Convergence of approximate solutions for Stationary compressible Stokes and Navier-Stokes equations

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Fisrt step for proving the convergence of approximate solutions for the evolution compressible Navier-Stokes equations (which gives, in particular, the existence of solutions, $d = 3$, $p = \rho^\gamma$, $\gamma > \frac{3}{2}$).

Stationary compressible Stokes equations

Ω is a bounded open set of $\mathbb{R}^d$, $d = 2$ or 3, with a Lipschitz continuous boundary, $\gamma \geq 1$, $f \in L^2(\Omega)^d$ and $M > 0$

$$-\Delta u + \nabla p = f \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega,$$

$$\operatorname{div}(\rho u) = 0 \text{ in } \Omega, \ \rho \geq 0 \text{ in } \Omega, \ \int_\Omega \rho(x) dx = M,$$

$$p = \rho^\gamma \text{ in } \Omega$$

Functional spaces : $u \in H_0^1(\Omega)$, $p \in L^2(\Omega)$, $\rho \in L^{2\gamma}(\Omega)$

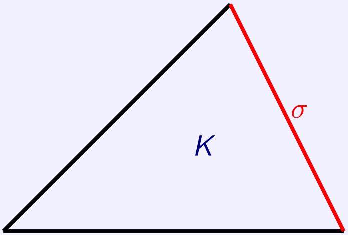
(different spaces for p and ρ in the case of Navier-Stokes if $d = 3$ and $\gamma < 3$)

Aim

Prove the existence of a weak solution to the compressible Stokes equations by the convergence of a sequence (up to a subsequence, since, up to now, no uniqueness result is available for this problem) of approximate solutions given by a numerical scheme as the mesh size goes to 0

Choice of the discrete unknowns

- Classical FE mesh of Ω with simplices ($d = 2$ or 3), the mesh size is called h
- Discretization of u by Non Conformal FE: Crouzeix-Raviart FE. $u_h \in H_h$, non conformal approx. of $(H_0^1(\Omega))^d$
- Discretization of p and ρ by piecewise constant functions. $p_h, \rho_h \in X_h$, approximation of $L^2(\Omega)$



Unknowns for u_h:
u_σ, $\sigma \in \{\text{interfaces}\}$($u_\sigma \in \mathbb{R}^d$)

Unknowns for p_h and ρ_h:
p_K, ρ_K, $K \in \{\text{simplices}\}$

Discretization of momentum equation

$$u_h \in H_h$$

$$\int_\Omega \nabla_h u_h : \nabla_h v \, dx - \int_\Omega p_h \mathrm{div}_h v \, dx = \int_\Omega f v \, dx, \text{ for all } v \in H_h$$

For $v \in H_h$ and for $K \in \{\text{simplices}\}$, one has in K:

- $\nabla_h v = \nabla v$
- $|K| \mathrm{div}_h v = \sum_{\sigma \in \{\text{interfaces of } K\}} v_\sigma \cdot n_{K,\sigma} |\sigma|$

$n_{K,\sigma}$ is the normal vector to σ, outward K

Discretization of mass equation

For all $K \in \{\text{simplices}\}$, $\mathrm{div}_K(\rho_h u_h) + M_K + S_K = 0$

- $|K|\mathrm{div}_K(\rho_h u_h) = \sum_{\sigma \in \{\text{interfaces of } K\}} |\sigma| \rho_\sigma u_\sigma \cdot n_{K,\sigma}$ with an upstream choice for ρ_σ, that is
  $\rho_\sigma = \rho_K \text{ if } u_\sigma \cdot n_{K,\sigma} \geq 0$
  $\rho_\sigma = \rho_L \text{ if } u_\sigma \cdot n_{K,\sigma} < 0, \ \sigma = K|L$
- $M_K = h^\alpha(\rho_K - \frac{M}{|\Omega|})$. It gives $\int_\Omega \rho_h dx = M$.
- $|K|S_K = \sum_{\sigma=K|L} h_\sigma^\xi \frac{|\sigma|}{h_\sigma} \left(|\rho_K| + |\rho_L|\right)^\zeta (\rho_K - \rho_L)$,
  $\zeta = \max(0, 2-\gamma)$

Two parameters: $0 < \alpha$, $0 < \xi < 2$.

Discretization of mass equation (2)

Upwinding and S replaces $\mathrm{div}(\rho u) = 0$ by
$\mathrm{div}(\rho u) - h\mathrm{div}(D|u|\nabla\rho) - h^{\xi}\mathrm{div}(D\rho^{\zeta}\nabla\rho) = 0$.

Upwinding is enough to ensure (with M) existence of a positive solution ρ_h, to the discrete mass equation, for a given u_h. It allows also to pass to the limit in the mass equation if ρ_h converges weakly in $L^2(\Omega)$ and u_h converges in $L^2(\Omega)^d$ as $h \to 0$.

The stabilization term S, which leads to a very small diffusion (taking ξ close to 2) but independent of u, is used for passing to the limit in the EOS ($p = \rho^{\gamma}$).

Discretization of the EOS: $p_K = \rho_K^{\gamma}$ for all K

Existence of an approximate solution, convergence result

Existence of a solution u_h, p_h and ρ_h of the scheme can be proven using the Brouwer Fixed Point Theorem.

For $\gamma > 1$, convergence of the approximate solution can be proven in the following sense, up to a subsequence:

- $u_h \to u$ in $L^2(\Omega)^d$, $u \in H_0^1(\Omega)^d$
- $p_h \to p$ in $L^q(\Omega)$ for any $1 \leq q < 2$ and weakly in $L^2(\Omega)$
- $\rho_h \to \rho$ in $L^q(\Omega)$ for any $1 \leq q < 2\gamma$ and weakly in $L^{2\gamma}(\Omega)$

where (u, p, ρ) is a weak solution of the compressible Stokes equations

For $\gamma = 1$, the same result holds, at least with only weak convergences of p_h and ρ_h

Proof of convergence, main steps

1. Estimate on the $H^1(\Omega)$-broken norm of u_h
2. $L^2(\Omega)$ estimate on p_h and $L^{2\gamma}(\Omega)$ estimate on ρ_h

   These two steps give (up to a subsequence), as $h \to 0$,
   - $u_h \to u$ in $L^2(\Omega)$ and $u \in H_0^1(\Omega)$
   - $p_h \to p$ weakly in $L^2(\Omega)$
   - $\rho_h \to \rho$ weakly in $L^2(\Omega)$
3. (u, p, ρ) is a weak solution of $-\Delta u + \nabla p = f$, $\mathrm{div}(\rho u) = 0$ $\rho \geq 0$, $\int_\Omega \rho dx = M$
4. Main difficulty, if $\gamma > 1$: $p = \rho^\gamma$ and "strong" convergence of p_h and ρ_h

Preliminary lemma

$\rho \in L^{2\gamma}(\Omega)$, $\gamma > 1$, $\rho \geq 0$ a.e. in Ω, $u \in (H_0^1(\Omega))^d$, $\mathrm{div}(\rho u) = 0$, then:

$$\int_\Omega \rho \mathrm{div}(u) dx = 0$$

$$\int_\Omega \rho^\gamma \mathrm{div}(u) dx = 0$$

The first result (and its discrete counterpart) is used for Step 4 (proof of $p = \rho^\gamma$)

The discrete counterpart (also true for $\gamma = 1$) of the second result is used for Step 1 (estimate for u_h)

Preliminary lemma for the approximate solution

Discretization of mass equation $\mathrm{div}(\rho u) = 0$ and $\int_\Omega \rho \, dx = M$:
For all $K \in \{\text{simplices}\}$, $\mathrm{div}_K(\rho_h u_h) + M_K + S_K = 0$

One proves:

$$\int_\Omega \rho_h^\gamma \mathrm{div}_h \, u_h dx \leq C h^\alpha,$$

$$\int_\Omega \rho_h \mathrm{div}_h \, u_h dx \leq C h^\alpha.$$

C depends on Ω, M and γ.

Ch^α is due to M_K
$\leq$ is due to upwinding and additionnal stabilization term S_K.

Estimate on u_h

Taking u_h as test function in the discrete momentum equation

$$\int_\Omega \nabla_h u_h : \nabla_h u_h \, dx - \int_\Omega p_h \mathrm{div}_h(u_h) \, dx = \int_\Omega f \cdot u_h \, dx.$$

But $p_h = \rho_h^\gamma$ a.e., Discrete mass equation and preliminary lemma gives $\int_\Omega p_h \mathrm{div}(u_h) \, dx \leq Ch^\alpha$.
This gives an estimate on u_h:

$$(\int_\Omega \nabla_h u_h \cdot \nabla_h u_h dx)^{\frac{1}{2}} = \|u_h\|_{1,b} \leq C_1.$$

Estimate on p_h (inf-sup condition)

Let $q \in L^2(\Omega)$ s.t. $\int_\Omega q dx = 0$.
Then, there exists $v \in (H_0^1(\Omega))^d$ s.t.

$$\mathrm{div}(v) = q \text{ a.e. in } \Omega,$$

$$\|v\|_{(H_0^1(\Omega))^d} \leq C_2 \|q\|_{L^2(\Omega)},$$

where C_2 only depends on Ω.

Estimate on p_h

$m_h = \frac{1}{|\Omega|}\int_\Omega p_h dx$, there exists $v_h \in H_h$, $\mathrm{div}_h(v_h) = p_h - m_h$.
Taking v_h as test function in the discrete momentum equation:

$$\int_\Omega \nabla_h u_h : \nabla_h v_h \, dx - \int_\Omega p_h \mathrm{div}_h(v_h)\, dx = \int_\Omega f \cdot v_h \, dx.$$

Using $\int_\Omega \mathrm{div}_h(v_h) dx = 0$:

$$\int_\Omega (p_h - m_h)^2 dx = \int_\Omega (f \cdot v_h - \nabla u_h : \nabla v_h) dx.$$

Since $\|v_h\|_{1,b} \leq C_2 \|p_h - m_h\|_{L^2(\Omega)}$ and $\|u_h\|_{1,b} \leq C_1$, the preceding inequality leads to:

$$\|p_h - m_h\|_{L^2(\Omega)} \leq C_3.$$

where C_3 only depends on f and on Ω.

Estimates on p_h and ρ_h

$$\|p_h - m_h\|_{L^2(\Omega)} \leq C_3.$$

$$\int_\Omega p_h^{\frac{1}{\gamma}} dx = \int_\Omega \rho_h dx = M$$

Then:

$$\|p_h\|_{L^2(\Omega)} \leq C_4;$$

where C_4 only depends on f, M, γ and Ω.

$p_h = \rho_h^\gamma$ a.e. in Ω, then:

$$\|\rho_h\|_{L^{2\gamma}(\Omega)} \leq C_5 = C_4^{\frac{1}{\gamma}}.$$

Convergence of u_h, p_h, ρ_h (weak for p_h and ρ_h)

Thanks to the estimates on u_h, p_h, ρ_h, it is possible to assume (up to a subsequence) that, as $h \to 0$:

$$u_h \to u \text{ in } L^2(\Omega)^d \text{ and } u \in H_0^1(\Omega)^d,$$

$$p_h \to p \text{ weakly in } L^2(\Omega),$$

$$\rho_h \to \rho \text{ weakly in } L^{2\gamma}(\Omega).$$

Passage to the limit on the equations, except EOS

momentum equation :

$$-\Delta u + \nabla p = f \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega,$$

mass equation (u_h converges in L^2 and ρ_h weakly in L^2):

$$\mathrm{div}(\rho u) = 0 \text{ in } \Omega,$$

L^1-weak convergence of ρ_h gives positivity of ρ and convergence of total mass:

$$\rho \geq 0 \text{ in } \Omega, \int_\Omega \rho(x)dx = M.$$

Question (if $\gamma > 1$):

$$p = \rho^\gamma \text{ in } \Omega \ ?$$

Idea : prove $\int_\Omega p_h \rho_h dx \to \int_\Omega p\rho dx$ and deduce a.e. convergence (of p_h and ρ_h) and $p = \rho^\gamma$.

$\nabla : \nabla = \mathrm{divdiv} + \mathrm{curl} \cdot \mathrm{curl}$

For all $\bar{u}, \bar{v}$ in $H_0^1(\Omega)^d$,

$$\int_\Omega \nabla \bar{u} : \nabla \bar{v} = \int_\Omega \mathrm{div}(\bar{u})\mathrm{div}(\bar{v}) + \int_\Omega \mathrm{curl}(\bar{u}) \cdot \mathrm{curl}(\bar{v}).$$

Assuming, for simplicity that $u_h \in H_0^1(\Omega)$ and $-\Delta u_h + \nabla p_h = f_h \in L^2(\Omega)$, $f_h \to f$ in L^2 as $h \to 0$. The weak form of $-\Delta u_h + \nabla p_h = f_h$ gives for all $\bar{v}$ in $H_0^1(\Omega)^d$

$$\int_\Omega \mathrm{div}(u_h)\mathrm{div}(\bar{v}) + \int_\Omega \mathrm{curl}(u_h) \cdot \mathrm{curl}(\bar{v}) - \int_\Omega p_h \mathrm{div}(\bar{v}) = \int_\Omega f_h \cdot \bar{v}.$$

Choice of $\bar{v}$? $\bar{v} = \bar{v}_h$ with $\mathrm{curl}(\bar{v}_h) = 0$, $\mathrm{div}(\bar{v}_h) = \rho_h$ and $\bar{v}_h$ bounded in H_0^1 (unfortunately, 0 is impossible).

Then, up to a subsequence,

$\bar{v}_h \to v$ in $L^2(\Omega)$ and weakly in $H_0^1(\Omega)$,

$\mathrm{curl}(v) = 0$, $\mathrm{div}(v) = \rho$.

Proof using $\bar{v}_h$ (1)

$$\int_\Omega \mathrm{div}(u_h)\mathrm{div}(\bar{v}_h) + \int_\Omega \mathrm{curl}(u_h)\cdot\mathrm{curl}(\bar{v}_h) - \int_\Omega p_h \mathrm{div}(\bar{v}_h) = \int_\Omega f_h \cdot \bar{v}_h.$$

But, $\mathrm{div}(\bar{v}_h) = \rho_h$ and $\mathrm{curl}(\bar{v}_h) = 0$. Then:

$$\int_\Omega (\mathrm{div}(u_h) - p_h)\rho_h = \int_\Omega f_h \cdot \bar{v}_h.$$

Convergence of f_h in $L^2(\Omega)^d$ to f and convergence of $\bar{v}_h$ in $L^2(\Omega)^d$ to v :

$$\lim_{h\to 0} \int_\Omega (\mathrm{div}(u_h) - p_h)\rho_h = \int_\Omega f \cdot v.$$

Proof using $\bar{v}_h$ (2)

But, since $-\Delta u + \nabla p = f$:

$$\int_\Omega \mathrm{div}(u)\mathrm{div}(v) + \int_\Omega \mathrm{curl}(u)\cdot\mathrm{curl}(v) - \int_\Omega p\,\mathrm{div}(v) = \int_\Omega f\cdot v.$$

which gives (using $\mathrm{div}(v)=\rho$ and $\mathrm{curl}(v)=0$):

$$\int_\Omega (\mathrm{div}(u)-p)\rho = \int_\Omega f\cdot v.$$

Then:

$$\lim_{h\to 0}\int_\Omega (p_h - \mathrm{div}(u_h))\rho_h dx = \int_\Omega (p-\mathrm{div}(u))\rho dx.$$

Finally, the preliminary lemma gives $\int_\Omega \rho_h \mathrm{div}(u_h) \le Ch^\alpha$ and $\int_\Omega \rho\,\mathrm{div}(u) = 0$ (since $\mathrm{div}_K(\rho_h u_h) - M_K - S_K = 0$ for all K and $\mathrm{div}(\rho u)=0$) at least for a subsequence

$$\lim_{h\to 0}\int_\Omega p_h\rho_h dx \le \int_\Omega p\rho dx.$$

Unfortunately, it is impossible to have $\bar{v}_h \in H_0^1$.

Curl-free test function

Let B be a ball containing Ω and $w_h \in H_0^1(B)$, $-\Delta w_h = \rho_h$,

$$v_h = \nabla w_h$$

- $v_h \in (H^1(\Omega))^d$,
- $\mathrm{div}(v_h) = \rho_h$ a.e. in Ω,
- $\mathrm{curl}(v_h) = 0$ a.e. in Ω,
- $\|v_h\|_{(H^1(\Omega))^d} \leq C_6 \|\rho_h\|_{L^2(\Omega)}$, where C_6 only depends on Ω.

Then, up to a subsequence,

$v_h \to v$ in $L^2(\Omega)$ and weakly in $H^1(\Omega)$,

$\mathrm{curl}(v) = 0$, $\mathrm{div}(v) = \rho$.

(Remark : $\|v_h\|_{(H^2(\Omega))^d} \leq C_6 \|\rho_h\|_{H^1(\Omega)}$)

Proving $\int_\Omega (p_h - \mathrm{div}(u_h))\rho_h \varphi dx \to \int_\Omega (p - \mathrm{div}(u))\rho\varphi dx$

Let $\varphi \in C_c^\infty(\Omega)$ (so that $v_h\varphi \in H_0^1(\Omega)^d$)). Taking $\bar{v} = v_h\varphi$:

$$\int_\Omega \mathrm{div}(u_h)\mathrm{div}(v_h\varphi)dx + \int_\Omega \mathrm{curl}(u_h)\cdot\mathrm{curl}(v_h\varphi)dx - \int_\Omega p_h \mathrm{div}(v_h\varphi)\,dx = \int_\Omega f_h \cdot (v_h\varphi)\,dx.$$

Using a proof smilar to that given if $\varphi = 1$ (with additionnal terms involving φ), we obtain :

$$\lim_{h\to 0} \int_\Omega (p_h - \mathrm{div}(u_h))\rho_h\varphi dx = \int_\Omega (p - \mathrm{div}(u))\rho\varphi dx,$$

Proving $\int_\Omega (p_h - \mathrm{div}(u_h))\rho_h\varphi dx \to \int_\Omega (p - \mathrm{div}(u))\rho\varphi dx$

Let $\varphi \in C_c^\infty(\Omega)$ (so that $v_h\varphi \in H_0^1(\Omega)^d$)). Taking $\bar{v} = v_h\varphi$:

$$\int_\Omega \mathrm{div}(u_h)\mathrm{div}(v_h\varphi)dx + \int_\Omega \mathrm{curl}(u_h)\cdot\mathrm{curl}(v_h\varphi)dx - \int_\Omega p_h\mathrm{div}(v_h\varphi)\,dx = \int_\Omega f_h\cdot(v_h\varphi)\,dx.$$

But, $\mathrm{div}(v_h\varphi) = \rho_h\varphi + v_h\cdot\nabla\varphi$ and $\mathrm{curl}(v_h\varphi) = L(\varphi)v_h$, where $L(\varphi)$ is a matrix involving the first order derivatives of φ. Then:

$$\int_\Omega (\mathrm{div}(u_h) - p_h)\rho_h\varphi dx = \int_\Omega f_h\cdot(v_h\varphi)dx - \int_\Omega \mathrm{div}(u_h)v_h\cdot\nabla\varphi dx - \int \mathrm{curl}(u_h)\cdot L(\varphi)v_h + \int_\Omega p_h v_h\cdot\nabla\varphi dx.$$

Weak convergence of u_h in $H_0^1(\Omega)^d$, weak convergence of p_h in $L^2(\Omega)$ and convergence of v_h and f_h in $L^2(\Omega)^d$:

$$\lim_{h\to 0}\int_\Omega (\mathrm{div}(u_h) - p_h)\rho_h\varphi dx = \int_\Omega f\cdot(v\varphi)dx - \int_\Omega \mathrm{div}(u)v\cdot\nabla\varphi dx - \int \mathrm{curl}(u)\cdot L(\varphi)v + \int_\Omega pv\cdot\nabla\varphi dx.$$

Proving $\int_\Omega (p_h - \mathrm{div}(u_h))\rho_h \varphi dx \to \int_\Omega (p - \mathrm{div}(u))\rho\varphi dx$

But, since $-\Delta u + \nabla p = f$:

$$\int_\Omega \mathrm{div}(u)\mathrm{div}(v\varphi)dx + \int_\Omega \mathrm{curl}(u)\cdot\mathrm{curl}(v\varphi)dx - \int_\Omega p\mathrm{div}(v\varphi)\,dx \\ = \int_\Omega f\cdot(v\varphi)\,dx.$$

which gives (using $\mathrm{div}(v) = \rho$ and $\mathrm{curl}(v) = 0$):

$$\int_\Omega(\mathrm{div}(u) - p)\rho\varphi dx = \int_\Omega f\cdot(v\varphi)dx \\ - \int_\Omega \mathrm{div}(u)v\cdot\nabla\varphi dx - \int \mathrm{curl}(u)\cdot L(\varphi)v + \int_\Omega pv\cdot\nabla\varphi dx.$$

Then:

$$\lim_{h\to 0}\int_\Omega (p_h - \mathrm{div}(u_h))\rho_h\varphi dx = \int_\Omega (p - \mathrm{div}(u))\rho\varphi dx.$$

Proving $\int_\Omega (p_h - \mathrm{div}(u_h))\rho_h dx \to \int_\Omega (p - \mathrm{div}(u))\rho dx$

Lemma : $F_n \to F$ in $D'(\Omega)$, $(F_n)_{n\in\mathbb{N}}$ bounded in L^q for some $q > 1$. Then $F_n \to F$ weakly in L^1.

With $F_n = (p_h - \mathrm{div}(u_h))\rho_h$, $F = (p - \mathrm{div}(u))\rho$ and since $\gamma > 1$, the lemma gives

$$\int_\Omega (p_h - \mathrm{div}(u_h))\rho_h dx \to \int_\Omega (p - \mathrm{div}(u))\rho dx.$$

Proving $\int_\Omega p_h \rho_h dx \to \int_\Omega p\rho dx$

$$\int_\Omega (p_h - \mathrm{div}(u_h))\rho_h dx \to \int_\Omega (p - \mathrm{div}(u))\rho dx.$$

But since $\mathrm{div}_K(\rho_h u_h) + M_K + S_K = 0$, $\mathrm{div}(\rho u) = 0$, the preliminary lemma gives:

$$\int_\Omega \mathrm{div}(u_h)\rho_h dx \le Ch^\alpha, \ \int_\Omega \mathrm{div}(u)\rho dx = 0;$$

Then:

$$\lim_{h\to 0} \int_\Omega p_h \rho_h dx \le \int_\Omega p\rho dx.$$

a.e. convergence of ρ_h and p_h

Let $G_h = (\rho_h^\gamma - \rho^\gamma)(\rho_h - \rho) \in L^1(\Omega)$ and $G_h \geq 0$ a.e. in Ω.
Futhermore $G_h = (p_h - \rho^\gamma)(\rho_h - \rho) = p_h\rho_h - p_h\rho - \rho^\gamma\rho_h + \rho^\gamma\rho$ and:

$$\int_\Omega G_h dx = \int_\Omega p_h\rho_h dx - \int_\Omega p_h\rho dx - \int_\Omega \rho^\gamma\rho_h dx + \int_\Omega \rho^\gamma\rho dx.$$

Using the weak convergence in $L^2(\Omega)$ of p_h and ρ_h and $\lim_{h\to 0}\int_\Omega p_h\rho_h dx \leq \int_\Omega p\rho dx$:

$$\lim_{h\to 0}\int_\Omega G_h dx \leq 0,$$

Then (up to a subsequence), $G_h \to 0$ a.e. and then $\rho_h \to \rho$ a.e. (since $y \mapsto y^\gamma$ is an increasing function on $\mathbb{R}_+$). Finally:

$\rho_h \to \rho$ in $L^q(\Omega)$ for all $1 \leq q < 2\gamma$,

$p_h = \rho_h^\gamma \to \rho^\gamma$ in $L^q(\Omega)$ for all $1 \leq q < 2$,

and $p = \rho^\gamma$.

Additional difficulty for stat. comp. NS equations

Ω is a bounded open set of $\mathbb{R}^d$, $d = 2$ or 3, with a Lipschitz continuous boundary, $\gamma > 1$, $f \in L^2(\Omega)^d$ and $M > 0$

$$-\Delta u + \mathrm{div}(\rho u \otimes u) + \nabla p = f \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega,$$

$$\mathrm{div}(\rho u) = 0 \text{ in } \Omega, \; \rho \geq 0 \text{ in } \Omega, \; \int_\Omega \rho(x) dx = M,$$

$$p = \rho^\gamma \text{ in } \Omega$$

$d = 2 :$ no aditional difficulty

$d = 3 :$ no additional difficulty if $\gamma \geq 3$. But for $\gamma < 3$, no estimate on p in $L^2(\Omega)$.

Estimates in the case of NS equations, $\frac{3}{2} < \gamma < 3$

Estimate on u : Taking u as test function in the momentum leads to an estimate on u in $(H_0^1(\Omega))^d$ since

$$\int_\Omega \rho u \otimes u : \nabla u dx = 0.$$

Then, we have also an estimate on u in $L^6(\Omega)^d$ (using Sobolev embedding).

Estimate on p in $L^q(\Omega)$, with some $1 < q < 2$ and $q = 1$ when $\gamma = \frac{3}{2}$ (using Nečas Lemma in some L^r instead of L^2).

Estimate on ρ in $L^q(\Omega)$, with some $\frac{3}{2} < q < 6$ and $q = \frac{3}{2}$ when $\gamma = \frac{3}{2}$ (since $p = \rho^\gamma$).

Remark : $\rho u \otimes u \in L^1(\Omega)$, since $u \in L^6(\Omega)^d$ and $\rho \in L^{\frac{3}{2}}(\Omega)$ (and $\frac{1}{6} + \frac{1}{6} + \frac{2}{3} = 1$).

NS equations, $\gamma < 3$, how to pass to the limit in the EOS

We prove

$$\lim_{h \to 0} \int_\Omega p_h \rho_h^\theta dx = \int_\Omega p \rho^\theta dx,$$

with some convenient choice of $\theta > 0$ instead of $\theta = 1$.

This gives, as for $\theta = 1$, the a.e. convergence (up to a subsequence) of p_h and ρ_h.