Convergence of Gradient Schemes for the Stefan problem

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$K\rho\acute{\eta}\tau\eta$, September 2012

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Stefan problem

$$\partial_t u - \Delta\varphi(u) = f \text{ in } \Omega \times (0, T),$$
$$u = 0 \text{ on } \partial\Omega \times (0, T),$$
$$u(\cdot, 0) = u_0 \text{ in } \Omega.$$

- Ω is a polygonal (for $d = 2$) or polyhedral (for $d = 3$) open subset of $\mathbb{R}^d$ ($d = 2$ or 3), $T > 0$
- φ is a non decreasing function from $\mathbb{R}$ to $\mathbb{R}$, Lipschitz continuous and $\liminf_{s\to+\infty} \varphi(s)/s > 0$
- $u_0 \in L^2(\Omega)$
- $f \in L^2(\Omega \times (0, T))$

Mail difficulty : φ may be constant on some interval of $\mathbb{R}$

Objective : To present a general framework to prove the convergence of many different schemes (FE, NCFE, FV, HFV...)

Discrete unknown

Discretization parameters, $\mathcal{D}$: spatial mesh, time step (δt)

Discrete unknown at time $t_k = k\delta t$: $u^{(k)} \in X_{\mathcal{D},0}$.

- values at the vextex of the mesh (FE)
- values at the edges of the mesh (NCFE)
- values in the cells (FV)
- values in the cells and in the edges (HFV)

With an element v of $X_{\mathcal{D},0}$ (for instance $v = u^{(k)}$ or $v = \varphi(u^{(k)})$), one defines two functions

- $\bar{v}$ (reconstruction of the approximate solution)
- $\nabla_{\mathcal{D}} v$ (reconstruction of an approximate gradient)

with some natural properties of consistency.

A crucial property is $\overline{\varphi(u)} = \varphi(\bar{u})$

N.B. the functions $\bar{v}$ and $\nabla_{\mathcal{D}} v$ are piecewise constant functions, but not necessarily on the same mesh

Numerical scheme (Gradient schemes)

$\bar{u}^{(0)}$ given by the initial condition and for $k \geq 0$,
$u^{(k+1)} \in X_{\mathcal{D},0}$

$$\int_\Omega \frac{\bar{u}^{(k+1)} - \bar{u}^{(k)}}{\delta t} \bar{v}\, dxdt + \int_\Omega \nabla_{\mathcal{D}} \varphi(u^{(k+1)}) \cdot \nabla_{\mathcal{D}} v dx = \frac{1}{\delta t} \int_{t_k}^{t_{k+1}} f \bar{v}\, dxdt,\ \forall v \in X_{\mathcal{D},0}$$

Classical examples : FE with mass lumping, FV on admissible meshes
but also many other schemes...

Steps of the proof of convergence

Let $(u_n)_{n\in\mathbb{N}}$ be a sequence of approximate solutions (associated to $\mathcal{D}_n$ and δt_n with $\lim_{n\to\infty} \mathrm{size}(\mathcal{D}_n) = 0$ and $\lim_{n\to\infty} \delta t_n = 0$)

1. Estimates on the approximate solution
2. Compactness result on the sequence of approximate solutions
3. Passage to the limit in the approximate equation

Steps 2 and 3 are tricky due to the fact that φ may be constant on some interval of $\mathbb{R}$

Estimates

One mimics the estimates for the continuous equation

$$\partial_t u - \Delta\varphi(u) = f \text{ in } \Omega \times (0,T),$$
$$u = 0 \text{ on } \partial\Omega \times (0,T),$$
$$u(\cdot,0) = u_0 \text{ in } \Omega.$$

Taking $\varphi(u)$ as test function one obtains

- an estimate on u in $L^\infty((0,T),L^2(\Omega))$
- an estimate on $\varphi(u)$ in $L^2((0,T),H_0^1(\Omega))$
- and therefore an estimate on $\partial_t u$ in $L^2((0,T),H^{-1}(\Omega))$

Estimates with corresponding discrete norms hold for the discrete setting of gradient schemes : $L^\infty((0,T),L^2(\Omega))$-estimate on $\bar{u}$, $L^2((0,T),L^2(\Omega))$-estimate on $\nabla_{\mathcal{D}}\varphi(u)$ and an estimate on the time discrete derivative for a dual norm

Estimates (2)

These estimates give only weak compactness on the sequences of approximate solutions $(u_n)_{n\in\mathbb{N}}$ and $(\varphi(u_n))_{n\in\mathbb{N}}$. Not sufficient to pass to the limit...

$$\lim_{n\to\infty} \varphi(u_n) = \varphi(\lim_{n\to\infty} u_n)?$$

Lions-Aubin-Simon Compactness Lemma

X, B, Y are three Banach spaces such that

- $X \subset B$ with compact embedding,
- $B \subset Y$ with continuous embedding.

Let $T > 0$, $1 \leq p < +\infty$ and $(v_n)_{n\in\mathbb{N}}$ be a sequence such that

- $(v_n)_{n\in\mathbb{N}}$ is bounded in $L^p((0,T),X)$,
- $(\partial_t v_n)_{n\in\mathbb{N}}$ is bounded in $L^p((0,T),Y)$.

Then there exists $v \in L^p((0,T),B)$ such that, up to a subsequence, $v_n \to v$ in $L^p((0,T),B)$.

Example: $p = 2$, $X = H_0^1(\Omega)$, $B = L^2(\Omega)$, $Y = H^{-1}(\Omega)$.

A dicrete version with a family a spaces $(X_n)_{n\in\mathbb{N}}$ and a family a spaces $(Y_n)_{n\in\mathbb{N}}$ is possible.

The Lions-Aubin-Simon lemma is of no use here

- $(\partial_t u_n)_{n\in\mathbb{N}}$ bounded in $L^2((0,T),H^{-1}(\Omega))$
- $\varphi(u_n)_{n\in\mathbb{N}}$ bounded in $L^2((0,T),H_0^1(\Omega))$

Unfortunately,

- the estimate on $(\varphi(u_n))_{n\in\mathbb{N}}$ does not give an analogue estimate on $(u_n)_{n\in\mathbb{N}}$ (since φ may be constant on some interval)
- the estimate on $(\partial_t u_n)_{n\in\mathbb{N}}$ does not give an analogue estimate on $(\partial_t\varphi(u_n))_{n\in\mathbb{N}}$ (the product of an $L^\infty(\Omega)$ function with a $H^{-1}(\Omega)$ element is not well defined).

One cannot use Lions-Aubin-Simon Compactness lemma on the sequence $(u_n)_{n\in\mathbb{N}}$ nor on the sequence $(\varphi(u_n))_{n\in\mathbb{N}}$

Alt-Luckhaus method

X, B are two Banach spaces such that

- $X \subset B$ with compact embedding,

Let $T > 0$, $1 \leq p < +\infty$ and $(v_n)_{n\in\mathbb{N}}$ be a sequence such that

- $(v_n)_{n\in\mathbb{N}}$ is bounded in $L^p((0,T),X)$,
- $\|v_n(\cdot + h) - v_n\|_{L^p((0,T),B)} \to 0$, as $h \to 0$, uniformly w.r.t. n.

Then there exists $v \in L^p((0,T),B)$ such that, up to a subsequence, $v_n \to v$ in $L^p((0,T),B)$.

Example: $p = 2$, $X = H_0^1(\Omega)$, $B = L^2(\Omega)$

Here also, a dicrete version with a family a spaces $(X_n)_{n\in\mathbb{N}}$ is possible.

Alt-Luckhaus method for the Stefan problem

One knows that $\varphi(u_n)_{n\in\mathbb{N}}$ is bounded in $L^2((0,T),H_0^1(\Omega))$. To obtain compactness of $\varphi(u_n)_{n\in\mathbb{N}}$ in $L^2((0,T),L^2(\Omega))$ one has to prove that $\|\varphi(u_n)(\cdot+h)-\varphi(u_n)\|_{L^2((0,T),L^2(\Omega))}\to 0$, as $h\to 0$, uniformly w.r.t. n. (For simplicity, $f=0$.)

$$\partial_t u_n(s)-\Delta\varphi(u_n(s))=0,\ s\in(t,t+h).$$

One multiplies by $\varphi(u_n(t+h))-\varphi(u_n(t))$ and integrate between t and $t+h$ and on Ω

$$\int_t^{t+h}\int_\Omega \partial_t u_n(s)(\varphi(u_n(t+h))-\varphi(u_n(t)))dxds$$
$$+\int_t^{t+h}\int_\Omega \nabla\varphi(u_n(s))\cdot(\nabla\varphi(u_n(t+h))-\nabla\varphi(u_n(t)))dxds.$$

AL method for the Stefan problem (2)

$$\int_t^{t+h}\int_\Omega \partial_t u_n(s)(\varphi(u_n(t+h))-\varphi(u_n(t)))dxds$$
$$+\int_t^{t+h}\int_\Omega \nabla\varphi(u_n(s))\cdot(\nabla\varphi(u_n(t+h))-\nabla\varphi(u_n(t)))dxds=0.$$
$$\int_\Omega (u_n(t+h))-u_n(t))(\varphi(u_n(t+h))-\varphi(u_n(t)))dx\leq$$
$$\int_t^{t+h}\int_\Omega |\nabla\varphi(u_n(s))||\nabla\varphi(u_n(t+h))|+|\nabla\varphi(u_n(s))||\nabla\varphi(u_n(t))|dxds.$$

One now integrates on $t\in(0,T-h)$, uses a Lipschitz constant for φ (denoted L) and $ab\leq(a^2+b^2)/2$

$$\int_0^{T-h}\int_\Omega(\varphi(u_n(t+h))-\varphi(u_n(t)))^2dx\leq$$
$$L\int_0^{T-h}\int_\Omega(u_n(t+h))-u_n(t))(\varphi(u_n(t+h))-\varphi(u_n(t)))dx\leq$$
$$L\sum_{i=1}^3 T_i$$

AL method for the Stefan problem (3)

$$\int_0^{T-h}\int_\Omega (\varphi(u_n(t+h)) - \varphi(u_n(t)))^2 dx \leq L(T_1 + T_2 + T_3)$$

$$T_1 = \int_0^{T-h}\int_t^{t+h}\int_\Omega |\nabla\varphi(u_n(s))|^2 dxdsdt \leq h \||\nabla\varphi(u_n)|\|_{L^2(Q)}^2$$

$$T_2 = \int_0^{T-h}\int_t^{t+h}\int_\Omega |\nabla\varphi(u_n(t+h))|^2 dxdsdt \leq h \||\nabla\varphi(u_n)|\|_{L^2(Q)}^2$$

$$T_3 = \int_0^{T-h}\int_t^{t+h}\int_\Omega |\nabla\varphi(u_n(t))|^2 dxdsdt \leq h \||\nabla\varphi(u_n)|\|_{L^2(Q)}^2$$

where $Q = \Omega \times (0, T)$.

Thanks to the $L^2((0,T), H_0^1(\Omega))$ estimate on $(\varphi(u_n))_{n\in\mathbb{N}}$, one obtains the relative compactness of this sequence in $L^2(Q)$.

Translation (in time) of $\varphi(u_n)$, at the discrete level

At the discrete level, let u_n be the approximate solution associated to mesh $\mathcal{D}_n$ and time step δt_n. A very similar proof gives

$$\int_0^{T-h} \int_\Omega (\varphi(\bar{u}_n(t+h)) - \varphi(\bar{u}_n(t)))^2 dx \leq h \||\nabla_{\mathcal{D}} \varphi(u_n)|\|^2_{L^2(Q)}$$

The only difference is due to the fact that $\partial_t u$ is replaced by a differential quotient.
For this proof, the crucial property $\overline{\varphi(u)} = \varphi(\bar{u})$ is used

Compactness, for a sequence of approximate solutions

X, B are two Banach spaces such that

- $X \subset B$ with compact embedding,

Let $T > 0$, $1 \le p < +\infty$ and $(v_n)_{n\in\mathbb{N}}$ be a sequence such that

- $(v_n)_{n\in\mathbb{N}}$ is bounded in $L^p((0,T),X)$,
- $\|v_n(\cdot + h) - v_n\|_{L^p((0,T),B)} \to 0$, as $h \to 0$, uniformly w.r.t. n.

Then there exists $v \in L^p((0,T),B)$ such that, up to a subsequence, $v_n \to v$ in $L^p((0,T),B)$.

Example: $p = 2$, $X = H_0^1(\Omega)$, $B = L^2(\Omega)$
One wants to take $v_n = \varphi(\bar{u}_n)$.

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Then there exists $v \in L^p((0,T),B)$ such that, up to a subsequence, $v_n \to v$ in $L^p((0,T),B)$.

Example: $p = 2$, $X = H^1_0(\Omega)$, $B = L^2(\Omega)$
One wants to take $v_n = \varphi(\bar{u}_n)$.
Everything is ok, except that there is no X-space...

Modified Compactness Lemma

B is a banach space ($B = L^2(Q)$)
X_n normed vector spaces ($X_n = X_{\mathcal{D}_n,0}$, $\|u\|_{X_n} = \||\nabla_{\mathcal{D}_n} u|\|_{L^2}$)
T_n a linear operator from X_n to B ($T_n(u) = \bar{u}$)
The hypothesis $X \subset B$ with compact embedding is replaced by "$u_n \in X_n$, if the sequence $(\|u_n\|_{X_n})_{n\in\mathbb{N}}$ is bounded, then the sequence $(T_n(u_n))_{n\in\mathbb{N}}$ is relatively compact in B".
With this hypothesis, let $T > 0$, $1 \leq p < +\infty$ and $(v_n)_{n\in\mathbb{N}}$ be a sequence such that $v_n \in L^p((0,T), X_n)$ for all n. Assume that

- There exists C such that $\|v_n\|_{L^p((0,T),X_n)} \leq C$ for all $n \in \mathbb{N}$
- $\|T_n(v_n)(\cdot + h) - T_n(v_n)\|_{L^p((0,T),B)} \to 0$, as $h \to 0$, uniformly w.r.t. n.

Then there exists $g \in L^p((0,T), B)$ such that, up to a subsequence, $T_n(v_n) \to g$ in $L^p((0,T), B)$.

$p = 2$, $v_n = \varphi(u_n)$. With this Compactness Lemma, one obtains that $\varphi(\bar{u}_n) \to g$ in $L^2(Q)$

Minty trick (simple version)

Let $(u_n)_{n\in\mathbb{N}}$ be a sequence of approximate solutions. One has, as $n \to \infty$,

$$\bar{u}_n \to u \text{ weakly in } L^2(Q),$$

$$\varphi(\bar{u}_n) \to g \text{ in } L^2(Q).$$

Then, the Minty trick (since φ is nondecreasing) gives $g = \varphi(u)$:
Let $w \in L^2(\Omega)$, $0 \le \int_Q (\varphi(\bar{u}_n) - \varphi(w))(\bar{u}_n - w)dxdt$ gives, as $n \to \infty$,

$$0 \le \int_Q (g - \varphi(w))(u - w)dxdt.$$

Taking $w = u + \varepsilon\psi$, with $\psi \in C_c^\infty(Q)$ and letting $\varepsilon \to 0^\pm$ leads to

$$\int_Q (g - \varphi(u))\psi dxdt = 0.$$

Then $g = \varphi(u)$ a.e.

Passing to the limit in the equation

It remains to pass to the limit in the approximate equation. This is possible thanks to some natural properties of consistency. That is to say, for any regular function ψ, as $\mathrm{size}(\mathcal{D}) \to 0$,

1. $\min_{v \in X_{\mathcal{D},0}} \|\bar{v} - \psi\|_{L^2(\Omega)} \to 0$
2. $\min_{v \in X_{\mathcal{D},0}} \||\nabla_{\mathcal{D}} v - \nabla\psi|\|_{L^2(\Omega)} \to 0$
3. $\max_{u \in X_{\mathcal{D},0}\setminus\{0\}} \frac{1}{\||\nabla_{\mathcal{D}} u|\|_{L^2(\Omega)}} \left|\int_\Omega \left(\nabla_{\mathcal{D}} u \cdot \psi + \bar{u}\,\mathrm{div}\psi\right) dx\right| \to 0$

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The hypothesis $X \subset B$ with compact embedding is replaced by "$u_n \in X_n$, if the sequence $(\|u_n\|_{X_n})_{n\in\mathbb{N}}$ is bounded, then the sequence $(T_n(u_n))_{n\in\mathbb{N}}$ is relatively compact in B".
With this hypothesis, let $T > 0$, $1 \leq p < +\infty$ and $(v_n)_{n\in\mathbb{N}}$ be a sequence such that $v_n \in L^p((0,T), X_n)$ for all n. Assume that

- There exists C such that $\|v_n\|_{L^p((0,T),X_n)} \leq C$ for all $n \in \mathbb{N}$
- $\|T_n(v_n)(\cdot + h) - T_n(v_n)\|_{L^p((0,T),B)} \to 0$, as $h \to 0$, uniformly w.r.t. n.

Then there exists $g \in L^p((0,T), B)$ such that, up to a subsequence, $T_n(v_n) \to g$ in $L^p((0,T), B)$.