Compactness of approximate solutions (for some evolution PDEs with diffusion)

T. Gallouët

$K\rho\acute{\eta}\tau\eta$, September 2010

with coauthors...

- Lucio Boccardo (continuous setting, 1989)
- Robert Eymard, Raphaèle Herbin (discrete setting, 2000)
- Aurélien Larcher, Jean-Claude Latché (discrete setting, submitted)

Example (coming from RANS model for turbulent flows)

$$\partial_t u + \mathrm{div}(vu) - \Delta u = f \text{ in } \Omega \times (0, T),$$
$$u = 0 \text{ on } \partial\Omega \times (0, T),$$
$$u(\cdot, 0) = u_0 \text{ in } \Omega.$$

- Ω is a bounded open subset of $\mathbb{R}^d$ ($d = 2$ or 3) with a Lipschitz continuous boundary
- $v \in C^1(\overline{\Omega} \times [0, T], \mathbb{R})$
- $u_0 \in L^1(\Omega)$ (or u_0 is a Radon measure on Ω)
- $f \in L^1(\Omega \times (0, T))$ (or f is a Radon measure on $\Omega \times (0, T)$)

with possible generalization to nonlinear problems.

Non smooth solutions.

Example, motivation

For this example, we have two objectives:

1. Existence of weak solution and (strong) convergence of "continuous approximate solutions", that is solutions of the continuous problem with regular data converging to f and u_0.
2. Existence of weak solution and (strong) convergence of the approximate solutions given by a full discretized problem.

In both case, we want to prove strong compactness of a sequence of approximate solutions. This is the main subject of this talk.

Continuous approximation

$(f_n)_{n\in\mathbb{N}}$ and $(u_{0,n})_{n\in\mathbb{N}}$ are two sequences of regular functions such that

$$\int_0^T\int_\Omega f_n\varphi dxdt \to \int_0^T\int_\Omega f\varphi dxdt,\ \forall\varphi\in C_c^\infty(\Omega\times(0,T),\mathbb{R}),$$
$$\int_\Omega u_{0,n}\varphi dx \to \int_\Omega u_0\varphi dx,\ \forall\varphi\in C_c^\infty(\Omega,\mathbb{R}).$$

For $n\in\mathbb{N}$, it is well known that there exist u_n solution of the regularized problem

$$\partial_t u_n + \mathrm{div}(vu_n) - \Delta u_n = f_n \text{ in } \Omega\times(0,T),$$
$$u_n = 0 \text{ on } \partial\Omega\times(0,T),$$
$$u_n(\cdot,0) = u_{0,n} \text{ in } \Omega.$$

One has, at least, $u_n\in L^2((0,T),H_0^1(\Omega))\cap C([0,T],L^2(\Omega))$ and $\partial_t u_n\in L^2((0,T),H^{-1}(\Omega))$.

Continuous approximation, steps of the proof of convergence

1. Estimate on u_n (not easy). One proves that the sequence $(u_n)_{n\in\mathbb{N}}$ is bounded in

$$L^q((0,T),W_0^{1,q}(\Omega)) \text{ for all } 1 \le q < \frac{d+2}{d+1}.$$

   (This gives, up to a subsequence, weak convergence in $L^q(\Omega\times(0,T))$ of u_n to some u and then, since the problem is linear, that u is a weak solution of the problem with f and u_0.)
2. Strong compactness of the sequence $(u_n)_{n\in\mathbb{N}}$
3. Regularity of the limit of the sequence $(u_n)_{n\in\mathbb{N}}$.
4. Passage to the limit in the approximate equation (easy).

Classical Lions' lemma

X, B, Y are three Banach spaces such that

- $X \subset B$ with compact embedding,
- $B \subset Y$ with continuous embedding.

Then, for any $\varepsilon > 0$, there exists C_ε such that, for $w \in X$,

$$\|w\|_B \leq \varepsilon \|w\|_X + C_\varepsilon \|w\|_Y.$$

Example: $X = W_0^{1,1}(\Omega)$, $B = L^1(\Omega)$,
$Y = W_\star^{-1,1}(\Omega) = (W_0^{1,\infty}(\Omega))'$. As usual, we identify an L^1-function with the corresponding linear form on $W_0^{1,\infty}(\Omega)$.

Classical Lions' lemma, another formulation

X, B, Y are three Banach spaces such that, $X \subset B \subset Y$,

- If $(\|w_n\|_X)_{n\in\mathbb{N}}$ is bounded, then, up to a subsequence, there exists $w \in B$ such that $w_n \to w$ in B.
- If $w_n \to w$ in B and $\|w_n\|_Y \to 0$, then $w = 0$.

Then, for any $\varepsilon > 0$, there exists C_ε such that, for $w \in X$,

$$\|w\|_B \leq \varepsilon\|w\|_X + C_\varepsilon\|w\|_Y.$$

The hypothesis $B \subset Y$ is not necessary.

Classical Lions' lemma, a particular case, simpler

B is a Hilbert space and X is a Banach space $X \subset B$. We define on X the dual norm of $\| \cdot \|_X$, with the scalar product of B, namely

$$\|u\|_Y = \sup\{(u/v)_B,\ v \in X, \|v\|_X \le 1\}.$$

Then, for any $\varepsilon > 0$ and $w \in X$,

$$\|w\|_B \le \varepsilon \|w\|_X + \frac{1}{\varepsilon}\|w\|_Y.$$

The proof is simple since

$$\|u\|_B = (u/u)_B^{\frac{1}{2}} \le (\|u\|_Y \|u\|_X)^{\frac{1}{2}} \le \varepsilon \|w\|_X + \frac{1}{\varepsilon}\|w\|_Y.$$

Compactness of X in B is not needed here (but this compactness is needed for Aubin-Simon' Lemma, next slide. . .).

Aubin-Simon' Compactness Lemma

X, B, Y are three Banach spaces such that

- $X \subset B$ with compact embedding,
- $B \subset Y$ with continuous embedding.

Let $T > 0$ and $(u_n)_{n \in \mathbb{N}}$ be a sequence such that

- $(u_n)_{n \in \mathbb{N}}$ is bounded in $L^1((0,T),X)$,
- $(\partial_t u_n)_{n \in \mathbb{N}}$ is bounded in $L^1((0,T),Y)$.

Then there exists $u \in L^1((0,T),B)$ such that, up to a subsequence, $u_n \to u$ in $L^1((0,T),B)$.

Example: $X = W_0^{1,1}(\Omega)$, $B = L^1(\Omega)$, $Y = W_\star^{-1,1}(\Omega)$.

Aubin-Simon' Compactness Lemma, another formulation

X, B, Y are three Banach spaces such that, $X \subset B \subset Y$,

- If $(\|w_n\|_X)_{n\in\mathbb{N}}$ is bounded, then, up to a subsequence, there exists $w \in B$ such that $w_n \to w$ in B.
- If $w_n \to w$ in B and $\|w_n\|_Y \to 0$, then $w = 0$.

Let $T > 0$ and $(u_n)_{n\in\mathbb{N}}$ be a sequence such that

- $(u_n)_{n\in\mathbb{N}}$ is bounded in $L^1((0,T),X)$,
- $(\partial_t u_n)_{n\in\mathbb{N}}$ is bounded in $L^1((0,T),Y)$.

Then there exists $u \in L^1((0,T),B)$ such that, up to a subsequence, $u_n \to u$ in $L^1((0,T),B)$.

Example: $X = W_0^{1,1}(\Omega)$, $B = L^1(\Omega)$, $Y = W_\star^{-1,1}(\Omega)$.

Continuous approx., compactness of the sequence $(u_n)_{n\in\mathbb{N}}$

u_n is solution of he continuous problem with data f_n and $u_{0,n}$.

$X = W_0^{1,1}(\Omega)$, $B = L^1(\Omega)$, $Y = W_\star^{-1,1}(\Omega)$.

In order to apply Aubin-Simon' lemma we need

- $(u_n)_{n\in\mathbb{N}}$ is bounded in $L^1((0,T),X)$,
- $(\partial_t u_n)_{n\in\mathbb{N}}$ is bounded in $L^1((0,T),Y)$.

The sequence $(u_n)_{n\in\mathbb{N}}$ is bounded in $L^q((0,T),W_0^{1,q}(\Omega))$ (for $1 \le q < (d+2)/(d+1)$) and then is bounded in $L^1((0,T),X)$, since $W_0^{1,q}(\Omega)$ is continuously embedded in $W_0^{1,1}(\Omega)$.

$\partial_t u_n = f_n - \mathrm{div}(vu_n) - \Delta u_n$. Is $(\partial_t u_n)_{n\in\mathbb{N}}$ bounded in $L^1((0,T),Y)$?

Continuous approx., Compactness of the sequence $(u_n)_{n\in\mathbb{N}}$

Bound of $(\partial_t u_n)_{n\in\mathbb{N}}$ in $L^1((0,T),W_\star^{-1,1}(\Omega))$?
$\partial_t u_n = f_n - \mathrm{div}(vu_n) - \Delta u_n$.

- $(f_n)_{n\in\mathbb{N}}$ is bounded in $L^1(0,T),L^1(\Omega))$ and then in $L^1((0,T),W_\star^{-1,1}(\Omega))$, since $L^1(\Omega)$ is continously embedded in $W_\star^{-1,1}(\Omega)$,
- $(\mathrm{div}(vu_n))_{n\in\mathbb{N}}$ is bounded in $L^1((0,T),W_\star^{-1,1}(\Omega))$ since $(vu_n)_{n\in\mathbb{N}}$ is bounded in $L^1((0,T),(L^1(\Omega))^d$ and div is a continuous operator from $(L^1(\Omega))^d$ to $W_\star^{-1,1}(\Omega)$,
- $(\Delta u_n)_{n\in\mathbb{N}}$ is bounded in $L^1((0,T),W_\star^{-1,1}(\Omega))$ since $(u_n)_{n\in\mathbb{N}}$ is bounded in $L^1((0,T),W_0^{1,1}(\Omega))$ and Δ is a continuous operator from $W_0^{1,1}(\Omega)$ to $W_\star^{-1,1}(\Omega)$.

Finally, $(\partial_t u_n)_{n\in\mathbb{N}}$ is bounded in $L^1((0,T),W_\star^{-1,1}(\Omega))$.
Aubin-Simon' lemma gives (up to a subsequence) $u_n \to u$ in $L^1((0,T),L^1(\Omega))$.

Regularity of the limit

$u_n \to u$ in $L^1(\Omega \times (0,T))$ and $(u_n)_{n\in\mathbb{N}}$ bounded in $L^q((0,T), W_0^{1,q}(\Omega))$ for $1 \le q < (d+2)/(d+1)$. Then

$$u_n \to u \text{ in } L^q(\Omega \times (0,T))) \text{ for } 1 \le q < \frac{d+2}{d+1},$$

$$\nabla u_n \to \nabla u \text{ weakly in } L^q(\Omega \times (0,T))^d \text{ for } 1 \le q < \frac{d+2}{d+1},$$

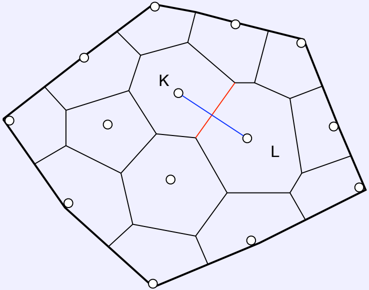
$$u \in L^q((0,T), W_0^{1,q}(\Omega)) \text{ for } 1 \le q < (d+2)/(d+1).$$

Remark: $L^q((0,T), L^q(\Omega)) = L^q(\Omega \times (0,T))$

An additional work is needed to prove the strong convergence of ∇u_n to ∇u.

Full approximation, FV scheme (or dG scheme)

Space discretization: Admissible mesh $\mathcal{M}$. Time step: k ($Nk = T$)



$T_{K,L}=m_{K,L}/d_{K,L}$

$\mathrm{size}(\mathcal{M}) = \sup\{\mathrm{diam}(K), K \in \mathcal{M}\}$
Unknowns: $u_K^{(p)} \in \mathbb{R}$, $K \in \mathcal{M}$, $p \in \{1, \ldots, N\}$.
Discretization: Implicit in time, upwind for convection, classical 2-points flux for diffusion. (Well known scheme.)

Full approximation, approximate solution

- $H_{\mathcal{M}}$ the space of functions from Ω to $\mathbb{R}$, constant on each K, $K \in \mathcal{M}$.
- The discrete solution u is constant on $K \times ((p-1)k, pk)$ with $K \in \mathcal{M}$ and $p \in \{1, \ldots, N\}$.
  $u(\cdot, t) = u^{(p)}$ for $t \in ((p-1)k, pk)$ and $u^{(p)} \in H_{\mathcal{M}}$.
- Discrete derivatives in time, $\partial_{t,k} u$, defined by:

$$\partial_{t,k} u(\cdot, t) = \partial_{t,k}^{(p)} u = \frac{1}{k}(u^{(p)} - u^{(p-1)}) \text{ for } t \in ((p-1)k, pk),$$

for $p \in \{2, \ldots, N\}$ (and $\partial_{t,k} u(\cdot, t) = 0$ for $t \in (0, k)$).

Full approximation, steps of the proof of convergence

Sequence of meshes and time steps, $(\mathcal{M}_n)_{n\in\mathbb{N}}$ and k_n.
$\mathrm{size}(\mathcal{M}_n) \to 0$, $k_n \to 0$, as $n \to \infty$.
For $n \in \mathbb{N}$, u_n is the solution of the FV scheme.

1. Estimate on u_n.
2. Strong compactness of the sequence $(u_n)_{n\in\mathbb{N}}$.
3. Regularity of the limit of the sequence $(u_n)_{n\in\mathbb{N}}$.
4. Passage to the limit in the approximate equation.

Discrete norms

Admissible mesh: $\mathcal{M}$.

$u \in H_{\mathcal{M}}$ (that is u is a function constant on each K, $K \in \mathcal{M}$).

- $1 \leq q < \infty$. Discrete $W_0^{1,q}$-norm:

$$\|u\|_{1,q,\mathcal{M}}^q = \sum_{\sigma\in\mathcal{E}_{int},\sigma=K|L} m_\sigma d_\sigma \left|\frac{u_K - u_L}{d_\sigma}\right|^q + \sum_{\sigma\in\mathcal{E}_{ext},\sigma\in\mathcal{E}_K} m_\sigma d_\sigma \left|\frac{u_K}{d_\sigma}\right|^q$$

- $q = \infty$. Discrete $W_0^{1,\infty}$-norm: $\|u\|_{1,\infty,\mathcal{M}}^q = \max\{M_i, M_e, M\}$ with

$$M_i = \max\{\frac{|u_K - u_L|}{d_\sigma},\ \sigma \in \mathcal{E}_{int}, \sigma = K|L\},$$

$$M_e = \max\{\frac{|u_K|}{d_\sigma},\ \sigma \in \mathcal{E}_{ext}, \sigma \in \mathcal{E}_K\},$$

$$M = \max\{|u_K|,\ K \in \mathcal{M}\}.$$

Discrete dual norms

Admissible mesh: $\mathcal{M}$.
For $r \in [1, \infty]$, $\| \cdot \|_{-1,r,\mathcal{M}}$ is the dual norm of the norm $\| \cdot \|_{1,q,\mathcal{M}}$ with $q = r/(r-1)$. That is, for $u \in H_{\mathcal{M}}$,

$$\|u\|_{-1,r,\mathcal{M}} = \max\{\int_\Omega uv\,dx,\ v \in H_{\mathcal{M}}, \|v\|_{1,q,\mathcal{M}} \leq 1\}.$$

Example: $r = 1$ $(q = \infty)$.

Full discretization, estimate on the discrete solution

For $1 \leq q < (d+2)/(d+1)$, the sequence $(u_n)_{n\in\mathbb{N}}$ is bounded in $L^q((0,T), W_{q,n})$, where $W_{q,n}$ is the space $H_{\mathcal{M}_n}$, endowed with the norm $\|\cdot\|_{1,q,\mathcal{M}_n}$. That is

$$\sum_{p=1}^{N_n} k\|u_n^{(p)}\|_{1,q,\mathcal{M}_n}^q \leq C.$$

Discrete Lions' lemma

B is a Banach space, $(B_n)_{n\in\mathbb{N}}$ is a sequence of finite dimensional subspaces of B. $\|\cdot\|_{X_n}$ and $\|\cdot\|_{Y_n}$ are two norms on B_n such that:

- If $(\|w_n\|_{X_n})_{n\in\mathbb{N}}$ is bounded, then, up to a subsequence, there exists $w \in B$ such that $w_n \to w$ in B.
- If $w_n \to w$ in B and $\|w_n\|_{Y_n} \to 0$, then $w = 0$.

Then, for any $\varepsilon > 0$, there exists C_ε such that, for $n \in \mathbb{N}$ and $w \in B_n$

$$\|w\|_B \leq \varepsilon\|w\|_{X_n} + C_\varepsilon\|w\|_{Y_n}.$$

Example: $B = L^1(\Omega)$. $B_n = H_{\mathcal{M}_n}$ (the finite dimensional space given by the mesh $\mathcal{M}_n$). We have to choose $\|\cdot\|_{X_n}$ and $\|\cdot\|_{Y_n}$.

Discrete Lions' lemma, proof

Proof by contradiction. There exists $\varepsilon > 0$ and $(w_n)_{n\in\mathbb{N}}$ such that, for all n, $w_n \in B_n$ and

$$\|w_n\|_B > \varepsilon\|w_n\|_{X_n} + C_n\|w_n\|_{Y_n},$$

with $\lim_{n\to\infty} C_n = +\infty$.

It is possible to assume that $\|w_n\|_B = 1$. Then $(\|w_n\|_{X_n})_{n\in\mathbb{N}}$ is bounded and, up to a subsequence, $w_n \to w$ in B (so that $\|w\|_B = 1$). But $\|w_n\|_{Y_n} \to 0$, so that $w = 0$, in contradiction with $\|w\|_B = 1$.

Discrete Aubin-Simon' Compactness Lemma

B a Banach, $(B_n)_{n\in\mathbb{N}}$ family of finite dimensional subspaces of B.
$\|\cdot\|_{X_n}$ and $\|\cdot\|_{Y_n}$ two norms on B_n such that:

- If $(\|w_n\|_{X_n})_{n\in\mathbb{N}}$ is bounded, then, up to a subsequence, there exists $w \in B$ such that $w_n \to w$ in B.
- If $w_n \to w$ in B and $\|w_n\|_{Y_n} \to 0$, then $w = 0$.

$X_n = B_n$ with norm $\|\cdot\|_{X_n}$, $Y_n = B_n$ with norm $\|\cdot\|_{Y_n}$. Let $T > 0$, $k_n > 0$ and $(u_n)_{n\in\mathbb{N}}$ be a sequence such that

- for all n, $u_n(\cdot, t) = u_n^{(p)} \in B_n$ for $t \in ((p-1)k_n, pk_n)$
- $(u_n)_{n\in\mathbb{N}}$ is bounded in $L^1((0,T), X_n)$,
- $(\partial_{t,k_n} u_n)_{n\in\mathbb{N}}$ is bounded in $L^1((0,T), Y_n)$.

Then there exists $u \in L^1((0,T), B)$ such that, up to a subsequence, $u_n \to u$ in $L^1((0,T), B)$.

Example: $B = L^1(\Omega)$. $B_n = H_{\mathcal{M}_n}$. What choice for $\|\cdot\|_{X_n}$, $\|\cdot\|_{Y_n}$?

Full approx., compactness of the sequence $(u_n)_{n\in\mathbb{N}}$

u_n is solution of the fully discretized problem with mesh $\mathcal{M}_n$ and time step k_n.

$B = L^1(\Omega)$, $B_n = H_{\mathcal{M}_n}$,
$\|\cdot\|_{X_n} = \|\cdot\|_{1,1,\mathcal{M}_n}$, $\|\cdot\|_{Y_n} = \|\cdot\|_{-1,1,\mathcal{M}_n}$

In order to apply the discrete Aubin-Simon' lemma we need to verify the hypotheses of the discrete Lions' lemma and that

- $(u_n)_{n\in\mathbb{N}}$ is bounded in $L^1((0,T),X_n)$,
- $(\partial_{t,k_n}u_n)_{n\in\mathbb{N}}$ is bounded in $L^1((0,T),Y_n)$.

The sequence $(u_n)_{n\in\mathbb{N}}$ is bounded in $L^q((0,T),W_{q,n}(\Omega))$ (for $1 \le q < (d+2)/(d+1)$) and then is bounded in $L^1((0,T),X_n)$ since $\|\cdot\|_{1,1,\mathcal{M}_n} \le C_q\|\cdot\|_{1,q,\mathcal{M}_n}$ for $q > 1$.

Using the scheme, it is quite easy to prove (similarly to the continuous approximation) that $(\partial_{t,k_n}u_n)_{n\in\mathbb{N}}$ is bounded in $L^1((0,T),Y_n)$.

Full approx., Compactness of the sequence $(u_n)_{n\in\mathbb{N}}$

It remains to verify the hypotheses of the discrete Lions' lemma.

- If $w_n \in H_{\mathcal{M}_n}$, $(\|w_n\|_{1,1,\mathcal{M}_n})_{n\in\mathbb{N}}$ is bounded, there exists $w \in L^1(\Omega)$ such that $w_n \to w$ in $L^1(\Omega)$?
  Yes, this is classical now...
- If $w_n \in H_{\mathcal{M}_n}$, $w_n \to w$ in $L^1(\Omega)$ and $\|w_n\|_{-1,1,\mathcal{M}_n} \to 0$, then $w = 0$? Yes... Proof :
  Let $\varphi \in W_0^{1,\infty}(\Omega)$ and its "projection" $\pi_n\varphi \in H_{\mathcal{M}_n}$. One has $\|\pi_n\varphi\|_{1,\infty,\mathcal{M}_n} \le \|\varphi\|_{W^{1,\infty}(\Omega)}$ and then

$$|\int_\Omega w_n(\pi_n\varphi)dx| \le \|w_n\|_{-1,1,\mathcal{M}_n}\|\varphi\|_{W^{1,\infty}(\Omega)} \to 0,$$

  and, since $w_n \to w$ in $L^1(\Omega)$ and $\pi_n\varphi \to \varphi$ uniformly,

$$\int_\Omega w_n(\pi_n\varphi)dx \to \int_\Omega w\varphi dx.$$

  This gives $\int_\Omega w\varphi dx = 0$ for all $\varphi \in W_0^{1,\infty}(\Omega)$ and then $w = 0$ a.e.

Regularity of the limit

As the continuous approximation,
$u_n \to u$ in $L^1(\Omega \times (0,T))$ and $(u_n)_{n\in\mathbb{N}}$ bounded in $L^q((0,T), W_{q,n}(\Omega))$ for $1 \leq q < (d+2)/(d+1)$. Then

$$u_n \to u \text{ in } L^q(\Omega \times (0,T))) \text{ for } 1 \leq q < \frac{d+2}{d+1},$$

$$u \in L^q((0,T), W_0^{1,q}(\Omega)) \text{ for } 1 \leq q < (d+2)/(d+1).$$