Boundary Condition for Hyperbolic and Degenerate Parabolic Equations

Thierry Gallouët
Université de Marseille, CMI, 13453

Let Ω be a bounded "polygonal" open subset of $\mathbf{R}^d$, ($d = 1, 2$ or 3) with boundary $\partial\Omega$ and let $T \in \mathbf{R}_+^*$. The aim of this talk is to present some results (obtained by several authors) on the convergence of numerical schemes for some hyperbolic or degenerate parabolic equations posed on $\Omega \times (0, T)$. A main difficulty is to adequately take into account the boundary conditions.

1 Hyperbolic Equations

In the case of a nonlinear hyperbolic equation with initial and boundary conditions, a first question is to understand the sense of the boundary condition. In some cases, using for instance finite volume schemes with a "monotone flux", the discretized values of the boundary condition are used, for the computation of the numerical solution, everywhere on the boundary and it is not so easy to understand what is the "remaining part" of this boundary condition as the mesh size goes to zero (and the approximate solution converges towards the exact solution). Indeed, the boundary condition has to be taken in the sense given by the pioneering work of [1] (the boundary conditions are "active" only on a part of the boundary). The convergence of some numerical schemes is given, for different cases, in some papers as [8], [4], [2] or [9]. This convergence result can be given (see [9]) when the data are only in L^∞ by using a very fine "weak formulation" of the problem which is introduced in the work of Otto (see [7]). I now give this formulation (which does not use the "trace" of the solution on the boundary).

Let $\overline{u} \in L^\infty(\partial\Omega \times (0,T))$, $u_0 \in L^\infty(\Omega)$ and $f \in C^2(\mathbf{R}^d \times \mathbf{R}_+ \times \mathbf{R}, \mathbf{R}^d)$ such that $\mathrm{div}_x f(x,t,s) = 0$ for all $(x,t,s) \in \mathbf{R}^d \times \mathbf{R}_+ \times \mathbf{R}$ (these conditions on f can be somewhat relaxed). Let A, $B \in \mathbf{R}$ such that $A \le \overline{u} \le B$ a.e. on $\partial\Omega \times (0,T)$ and $A \le u_0 \le B$ a.e. on Ω. Let $M \in \mathbf{R}$ such that $|f(x,t,s_1) - f(x,t,s_2)| \le M|s_1 - s_2|$ for a.e. $(x,t) \in \Omega \times (0,T)$ and all s_1, $s_2 \in [A,B]$ ($|\cdot|$ is the Euclidean norm in $\mathbf{R}^d$ and $\mathbf{R}$). One considers the following problem :

$$\begin{aligned} &u_t(x,t) + \mathrm{div} f(x,t,u(x,t)) = 0, \text{ for } (x,t) \in \Omega \times (0,T), \\ &u(x,0) = u_0(x), \text{ for } x \in \Omega, \\ &\text{“}u(x,t) = \overline{u}(x,t)\text{”}, \text{ for } (x,t) \in \partial\Omega \times (0,T). \end{aligned} \tag{1}$$

An entropy weak solution to (1) is a function $u \in L^\infty(\Omega \times (0,T)$ satisfying the following inequalities for all $\kappa \in [A,B]$ and all $\psi \in C_c^1(\overline{\Omega} \times [0,T), \mathbf{R}_+)$:

$$\begin{aligned} &\int_0^T \int_\Omega ((u-\kappa)^\pm \psi_t + \mathrm{sign}^\pm(u-\kappa)(f(x,t,u) - f(x,t,\kappa)) \cdot \nabla\psi)\, dxdt \\ &+ \int_\Omega (u_0 - \kappa)^\pm \psi(x,0)\, dx + M \int_0^T \int_{\partial\Omega} (\overline{u} - \kappa)^\pm \psi(x,t)\, d\sigma(x) dt \ge 0. \end{aligned} \tag{2}$$

One can prove existence and uniqueness of the solution of (2). This solution satisfies the boundary condition (at least, if u is regular enough), $u = \overline{u}$, only on a part of the boundary (for instance, if $f(x,t,u) = q(x,t)f(u)$ with $f' \ge 0$ the boundary condition is satisfied when $q \cdot n < 0$, where n is an "outward normal vector" to $\partial\Omega$). Note that, if in (2) one replaces the two entropies $(u-\kappa)^\pm$ by the sole $|u - \kappa|$, one still has existence of a solution but not uniqueness (see [9], uniqueness remains true if $f(x,t,s)$ is a monotonous function w.r.t. s). Note also that it is possible to replace, in (2), $(u_0 - \kappa)^\pm$ by $N(u_0 - \kappa)^\pm$ with a given $N \ge 1$. It does not change the unique solution of (2) and leads to a more "symmetric" treatment of the boundary condition and the initial condition (indeed, in order to obtain more symmetry between t and x, it is also possible to add an artificial condition in T and to consider test functions which do not vanish in T).

One can also prove the convergence of the approximate solutions given by any finite volume scheme (on possibly unstructured meshes) with a monotone flux (and under a classical CFL condition in the case of explicit schemes). Indeed, an L^∞ estimate on the approximate solutions is easy to obtain. Then, passing to the limit when the mesh size goes to zero, one obtains (as limits of approximate solutions) "entropy process solutions" (or "measure-valued solutions") of (1). A uniqueness result for these entropy process solutions yields to the desired result (namely the convergence in L^p, for any $p < \infty$, of the approximate solution towards the solution of (2)).

2 Degenerate Parabolic Equations

Keeping the hypotheses of the first section, a diffusion term is now added to (1). Let $\varphi \in C(\mathbf{R}, \mathbf{R})$ be a nondecreasing Lipschitz continuous function. The new problem is :

$$\begin{aligned} &u_t(x,t) + \operatorname{div} f(x,t,u(x,t)) - \Delta(\varphi(u(x,t))) = 0, \text{ for } (x,t) \in \Omega \times (0,T), \\ &u(x,0) = u_0(x), \text{ for } x \in \Omega, \\ &\text{“}u(x,t) = \overline{u}(x,t)\text{”}, \text{ for } (x,t) \in \partial\Omega \times (0,T). \end{aligned} \tag{3}$$

Note that that Problem (3) contains Problem (1) since φ can be a constant function. But, working with the methods of the first section, the diffusion term introduces additional difficulties.

Let us consider first the "simple" case where $f(x,t,u) = q(x,t)f(u)$ with $f' \geq 0$ (in this case, it is sufficient to work with "Kruskov entropies", namely $|u-\kappa|$ for all κ) and $q \cdot n = 0$ on $\partial\Omega$, where n is a normal vector to $\partial\Omega$ (in this case the boundary condition is quite "simple"). Assume furthermore that $\overline{u}$ is the "trace" on $\partial\Omega \times (0,T)$ of an element, denoted also by $\overline{u}$, of $H^1(\Omega \times (0,T))$. Then, one says that $u \in L^\infty(\Omega \times (0,T))$ is an entropy weak solution of (3) if $(\varphi(u) - \varphi(\overline{u})) \in L^2(0,T; H_0^1(\Omega))$ and if u satisfies the following inequalities for all $\kappa \in [A,B]$ and all $\psi \in C_c^1(\Omega \times [0,T), \mathbf{R}_+)$:

$$\begin{aligned} &\int_0^T \int_\Omega (|u-\kappa|\psi_t + |f(x,t,u) - f(x,t,\kappa)| q \cdot \nabla\psi)\, dx dt \\ &- \int_0^T (\int_\Omega \nabla|\varphi(u) - \varphi(\kappa)| \cdot \nabla\psi\, dx dt + \int_\Omega |u_0 - \kappa|\psi(x,0)\, dx \geq 0. \end{aligned} \tag{4}$$

It is possible to prove the convergence of approximate solutions given by finite volumes schemes towards the unique solution of (4) (see, for instance, [5]). Indeed, one first proves the convergence (up to a subsequence) of sequences of approximate solutions towards some "entropy process solutions" of (3). Then, one proves a uniqueness result of the entropy process solution which yields the result. A main difficulty, with the diffusion term, in this uniqueness result (where we use the technique of dedoubling variables of Kruskov) is overcome with a very fine "trick" of J. Carillo (see [3]) for which we first work with some regularizations of the Kruskov entropies.

In the case of a general function f, it is much more difficult to handle the boundary condition (and its "interaction" with the diffusion term). Assuming that $\overline{u}$ is continuous and is the "trace" on $\partial\Omega \times (0,T)$ of an element, also denoted by $\overline{u}$, of $H^1(\Omega \times (0,T))$, one says (see [6]) that $u \in L^\infty(\Omega \times (0,T))$ is an entropy weak solution of (3) if $(\varphi(u) - \varphi(\overline{u})) \in L^2(0,T; H_0^1(\Omega))$ and if u satisfies the following inequalities for all $\kappa \in [A,B]$ and all $\psi \in C_c^1(\overline{\Omega} \times [0,T), \mathbf{R}_+)$ such that $\operatorname{sign}^\pm(\overline{u} - \kappa)\psi = 0$ on $\partial\Omega \times (0,T)$:

$$\begin{aligned}&\int_0^T\int_\Omega ((u-\kappa)^\pm\psi_t+\operatorname{sign}^\pm(u-\kappa)(f(x,t,u)-f(x,t,\kappa))\cdot\nabla\psi)\,dxdt\\&-\int_0^T\int_\Omega\nabla(\varphi(u)-\varphi(\kappa))^\pm\cdot\nabla\psi\,dxdt-\int_0^T\int_{\partial\Omega}(\varphi(\overline{u})-\varphi(\kappa))^\pm(\partial\psi/\partial n)\,d\sigma(x)dt\\&+\int_\Omega(u_0-\kappa)^\pm\psi(x,0)\,dx\geq 0.\end{aligned}\tag{5}$$

Indeed, in this latter case, the "complexity" of the boundary condition does not allow to consider only test functions vanishing at the boundary (as it was done for (4)). Furthermore, due to the diffusion term, it is not easy to find a "good" definition of entropy weak solution using general test functions nonvanishing at the boundary and a Lipschitz constant for f (as it was done in the purely hyperbolic case, see (2)). In (5), the test functions vanish on a part of the boundary which depends on $\overline{u}$ (and κ). Thus, the space of test functions depends on $\overline{u}$. For this reason, one needs some regularity on $\overline{u}$. Note also that the "trace" of u does not appear in this definition of solution of (3), as in the previous cases.

In order to prove existence and uniqueness of the solution to (5) and to prove the convergence of approximate solutions, given by finite volumes schemes, towards the unique solution of (5), one follows the same lines as in the previous cases (with additional difficulties). One first proves the convergence of approximate solutions towards "entropy process solutions" (a generalization of (5)) and then proves the uniqueness of the entropy process solution, see [6].

References

[1] C. Bardos, A. Y. Leroux, J. C. Nedelec. *First order quasilinear equations with boundary conditions*, *Comm. P. D. E.*, **4**, 9 (1979), 1017-1034.

[2] S. Benharbi, A. Chalabi, J. P. Vila. *Numerical viscosity and convergence of finite volume methods for conservation laws with boundary conditions*, *SIAM J. Num. Anal.*, **32**, 3 (1995), 775-796.

[3] J. Carillo. *Entropy solutions for nonlinear degenerate problems*, *Arch. Rat. Mech. Anal.*, **147**, n^o4, 269-361, 1999.

[4] B. Cockburn, F. Coquel, P. G. LeFloch. *Convergence of the finite volume method for multidimensional conservation laws*, *SIAM J. Num. Anal.*, **32**, 3 (1995), 687-705.

[5] R. Eymard, T. Gallouët, R. Herbin, A. Michel. *Convergence of a finite volume scheme for nonlinear degenerate parabolic equations*, submitted. Preprint LATP 00-40.

[6] A. Michel, J. Vovelle. *A finite volume method for parabolic degenerate problems with general Dirichlet boundary conditions.* In preparation.

[7] J. Malek, J. Necas, M. Rokyta, M. Ruzicka. *Weak and measure-valued solutions to evolutionary PDE's*, *Chapman & Hall*, 1996.

[8] A. Szepessy. *Measure-valued solutions of scalar conservation laws with boundary conditions*, *Arch. Rat. Mech. Anal.*, **107**, n^o2, 181-193, 1989.

[9] J. Vovelle. *Convergence of Finite Volume Monotone Schemes for Scalar Conservation Laws on Bounded Domains*, accepted for publication in *Numer. Math.*. Preprint LATP 00-09.