for all $K\in\mathcal{T}$ and for all $n\in\mathbb{N}$ by

$$\left.\begin{aligned}
&\bar u_K^{n+1} = \frac{1}{k\,m(K)}\int_{nk}^{(n+1)k}\int_K \bar u(x,t)\,dx\,dt,\\
&\bar u_\sigma^{n+1} = \frac{1}{k\,m(\sigma)}\int_{nk}^{(n+1)k}\int_\sigma \bar u(s,t)\,ds\,dt,\\
&\quad\text{for all } \sigma\in\mathcal{E}_{ext},\\
&\tau_{K,\sigma}\,\Lambda(w_K^n)\,(\bar u_\sigma^{n+1}-\bar u_K^{n+1}) + \tau_{L,\sigma}\,\Lambda(w_L^n)\,(\bar u_\sigma^{n+1}-\bar u_L^{n+1}) = 0,\\
&\quad\text{for all } \sigma\in\mathcal{E}_{int} \text{ with } \sigma = K|L,
\end{aligned}\right\} \tag{4.36}$$

where we recall that the function $\bar u$ denotes a function of $H^1(\Omega\times(0,T))$ whose trace on $\partial\Omega$ is the function $\bar u$ given by the boundary condition (1.6); let $\bar u_{\mathcal{T},k}$ be the piecewise constant function defined by

$$\bar u_{\mathcal{T},k}(x,t) = \bar u_K^{n+1} \text{ for } x\in K \text{ and } t\in[t,t_{n+1}). \tag{4.37}$$

(Note that, for $\sigma\in\mathcal{E}_{int}$, the values $\bar u_\sigma^{n+1}$ are defined in order to satisfy the continuity of the fluxes, and are not the average values of the trace of $\bar u$ on σ). Let us define for all $n\in\mathbb{N}$

$$\left.\begin{aligned}
&v_\sigma^{n+1} = u_\sigma^{n+1} - \bar u_\sigma^{n+1} = 0, && \text{for all } \sigma\in\mathcal{E}_{ext},\\
&v_\sigma^{n+1} = u_\sigma^{n+1} - \bar u_\sigma^{n+1}, && \text{for all } \sigma\in\mathcal{E}_{int},\\
&v_K^{n+1} = u_K^{n+1} - \bar u_K^{n+1}, && \text{for all } K\in\mathcal{T},\\
&v_{\mathcal{T},k}(x,t) = v_K^{n+1}, && \text{for all } K\in\mathcal{T} \text{ and } t\in[nk,(n+1)k).
\end{aligned}\right\} \tag{4.38}$$

Then there exists C_1 depending only on Ω, T, ζ, M, u_0, $\bar u$, Λ and β such that

$$\sum_{n=0}^{[T/k]} k \sum_{K\in\mathcal{T}}\sum_{\sigma\in\mathcal{E}_K} \tau_{K,\sigma}(v_\sigma^{n+1}-v_K^{n+1})^2 \le C_1, \tag{4.39}$$

and

$$\sum_{n=0}^{[T/k]} k \sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}_K} \tau_{K,\sigma}(u_\sigma^{n+1} - u_K^{n+1})^2 \le C_1, \tag{4.40}$$

where we set, for all $a \in \mathbb{R}^+$, $[a] = \max\{n \in \mathbb{N},\, n \le a\}$.

Proof. First we replace u_K^{n+1} (resp. u_σ^{n+1}) by $v_K^{n+1} + \bar{u}_K^{n+1}$ (resp. $v_\sigma^{n+1} + \bar{u}_\sigma^{n+1}$) in (2.10). This yields

$$\begin{array}{l} m(K)\dfrac{w_K^{n+1} - w_K^n}{k} - \displaystyle\sum_{\sigma\in\mathcal{E}_K} \tau_{K,\sigma}\Lambda(w_K^n)(v_\sigma^{n+1} - v_K^{n+1}) \quad = \\ \displaystyle\sum_{\sigma\in\mathcal{E}_K} \tau_{K,\sigma}\Lambda(w_K^n)(\bar{u}_\sigma^{n+1} - \bar{u}_K^{n+1}), \end{array} \tag{4.41}$$

and

$$\left.\begin{array}{l} v_\sigma^{n+1} = 0, \\ \quad \text{for all } \sigma \in \mathcal{E}_{ext}, \\ \tau_{K,\sigma}\Lambda(w_K^n)(v_\sigma^{n+1} - v_K^{n+1}) + \tau_{L,\sigma}\Lambda(w_L^n)(v_\sigma^{n+1} - v_L^{n+1}) = 0, \\ \quad \text{for all } \sigma \in \mathcal{E}_{int} \text{ with } \sigma = K|L. \end{array}\right\} \tag{4.42}$$

Multiplying (4.41) by $k\, v_K^{n+1}$, summing over $K \in \mathcal{T}$ and over $n = 0, ...[T/k]$ and using a discrete integration by parts (see Remark 2.2), thanks to (4.42), we obtain

$$B_1 + B_2 = B_3 \tag{4.43}$$

where

$$\left.\begin{array}{rl} B_1 &= \displaystyle\sum_{n=0}^{[T/k]} \sum_{K\in\mathcal{T}} m(K)(w_K^{n+1} - w_K^n)v_K^{n+1}, \\ B_2 &= \displaystyle\sum_{n=0}^{[T/k]} k \sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}_K} \tau_{K,\sigma}\Lambda(w_K^n)(v_\sigma^{n+1} - v_K^{n+1})^2, \\ B_3 &= \displaystyle\sum_{n=0}^{[T/k]} k \sum_{K\in\mathcal{T}} v_K^{n+1} \sum_{\sigma\in\mathcal{E}_K} \tau_{K,\sigma}\Lambda(w_K^n)(\bar{u}_\sigma^{n+1} - \bar{u}_K^{n+1}). \end{array}\right\} \tag{4.44}$$

Let us first study B_1. We remark that

$$\begin{array}{rl} (w_K^{n+1} - w_K^n)\,(u_K^{n+1} - \bar{u}_K^{n+1}) = & w_K^{n+1}u_K^{n+1} - w_K^n u_K^n + w_K^n u_K^n - w_K^n u_K^{n+1} \\ + & w_K^n \bar{u}_K^{n+1} - w_K^{n+1}\bar{u}_K^{n+1}. \end{array} \tag{4.45}$$

Since

$$\begin{array}{rl} \displaystyle\sum_{n=0}^{[T/k]} (w_K^n \bar{u}_K^{n+1} - w_K^{n+1}\bar{u}_K^{n+1}) = & \displaystyle\sum_{n=1}^{[T/k]} w_K^n \bar{u}_K^{n+1} - \sum_{n=1}^{[T/k]} w_K^n \bar{u}_K^n \\ + & w_K^0 \bar{u}_K^1 - w_K^{[T/k]+1}\bar{u}_K^{[T/k]+1}, \end{array} \tag{4.46}$$

we finally have that $B_1 = B_{11} - B_{12}$ where

$$\begin{aligned} B_{11} = &\sum_{K\in\mathcal{T}} m(K)(w_K^{[T/k]+1} u_K^{[T/k]+1} &+\sum_{n=0}^{[T/k]} w_K^n(u_K^n - u_K^{n+1}) - w_K^0 u_K^0), \\ B_{12} = &\sum_{K\in\mathcal{T}} m(K) w_K^{[T/k]+1} \bar{u}_K^{[T/k]+1} &+\textstyle\sum_{n=1}^{[T/k]} k \displaystyle\sum_{K\in\mathcal{T}} m(K) w_K^n \frac{\bar{u}_K^n - \bar{u}_K^{n+1}}{k} \\ &-\sum_{K\in\mathcal{T}} m(K) w_K^0 \bar{u}_K^1. \end{aligned}$$

Let us now turn to B_{11}. From the definition of w_K^n, we have that

$$w_K^n(u_K^{n+1} - u_K^n) = \int_{u_K^n}^{u_K^{n+1}} \beta(u_K^n + z_K)\, d\xi. \tag{4.47}$$

We introduce the integral function $q_K(\xi) = \int_0^\xi \beta(s + z_K) ds$ and add and subtract $q_K(u_K^{n+1}) - q_K(u_K^n)$ to the right-hand-side of (4.47). We obtain

$$w_K^n(u_K^{n+1} - u_K^n) = q_K(u_K^{n+1}) - q_K(u_K^n) - \int_{u_K^n}^{u_K^{n+1}} (\beta(\xi + z_K) - \beta(u_K^n + z_K)) d\xi. \tag{4.48}$$

Since β is nondecreasing, the term $\int_{u_K^n}^{u_K^{n+1}} (\beta(\xi+z_K) - \beta(u_K^n + z_K)) d\xi$ is nonnegative. Therefore $w_K^n(u_K^{n+1} - u_K^n) \le q_K(u_K^{n+1}) - q_K(u_K^n)$ and we obtain

$$B_{11} \ge \sum_{K\in\mathcal{T}} m(K) \left(w_K^{[T/k]+1} u_K^{[T/k]+1} - q_K(u_K^{[T/k]+1}) + q_K(u_K^0) - w_K^0 u_K^0 \right). \tag{4.49}$$

Using Hypothesis (H4) and the definition of q_K, we have that

$$0 \le \xi\beta(\xi + z_K) - q_K(\xi) = \int_0^\xi \Big(\beta(\xi + z_K) - \beta(s + z_K)\Big)\, ds \le L_\beta\, \xi^2/2.$$

Using these equalities with $\xi = u_K^{[T/k]+1}$ and $\xi = u_K^0$, one obtains from (4.49) that:

$$B_{11} \ge 0 - \frac{L_\beta}{2} \sum_{K\in\mathcal{T}} m(K)(u_K^0)^2 \ge -\frac{L_\beta}{2} \|u_0\|_{L^2(\Omega)}^2. \tag{4.50}$$

Let us now handle B_{12}. We set $\beta_M = \max(-\beta_I, \beta_S)$. Using Hypothesis (H4), we apply the Cauchy-Schwarz inequality to the first term of B_{12} and obtain

$$\left(\sum_{K\in\mathcal{T}} m(K) w_K^{[T/k]+1} \bar{u}_K^{[T/k]+1} \right)^2 \le \beta_M^2 \sum_{K\in\mathcal{T}} m(K) \sum_{K\in\mathcal{T}} m(K) (\bar{u}_K^{[T/k]+1})^2.$$

From the definition of $\bar{u}_K^{[T/k]+1}$, we have that

$$\sum_{K\in\mathcal{T}} m(K)\left(\bar{u}_K^{[T/k]+1}\right)^2 \le \sum_{K\in\mathcal{T}} \int_K \left(\frac{1}{k}\int_{k[T/k]}^{k[T/k]+k} \bar{u}(x,t)\,dt\right)^2 dx.$$

Since $\bar{u} \in H^1(\Omega\times(0,T))$, it may be shown that the function $t \mapsto \int_K \bar{u}(x,t)dx$ belongs to $H^1(0,T)$. Therefore, since $H^1(0,T)$ is continuously imbedded in $C([0,T])$ (see e.g. Brezis[4]), it is easily shown that there exists $C(T) \in \mathbb{R}^+$, depending only on T, such that

$$\sum_{K\in\mathcal{T}} m(K)\left(\bar{u}_K^{[T/k]+1}\right)^2 \le C(T)\int_\Omega\int_0^T \left(\bar{u}^2(x,t) + \bar{u}_t^2(x,t)\right)\,dx\,dt.$$

Therefore

$$\left(\sum_{K\in\mathcal{T}} m(K) w_K^{[T/k]+1}\bar{u}_K^{[T/k]+1}\right)^2 \le \beta_M^2\, m(\Omega)\, C(T)\, \|\bar{u}\|^2_{H^1(\Omega\times(0,T))}. \tag{4.51}$$

In the same way, we have that

$$\left(\sum_{K\in\mathcal{T}} m(K) w_K^0 \bar{u}_K^1\right)^2 \le \beta_M^2\, m(\Omega)\, C(T)\, \|\bar{u}\|^2_{H^1(\Omega\times(0,T))}. \tag{4.52}$$

Applying the Cauchy-Schwarz inequality to the second term of B_{12} gives

$$\left(\sum_{n=1}^{[T/k]} k \sum_{K\in\mathcal{T}} m(K) w_K^n \frac{\bar{u}_K^n - \bar{u}_K^{n+1}}{k}\right)^2 \le \beta_M^2\, T\, m(\Omega) \sum_{K\in\mathcal{T}} m(K) \sum_{n=1}^{[T/k]} \frac{(\bar{u}_K^n - \bar{u}_K^{n+1})^2}{k}.$$

Since the function $\tilde{u}_K : t \mapsto \int_K \bar{u}(x,t)dx$ belongs to $H^1(0,T)$, it is Hölder continuous with exponent $1/2$; more precisely (see Brezis[4]), one has, for any $t > 0$:

$$|\tilde{u}_K(t) - \tilde{u}_K(t+k)| \le \sqrt{k}\|\tilde{u}_K\|_{H^1(t,t+k)}.$$

Hence, thanks to $k \in (0,T)$ and to hypothesis $(H5)$, we have

$$\left(\sum_{n=1}^{[T/k]} k \sum_{K\in\mathcal{T}} m(K) w_K^n \frac{\bar{u}_K^n - \bar{u}_K^{n+1}}{k}\right)^2 \le \beta_M^2\, T\, m(\Omega)\, \|\bar{u}\|^2_{H^1(\Omega\times(0,2T))}. \tag{4.53}$$

Therefore, from (4.51), (4.52) and (4.53) there exists $C(\beta,\Omega,T)$, depending only on β, Ω and T, such that

$$(B_{12})^2 \le C(\beta,\Omega,T)\, \|\bar{u}\|^2_{H^1(\Omega\times(0,2T))}. \tag{4.54}$$

Finally, we get the existence of $C_2 > 0$, depending only on Ω, T, u_0, $\bar{u}$ and β such that

$$B_1 \ge -C_2. \tag{4.55}$$

We now turn to the study of B_3. Since $v_\sigma^{n+1} = 0$ for all $\sigma \in \mathcal{E}_{ext}$ and $\tau_{K,\sigma}\Lambda(w_K^n)(\bar{u}_\sigma^{n+1} - \bar{u}_K^{n+1}) + \tau_{L,\sigma}\Lambda(w_L^n)(\bar{u}_\sigma^{n+1} - \bar{u}_L^{n+1}) = 0$ for all $\sigma \in \mathcal{E}_{int}$ with $\sigma = K|L$, we have using Remark 2.2 that

$$B_3 = \sum_{n=0}^{[T/k]} k \sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}_K} \tau_{K,\sigma}\Lambda(w_K^n)(\bar{u}_\sigma^{n+1} - \bar{u}_K^{n+1})(v_K^{n+1} - v_\sigma^{n+1}). \tag{4.56}$$

We then apply the Cauchy-Schwarz inequality to obtain

$$(B_3)^2 \leq B_2 \sum_{n=0}^{[T/k]} k \sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}_K} \tau_{K,\sigma}\Lambda(w_K^n)(\bar{u}_\sigma^{n+1} - \bar{u}_K^{n+1})^2. \tag{4.57}$$

Since β is bounded (Hypothesis (H4)) and since Λ is a continuous function, there exists a constant Λ_M which is independent of size$(\mathcal{T})$ and k such that

$$\Lambda(w_K^n) \leq \Lambda_M, \quad \text{for all } K \in \mathcal{T} \text{ and } n = 0, .., [T/k]. \tag{4.58}$$

Therefore in view of Lemma 2.2, we have that

$$(B_3)^2 \leq B_2\, \Lambda_M\, C(\zeta, M, d)\, \|\bar{u}\|^2_{L^2(0,T+1;H^1(\Omega))}. \tag{4.59}$$

We return to (4.43). Substituting inequality (4.55) yields

$$B_2 \leq B_3 + C_2. \tag{4.60}$$

In view of (4.59) and since $x^2 \leq ax + b$ implies $x \leq a + \sqrt{b}$, we have that

$$B_2 \leq C. \tag{4.61}$$

Hypothesis (H3) states that $\Lambda \geq \Lambda_m > 0$. Hence (4.61) implies exactly (4.39). Moreover, from the definition of $v_{\mathcal{T},k}$ (see (4.38)), we have that

$$\begin{aligned}\sum_{n=0}^{[T/k]} k \sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}_K} \tau_{K,\sigma}(u_\sigma^{n+1} - u_K^{n+1})^2 \leq\ & 2\sum_{n=0}^{[T/k]} k \sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}_K} \tau_{K,\sigma}(v_\sigma^{n+1} - v_K^{n+1})^2 \\ +\ & 2\sum_{n=0}^{[T/k]} k \sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}_K} \tau_{K,\sigma}(\bar{u}_\sigma^{n+1} - \bar{u}_K^{n+1})^2.\end{aligned} \tag{4.62}$$

In view of Lemma 2.2, (4.40) immediately follows from (4.62) which concludes the proof of Lemma 4.4.□

Several consequences can be drawn from Lemma 4.4.

Corollary 4.1 Under hypotheses (H), let $T > 0$ and $\mathcal{T}$ be an admissible mesh of Ω in the sense of Definition 2.2, $k \in (0,T)$ be the time step. Let $\zeta > 0$ and $M > 0$ be such that (4.34) and (4.35) hold. Let $u_{\mathcal{T},k}$ be given by (2.9), (2.10), (2.11) and (2.14) and let $v_{\mathcal{T},k}$ be given by (4.36), (4.37) and (4.38).

Then there exists C depending only on Ω, T, ζ, M, u_0, $\bar{u}$, Λ and β such that

$$\|v_{\mathcal{T},k}\|_{L^2(0,T;L^2(\Omega))} \leq C, \tag{4.63}$$

and that

$$\|u_{\mathcal{T},k}\|_{L^2(0,T;L^2(\Omega))} \leq C. \tag{4.64}$$

Proof. We first deduce (4.63) from Lemma 4.4 and the discrete Poincaré inequality (Lemma 2.1). Therefore we deduce estimate (4.64) from the definition (4.38) of $v_{\mathcal{T},k}$ and from Hypothesis (H5). □

Corollary 4.2 Under hypotheses (H), let $T > 0$; let $(\mathcal{T}_\ell)_{\ell\in\mathbb{N}}$ be a sequence of admissible meshes of Ω in the sense of Definition 2.2 such that $size(\mathcal{T}_\ell) \longrightarrow 0$ as $\ell \longrightarrow +\infty$, $(k_\ell)_{\ell\in\mathbb{N}\subset\mathbb{R}^{+*}}$ such that $k_\ell \longrightarrow 0$ as $\ell \longrightarrow +\infty$; assume that there exist $\zeta > 0$ and $M > 0$ such that

$$\zeta \leq \inf_{\ell\in\mathbb{N}} \inf_{\substack{K\in\mathcal{T}_l,\\ \sigma\in\mathcal{E}_K}} \frac{d_{K,\sigma}}{\delta(K)},$$

and

$$M \geq \sup_{\ell\in\mathbb{N}} \max_{K\in\mathcal{T}_l} \operatorname{card}(\mathcal{E}_{\mathrm{K}}).$$

Let $u_{\mathcal{T}_\ell,k_\ell}$ and $v_{\mathcal{T}_\ell,k_\ell}$ be the piecewise constant functions defined by (2.14), (2.9), (2.10) and (2.11) and (4.38) with $\mathcal{T} = \mathcal{T}_\ell$ and $k = k_\ell$. Then there exist subsequences of $(\mathcal{T}_\ell)_{\ell\in\mathbb{N}}$ and $(k_\ell)_{\ell\in\mathbb{N}\subset\mathbb{R}^{+*}}$, still denoted $(\mathcal{T}_\ell)_{\ell\in\mathbb{N}}$ and $(k_\ell)_{\ell\in\mathbb{N}}$, such that $u_{\mathcal{T}_\ell,k_\ell}$ (resp. $v_{\mathcal{T}_\ell,k_\ell}$) converges weakly in $L^2(0,T;L^2(\Omega))$ to a limit u (resp. v) as ℓ tends to $+\infty$.

We also deduce from Lemma 4.4 the following results on differences of space translates.

Lemma 4.5 Under hypotheses (H), let $T > 0$ and $\mathcal{T}$ be an admissible mesh of Ω in the sense of definition 2.2, $k \in (0,T)$ be the time step. Let $\zeta > 0$ be given by (4.34) and $M > 0$ be given by (4.35). Let $(u^n_K)_{K\in\mathcal{T},n\in\mathbb{N}}$ and $(u^{n+1}_\sigma)_{\sigma\in\mathcal{E},n\in\mathbb{N}}$ be the unique solution of equations (2.9), (2.10), (2.11). Let $(\bar{u}^{n+1}_K)$, $(\bar{u}^{n+1}_\sigma)$, (v^{n+1}_K), (v^{n+1}_σ), (w^n_K), for all $K \in \mathcal{T}$, $\sigma \in \mathcal{E}$, $n \in \mathbb{N}$ be given by (4.36), (4.38). Let $u_{\mathcal{T},k}$, $v_{\mathcal{T},k}$ and $w_{\mathcal{T},k}$ be respectively defined by (2.14), (4.38), (2.15), these definitions being extended to $\mathbb{R}^d \times (0,T)$ by zero outside of Ω. Then there exists C_2 depending only on Ω, T, ζ, M, u_0, $\bar{u}$, Λ and β, such that

$$\|v_{\mathcal{T},k}(\cdot+\eta,\cdot) - v_{\mathcal{T},k}(\cdot,\cdot)\|^2_{L^2(\Omega\times(0,T))} \leq C_2\,|\eta|\,\big(|\eta| + 2\,\mathrm{size}(\mathcal{T})\big), \tag{4.65}$$

$$\|u_{\mathcal{T},k}(\cdot+\eta,\cdot) - u_{\mathcal{T},k}(\cdot,\cdot)\|^2_{L^2(\Omega\times(0,T))} \leq C_2\,|\eta|\,\big(|\eta| + 2\,\mathrm{size}(\mathcal{T})\big), \tag{4.66}$$

and

$$\|w_{\mathcal{T},k}(\cdot+\eta,\cdot)-w_{\mathcal{T},k}(\cdot,\cdot)\|^2_{L^2(\Omega\times(0,T))}\le C_2\,|\eta|\,\big(|\eta|+2\,\mathrm{size}(\mathcal{T})\big), \qquad (4.67)$$

for all $\eta \in \mathbb{R}^d$.

Proof. The proofs of (4.65) and (4.66) are identical to those given in Eymard *et al* [6] whereas (4.67) immediatly follows from (4.66) and the Lipschitz continuity of β.

We deduce from this lemma the following regularity on the limits of the subsequences of approximate functions.

Corollary 4.3 (Regularity of the limits) Under the assumptions of Corollary 4.2 and using the same notations, the weak limits u and v of the subsequences $(u_{\mathcal{T}_\ell,k_\ell})_{\ell\in\mathbb{N}}$ and $(v_{\mathcal{T}_\ell,k_\ell})_{\ell\in\mathbb{N}}$ satisfy

$$v \in L^2(0,T;H^1_0(\Omega)), \qquad (4.68)$$

and

$$u = v+\bar{u} \in L^2(0,T;H^1(\Omega)). \qquad (4.69)$$

Proof. The assertion (4.68) immediately follows from Lemma 4.5 and Theorem 1 of Eymard *et al* [6]. It is clear from the definition of $\bar{u}_{\mathcal{T},k}$ (see (4.37)), that $\bar{u}_{\mathcal{T}_\ell,k_\ell} \longrightarrow \bar{u}$ in $L^2(\Omega\times(0,T))$ as $\ell \longrightarrow +\infty$. Since $u_{\mathcal{T}_\ell,k_\ell} = v_{\mathcal{T}_\ell,k_\ell} + \bar{u}_{\mathcal{T}_\ell,k_\ell}$, (4.69) follows. □

4.2. *Time translates estimate*

We first give a technical lemma which we use in the proof of the time translates estimate.

Lemma 4.6 Let $T>0$, $\tau\in(0,T)$, $k\in(0,T)$ and $(a^n)_{n\in\mathbb{N}}$ be a family of non negative real values. Then

$$\int_0^{T-\tau}\sum_{n=[t/k]+1}^{[(t+\tau)/k]} a^{n+1}dt \le \tau\sum_{n=0}^{[T/k]} a^{n+1}, \qquad (4.70)$$

and for any $\zeta\in[0,\tau]$

$$\int_0^{T-\tau}\sum_{n=[t/k]+1}^{[(t+\tau)/k]} a^{[(t+\zeta)/k]+1}dt \le \tau\sum_{n=0}^{[T/k]} a^{n+1}. \qquad (4.71)$$

Proof. Let χ be a function from $\mathbb{N}\times\mathbb{R}^+\times\mathbb{R}^+$ into $\{0,1\}$ defined by $\chi(n,t_1,t_2)=1$ if $t_1<nk\le t_2$ and else $\chi(n,t_1,t_2)=0$. Then we have that

$$\int_0^{T-\tau} \sum_{n=[t/k]+1}^{[(t+\tau)/k]} a^{n+1}\, dt \le \sum_{n=1}^{[T/k]} a^{n+1} \int_0^{T-\tau} \chi(n,t,t+\tau)\, dt \le \sum_{n=1}^{[T/k]} a^{n+1} \int_{nk-\tau}^{nk} dt \le \tau \sum_{n=1}^{[T/k]} a^{n+1},$$

which yields (4.70).
In order to prove (4.71), we remark that:

$$\int_0^{T-\tau} \sum_{n=[t/k]+1}^{[(t+\tau)/k]} a^{[(t+\zeta)/k]+1}\, dt \le \sum_{m=0}^{[T/k]} \int_{mk}^{(m+1)k} a^{m+1} \sum_{n=0}^{[T/k]} \chi(n,t-\zeta,t-\zeta+\tau)\, dt,$$

and for all $m \in \mathbb{N}$

$$\begin{aligned}\int_{mk}^{(m+1)k} \sum_{n=0}^{[T/k]} \chi(n,t-\zeta,t-\zeta+\tau)dt &= \int_0^k \sum_{n=0}^{[T/k]} \chi(n-m,t-\zeta,t-\zeta+\tau)dt\\ &= \sum_{n=0}^{[T/k]} \int_{-nk}^{k-nk} \chi(-m,t-\zeta,t-\zeta+\tau)dt\\ &\le \int_{\mathbb{R}} \chi(-m,t-\zeta,t-\zeta+\tau)dt\\ &\le \int_{\zeta+\tau-mk}^{\zeta-mk} dt\\ &\le \tau,\end{aligned}$$

which concludes the proof. □

Lemma 4.7 Under hypotheses (H), let $T > 0$ and $\mathcal{T}$ be an admissible mesh of Ω in the sense of definition 2.2, $k \in (0,T)$ be the time step. Let $\zeta > 0$ be given by (4.34) and $M > 0$ be given by (4.35). Let $w_{\mathcal{T},k}$ be defined by (2.15), (2.9), (2.10), (2.11), and extended by 0 outside of Ω. Then there exists C_3 depending only on Ω, T, ζ, M, u_0, $\bar{u}$, Λ and β such that

$$\|w_{\mathcal{T},k}(\cdot,\cdot+\tau) - w_{\mathcal{T},k}(\cdot,\cdot)\|^2_{L^2(\Omega\times(0,T-\tau))} \le C_3\,\tau, \quad \forall \tau \in (0,T). \tag{4.72}$$

Proof. Let $\tau \in (0,T)$; by definition of $w_{\mathcal{T},k}$, one has

$$\|w_{\mathcal{T},k}(\cdot,\cdot+\tau) - w_{\mathcal{T},k}(\cdot,\cdot)\|^2_{L^2(\Omega\times(0,T-\tau))} = \int_0^{T-\tau} \sum_{K\in\mathcal{T}} m(K)\left(w_K^{[(t+\tau)/k]+1} - w_K^{[t/k]+1}\right)^2 dt.$$

Since β is Lipschitz continuous with constant L_β (Hypothesis (H4)), one has

$$\|w_{\mathcal{T},k}(\cdot,\cdot+\tau) - w_{\mathcal{T},k}(\cdot,\cdot)\|^2_{L^2(\Omega\times(0,T-\tau))} \le \int_0^{T-\tau} A(t)\, dt,$$

with

$$A(t) = L_\beta \sum_{K\in\mathcal{T}} m(K) \left(u_K^{[(t+\tau)/k]+1} - u_K^{[t/k]+1}\right)\left(w_K^{[(t+\tau)/k]+1} - w_K^{[t/k]+1}\right).$$

Noting that

$$w_K^{[(t+\tau)/k]+1} - w_K^{[t/k]+1} = \sum_{n=[t/k]+1}^{[(t+\tau)/k]} \left(w_K^{n+1} - w_K^n\right),$$

using (2.10) and the definition of $v_{\mathcal{T},k}$ (see (4.38)), one has

$$\begin{aligned} A(t) \quad = \\ L_\beta \sum_{K\in\mathcal{T}} \left(\bar u_K^{[(t+\tau)/k]+1} - \bar u_K^{[t/k]+1}\right) \sum_{n=[t/k]+1}^{[(t+\tau)/k]} k \sum_{\sigma\in\mathcal{E}_K} \tau_{K,\sigma}\,\Lambda(w_K^n)\,(u_\sigma^{n+1} - u_K^{n+1}) \quad + \\ L_\beta \sum_{K\in\mathcal{T}} \left(v_K^{[(t+\tau)/k]+1} - v_K^{[t/k]+1}\right) \sum_{n=[t/k]+1}^{[(t+\tau)/k]} k \sum_{\sigma\in\mathcal{E}_K} \tau_{K,\sigma}\,\Lambda(w_K^n)\,(u_\sigma^{n+1} - u_K^{n+1}). \end{aligned} \tag{4.73}$$

We perform a discrete integration by parts (see Remark 2.2) to both sums in the right-hand side of (4.73). It leads to

$$\begin{aligned} A(t) = \;& L_\beta \sum_{K\in\mathcal{T}} \sum_{n=[t/k]+1}^{[(t+\tau)/k]} k \sum_{\sigma\in\mathcal{E}_K} \tau_{K,\sigma}\,\Lambda(w_K^n) \begin{pmatrix} (\bar u_K^{[(t+\tau)/k]+1} - \bar u_\sigma^{[(t+\tau)/k]+1}). \\ (u_\sigma^{n+1} - u_K^{n+1}) \end{pmatrix} \\ -\;& L_\beta \sum_{K\in\mathcal{T}} \sum_{n=[t/k]+1}^{[(t+\tau)/k]} k \sum_{\sigma\in\mathcal{E}_K} \tau_{K,\sigma}\,\Lambda(w_K^n) \begin{pmatrix} (\bar u_K^{[t/k]+1} - \bar u_\sigma^{[t/k]+1}). \\ (u_\sigma^{n+1} - u_K^{n+1}) \end{pmatrix} \\ +\;& L_\beta \sum_{K\in\mathcal{T}} \sum_{n=[t/k]+1}^{[(t+\tau)/k]} k \sum_{\sigma\in\mathcal{E}_K} \tau_{K,\sigma}\,\Lambda(w_K^n) \begin{pmatrix} (v_K^{[(t+\tau)/k]+1} - v_\sigma^{[(t+\tau)/k]+1}). \\ (u_\sigma^{n+1} - u_K^{n+1}) \end{pmatrix} \\ -\;& L_\beta \sum_{K\in\mathcal{T}} \sum_{n=[t/k]+1}^{[(t+\tau)/k]} k \sum_{\sigma\in\mathcal{E}_K} \tau_{K,\sigma}\,\Lambda(w_K^n)\,(v_K^{[t/k]+1} - v_\sigma^{[t/k]+1})\,(u_\sigma^{n+1} - u_K^{n+1}). \end{aligned} \tag{4.74}$$

Applying four times the inequality $\pm 2ab \le a^2 + b^2$, we obtain

$$A(t) \le \frac{L_\beta}{2}\Big(A_{\bar u}(t,\tau) + A_{\bar u}(t,0) + A_v(t,\tau) + A_v(t,0)\Big) + 2\,L_\beta\,A_u(t), \tag{4.75}$$

where

$$A_{\bar u}(t,\theta) = \sum_{n=[t/k]+1}^{[(t+\tau)/k]} k \sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}_K} \tau_{K,\sigma}\,\Lambda(w_K^n)\left(\bar u_\sigma^{[(t+\theta)/k]+1} - \bar u_K^{[(t+\theta/k]+1}\right)^2,$$

and

$$A_v(t,\theta) = \sum_{n=[t/k]+1}^{[(t+\tau)/k]} k \sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}_K} \tau_{K,\sigma}\,\Lambda(w_K^n)\left(v_\sigma^{[(t+\theta)/k]+1} - v_K^{[(t+\theta/k]+1}\right)^2,$$

for $\theta = 0$ or $\theta = \tau$ and where

$$A_u(t) = \sum_{n=[t/k]+1}^{[(t+\tau)/k]} k \sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}_K} \tau_{K,\sigma}\, \Lambda(w_K^n)\, (u_\sigma^{n+1} - u_K^{n+1})^2.$$

Next we integrate $A_{\bar{u}}(t,\theta)$ and $A_v(t,\theta)$ from 0 to $T-\tau$. In view of (4.71) and of the upper bound Λ_M of $\Lambda(w_K^n)$ (cf. (4.58)), we obtain

$$\int_0^{T-\tau} A_{\bar{u}}(t,\theta)\, dt \le \tau\, \Lambda_M \sum_{n=0}^{[T/k]} k \sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}_K} \tau_{K,\sigma}\, \left(\bar{u}_\sigma^{n+1} - \bar{u}_K^{n+1}\right)^2,$$

and

$$\int_0^{T-\tau} A_v(t,\theta)\, dt \le \tau\, \Lambda_M \sum_{n=0}^{[T/k]} k \sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}_K} \tau_{K,\sigma}\, \left(v_\sigma^{n+1} - v_K^{n+1}\right)^2.$$

From Lemma 2.2, we finally deduce that

$$\int_0^{T-\tau} A_{\bar{u}}(t,\theta)\, dt \le \tau\, \Lambda_M\, C(\zeta, M, d)\, \|\bar{u}\|^2_{L^2(0,2T;H^1(\Omega))}, \tag{4.76}$$

and from Lemma 4.4, we deduce that

$$\int_0^{T-\tau} A_v(t,\theta)\, dt \le \tau\, \Lambda_M\, C_1. \tag{4.77}$$

Next we integrate $A_u(t)$ from 0 to $T-\tau$. In view of (4.70) and of (4.58), we obtain

$$\int_0^{T-\tau} A_u(t)\, dt \le \tau\, \Lambda_M \sum_{n=0}^{[T/k]} k \sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}_K} \tau_{K,\sigma}\, \left(u_\sigma^{n+1} - u_K^{n+1}\right)^2.$$

From Lemma 4.4, we deduce that

$$\int_0^{T-\tau} A_u(t)\, dt \le \tau\, \Lambda_M\, C_1. \tag{4.78}$$

Finally we integrate inequality (4.75) and substitute estimates (4.76), (4.77) and (4.78). This yields exactly (4.72).□

We then obtain the following result.

Corollary 4.4 Under the assumptions of Corollary 4.2, let $w_{\mathcal{T}_\ell,k_\ell}$ be the piecewise constant function defined by (2.15), (2.9), (2.10) and (2.11) with $\mathcal{T} = \mathcal{T}_\ell$ and $k = k_\ell$. Then there exist a subsequence of $(\mathcal{T}_\ell)_{\ell\in\mathbb{N}}$ and $(k_\ell)_{\ell\in\mathbb{N}\subset\mathbb{R}^{+*}}$, still denoted $(\mathcal{T}_\ell)_{\ell\in\mathbb{N}}$ and $(k_\ell)_{\ell\in\mathbb{N}}$, and a function $\bar{\beta} \in L^2(0,T;L^2(\Omega))$ such that

$w_{\mathcal{T}_\ell,k_\ell}$ converges to $\bar{\beta}$ strongly in $L^2(0,T;L^2(\Omega))$ and a.e. in $\Omega\times(0,T)$.

Proof. In view of the space translates estimate (4.67) and of the time translates estimate (4.72), this result is a direct consequence of Kolmogorov's theorem (see Brezis[4], Theorem IV.25, page 72 and Eymard *et al* [9], Lemma 3.5).

5. Convergence

Theorem 5.1 Under the assumptions of the corollaries 4.2 and 4.4 and with the same notations, the sequence $(u_{\mathcal{T}_\ell,k_\ell})_{\ell\in\mathbb{N}}$ converges to a weak solution u of (1.5),(1.6) and (1.7) weakly in $L^2(\Omega\times(0,T))$ as $\ell\longrightarrow+\infty$ and the sequence $(w_{\mathcal{T}_\ell,k_\ell})_{\ell\in\mathbb{N}}$ converges to $w=\beta(u+z)$ strongly in $L^2(\Omega\times(0,T))$ as $\ell\longrightarrow+\infty$.

Proof. Let $(\mathcal{T}_\ell,k_\ell)_{\ell\in\mathbb{N}}$ be a sequence of admissible meshes and time steps satisfying the assumptions of Corollary 4.2. From the lemmas 4.4, 4.1, 4.5 and 4.7, and the corollaries 4.2 and 4.4, we deduce that under the given assumptions, there exists a subsequence of meshes and time steps that we denote again $(\mathcal{T}_\ell,k_\ell)_{\ell\in\mathbb{N}}$, such that

$$\left.\begin{array}{llll}\text{(i)} & u_{\mathcal{T}_\ell,k_\ell} & \text{converges to} \quad u & \text{weakly in } L^2(\Omega\times(0,T)),\\ \text{(ii)} & w_{\mathcal{T}_\ell,k_\ell} & \text{converges to} \quad \bar\beta & \text{strongly in } L^2(\Omega\times(0,T))\end{array}\right\}\text{ as } \ell\longrightarrow+\infty. \tag{5.79}$$

Using a result of Eymard *et al* [10], it is easily shown that

$$\bar\beta=\beta(u+z)\qquad\text{a.e. in }\Omega\times(0,T). \tag{5.80}$$

It remains to show that the limit u is a weak solution of Problem (P). For the sake of simplicity, we set $\mathcal{T}=\mathcal{T}_\ell$ and $k=k_\ell$. Let T be a fixed positive constant and $\psi\in\Psi$ where Ψ is defined by

$$\Psi=\left\{\psi\in C^{2,1}(\overline{\Omega}\times[0,T]),\ \psi=0\text{ on }\partial\Omega\times[0,T],\ \psi=0\text{ on }\Omega\times\{T\}\right\}, \tag{5.81}$$

where $C^{2,1}(\overline{\Omega}\times[0,T])$ is the set of functions with continuous second derivatives with respect to x and continuous first derivative with respect to t. We multiply Equation (2.10) by $k\,\psi(x_K,nk)$, and sum the result over $n=0,...[T/k]$ and $K\in\mathcal{T}$. We deduce that

$$T_\ell+A_{0\ell}=0, \tag{5.82}$$

where

$$T_\ell=\sum_{n=0}^{[T/k]}\sum_{K\in\mathcal{T}}m(K)\left(w_K^{n+1}-w_K^n\right)\psi(x_K,nk), \tag{5.83}$$

and

$$A_{0\ell}=\sum_{n=0}^{[T/k]}k\sum_{K\in\mathcal{T}}\sum_{\sigma\in\mathcal{E}_K}\tau_{K,\sigma}\,\Lambda(w_K^n)\left(u_\sigma^{n+1}-u_K^{n+1}\right)\psi(x_K,nk). \tag{5.84}$$

The study of the discrete time derivative T_ℓ is similar to that of Eymard *et al* [10] and yields that

$$T_{1\ell}\longrightarrow-\int_\Omega\beta(u_0(x)+z(x))\,\psi(x,0)\,dx-\int_0^T\int_\Omega\beta(u(x,t)+z(x))\,\psi_t(x,t)\,dx\,dt, \tag{5.85}$$

as $\ell \longrightarrow \infty$. Let us now study the space term $A_{0\ell}$ of the definition. For $(x,t) \in \overline{\Omega} \times \mathbb{R}^+$, we set $\bar{\beta} = \beta(u(x,t) + z(x))$ and consider the terms

$$A_{1\ell} = \int_0^T \int_\Omega u_{\mathcal{T},k}(x,t)\, \mathrm{div}(\Lambda(\bar{\beta})\nabla\psi)(x,t)\, dx\, dt,$$

and

$$A_{2\ell} = \sum_{n=0}^{[T/k]} \sum_{K\in\mathcal{T}} u_K^{n+1} \int_{nk}^{(n+1)k} \int_K \mathrm{div}(\Lambda(\bar{\beta})\nabla\psi)(x,t)\, dx\, dt.$$

Since $u_{\mathcal{T},k}$ converges weakly to u in $L^2(\Omega \times (0,T))$, we deduce that

$$\lim_{\ell\longrightarrow\infty} A_{1\ell} = \int_0^T \int_\Omega u(x,t)\, \mathrm{div}(\Lambda(\bar{\beta})\nabla\psi)(x,t)\, dx\, dt,$$

and that

$$\lim_{\ell\longrightarrow\infty} |A_{1\ell} - A_{2\ell}| = 0.$$

Moreover we have that

$$A_{2\ell} = \sum_{n=0}^{[T/k]} \sum_{K\in\mathcal{T}} u_K^{n+1} \sum_{\sigma\in\mathcal{E}_K} \int_{nk}^{(n+1)k} \int_\sigma \Lambda(\bar{\beta}(x,t))\nabla\psi(x,t)\cdot \mathbf{n}_{K,\sigma}(x) d\gamma(x)\, dt,$$

and therefore, performing a discrete integration by parts (see Remark 2.2), we obtain

$$A_{2\ell} = \sum_{n=0}^{[T/k]} \sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}_K} (u_K^{n+1} - u_\sigma^{n+1}) \int_{nk}^{(n+1)k} \int_\sigma \Lambda(\bar{\beta}(x,t))\nabla\psi(x,t)\cdot \mathbf{n}_{K,\sigma}(x) d\gamma(x)\, dt.$$

We set

$$\Lambda_{K,\sigma}^n = \frac{1}{km(\sigma)} \int_{nk}^{(n+1)k} \int_\sigma \Lambda(\bar{\beta}(x,t)) d\gamma(x)\, dt,$$

and

$$G_{K,\sigma}^n = \frac{\psi(x_K, nk) - \psi(x_\sigma, nk)}{d_{K,\sigma}},$$

where x_σ is the orthogonal projection of x_K on σ. Next we compare $A_{2\ell}$ to the term

$$A_{3\ell} = \sum_{n=0}^{[T/k]} k \sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}_K} m(\sigma)(u_K^{n+1} - u_\sigma^{n+1})\Lambda_{K,\sigma}^n G_{K,\sigma}^n.$$

Applying the Cauchy-Schwarz inequality gives

$$(A_{2\ell} - A_{3\ell})^2 \leq \sum_{n=0}^{[T/k]} k \sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}_K} \tau_{K,\sigma}(u_K^{n+1} - u_\sigma^{n+1})^2 \sum_{n=0}^{[T/k]} k \sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}_K} m(\sigma) d_{K,\sigma} R_{K,\sigma}^n,$$

with

$$R_{K,\sigma}^n = \left(\tfrac{1}{km(\sigma)} \int_{nk}^{(n+1)k} \int_\sigma \Lambda(\bar{\beta}(x,t)) \left(\nabla\psi(x,t)\cdot \mathbf{n}_{K,\sigma}(x) - G_{K,\sigma}^n\right) d\gamma(x) dt\right)^2.$$

Since $\psi \in \Psi$, there exists a positive real value C_ψ such that, for all $K \in \mathcal{T}$, for all $\sigma \in \mathcal{E}_K$ and for all $n \in \mathbb{N}$, we have that

$$|\nabla\psi(x,t)\cdot \mathbf{n}_{K,\sigma}(x) - G_{K,\sigma}^n| \leq C_\psi(\text{size}(\mathcal{T}) + \text{k}).$$

Therefore, thanks to (4.40), we deduce that

$$\lim_{\ell\longrightarrow\infty} (A_{2\ell} - A_{3\ell}) = 0.$$

Finally we prove that

$$\lim_{\ell\longrightarrow\infty} (A_{3\ell} - A_{0\ell}) = 0.$$

First we use a discrete integration by parts (see Remark 2.2) and the definition of $\tau_{K,\sigma}$ to rewrite $A_{0\ell}$ in the form

$$A_{0\ell} = \sum_{n=0}^{[T/k]} k \sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}_K} (u_K^{n+1} - u_\sigma^{n+1}) \Lambda(w_K^n) m(\sigma) G_{K,\sigma}^n.$$

Since $\psi \in \Psi$, there exists a bound G of $|\nabla\psi|$ so that $\left|G_{K,\sigma}^n\right| \leq G$ for all $K \in \mathcal{T}$, $\sigma \in \mathcal{E}_K$, $n \in \mathbb{N}$. Using the Cauchy-Schwarz inequality , we obtain

$$(A_{0\ell} - A_{3\ell})^2 \leq \quad G^2 B_{1\ell} \sum_{n=0}^{[T/k]} k \sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}_K} \tau_{K,\sigma}(u_K^{n+1} - u_\sigma^{n+1})^2, \tag{5.86}$$

where we define $B_{1\ell}$ by

$$B_{1\ell} = \sum_{n=0}^{[T/k]} k \sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}_K} m(\sigma) d_{K,\sigma} (\Lambda_{K,\sigma}^n - \Lambda(w_K^n))^2.$$

Using (4.40), it now suffices to prove that $\lim_{\ell\longrightarrow\infty} B_{1\ell} = 0$. We set

$$\Lambda_K^n = \frac{1}{km(K)} \int_{nk}^{(n+1)k} \int_K \Lambda(\bar{\beta}(x,t))\, dx\, dt.$$

Then

$$B_{1\ell} \leq 2(B_{2\ell} + B_{3\ell}),$$

where

$$B_{2\ell} = \sum_{n=0}^{[T/k]} k \sum_{K \in \mathcal{T}} \sum_{\sigma \in \mathcal{E}_K} m(\sigma) d_{K,\sigma} (\Lambda^n_{K,\sigma} - \Lambda^n_K)^2,$$

and

$$B_{3\ell} = \sum_{n=0}^{[T/k]} k \sum_{K \in \mathcal{T}} \sum_{\sigma \in \mathcal{E}_K} m(\sigma) d_{K,\sigma} (\Lambda^n_K - \Lambda(w^n_K))^2.$$

We have that $\sum_{\sigma \in \mathcal{E}_K} m(\sigma) d_{K,\sigma} = d\ m(K)$. Therefore the convergence of $\Lambda(\beta(u+z))$ to $\Lambda(\bar{\beta})$ for the L^2 norm implies that $\lim_{\ell \longrightarrow \infty} |B_{3\ell}| = 0$.

Using the properties of H^1 functions recalled in Lemma 2.2 on the function $\Lambda(\bar{\beta})$, we finally obtain $B_{2\ell} \le \mathrm{size}(\mathcal{T})^2 \mathrm{C}_\Lambda \|\Lambda(\bar{\beta})\|_{\mathrm{L}^2(0,\mathrm{T};\mathrm{H}^1(\Omega))}$ and hence $\lim_{\ell \longrightarrow \infty} |B_{2\ell}| = 0$. The density of the set Ψ in the set of test functions $\{\psi \in L^2(0,T; H^1_0(\Omega)),\ \psi_t \in L^\infty(\Omega \times (0,T)),\ \psi(.,T) = 0\}$ completes the proof of Theorem 5.1. □

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