Convergence of approximate solutions for Stationary compressible Stokes equations

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Fisrt step for proving the convergence of approximate solutions for the evolution compressible Navier-Stokes equations (which gives, in particular, the existence of solutions, $d = 3$, $p = \rho^\gamma$, $\gamma > \frac{3}{2}$).

Stationary compressible Stokes equations

Ω is a bounded open set of $\mathbb{R}^d$, $d = 2$ or 3, with a Lipschitz continuous boundary, $\gamma > 1$, $f \in L^2(\Omega)^d$ and $M > 0$

$$-\Delta u + \nabla p = f \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega,$$

$$\operatorname{div}(\rho u) = 0 \text{ in } \Omega, \ \rho \geq 0 \text{ in } \Omega, \ \int_\Omega \rho(x) dx = M,$$

$$p = \rho^\gamma \text{ in } \Omega$$

Functional spaces : $u \in H_0^1(\Omega)$, $p \in L^2(\Omega)$, $\rho \in L^{2\gamma}(\Omega)$

(different spaces for p and ρ in the case of Navier-Stokes if $d = 3$ and $\gamma < 3$)

Aim

Prove the existence of a weak solution to the compressible Stokes equations by the convergence of a sequence (up to a subsequence, since, up to now, no uniqueness result is available for this problem) of approximate solutions given by a numerical scheme as the mesh size goes to 0

Simpler result: "continuity" with respect to the data

$$-\Delta u_n + \nabla p_n = f_n \text{ in } \Omega, \quad u_n = 0 \text{ on } \partial\Omega,$$

$$\operatorname{div}(\rho_n u_n) = 0 \text{ in } \Omega, \ \rho_n \geq 0 \text{ in } \Omega, \ \int_\Omega \rho_n(x)dx = M_n,$$

$$p_n = \rho_n^\gamma \text{ in } \Omega$$

$f_n \to f$ in $(L^2(\Omega))^d$ and $M_n \to M$. Then, up to a subsequence,

- $u_n \to u$ in $L^2(\Omega)^d$ and weakly in $H_0^1(\Omega)^d$,
- $p_n \to p$ in $L^q(\Omega)$ for any $1 \leq q < 2$ and weakly in $L^2(\Omega)$,
- $\rho_n \to \rho$ in $L^q(\Omega)$ for any $1 \leq q < 2\gamma$ and weakly in $L^{2\gamma}(\Omega)$,

where (u, p, ρ) is a weak solution of the compressible Stokes equations (with f and M as data)

The case $\gamma = 1$ is also possible, but we obtain only weak convergence of p_n and ρ_n in $L^2(\Omega)$ (strong conv. are not needed).

Preliminary lemma

$\rho \in L^{2\gamma}(\Omega)$, $\rho \geq 0$ a.e. in Ω, $u \in (H_0^1(\Omega))^d$, $\mathrm{div}(\rho u) = 0$, then:

$$\int_\Omega \rho \mathrm{div}(u) dx = 0$$

$$\int_\Omega \rho^\gamma \mathrm{div}(u) dx = 0$$

Proof of the preliminary result

For simplicity : $\rho \in C^1(\bar{\Omega})$, $\rho \geq \alpha$ a.e. in Ω, $\alpha > 0$, $1 < \beta \leq \gamma$. Take $\varphi = \rho^{\beta-1}$ as test function in $\mathrm{div}(\rho u) = 0$:

$$\int_\Omega \rho u \cdot \nabla \rho^{\beta-1} dx = (\beta - 1) \int_\Omega \rho^{\beta-1} u \cdot \nabla \rho dx = 0.$$

Then

$$\frac{\beta - 1}{\beta} \int_\Omega u \cdot \nabla \rho^\beta dx = 0,$$

and finally

$$\int_\Omega \rho^\beta \mathrm{div}(u) dx = 0.$$

Two cases :

$\beta = \gamma$

$\beta = 1 + \frac{1}{k}$ and $k \to \infty$ (or $\varphi = \ln(\rho)$)

Variant of the preliminary lemma, for numerical schemes

In the case of the approximation by a numerical scheme, we will have a sequence (ρ_n, u_n) satisfying an approximation of $\mathrm{div}(\rho_n u_n) = 0$ and taking also into account the condition $\int_\Omega \rho_n dx = M_n$. We will use a weak version of the preceding lemma, namely :

$$\liminf_{n\to\infty} \int_\Omega \rho_n^\gamma \mathrm{div}\, u_n dx \le 0,$$

$$\liminf_{n\to\infty} \int_\Omega \rho_n \mathrm{div}\, u_n dx \le 0.$$

Estimate on u_n

Taking u_n as test function in $-\Delta u_n + \nabla p_n = f_n$:

$$\int_\Omega \nabla u_n : \nabla u_n \, dx - \int_\Omega p_n \mathrm{div}(u_n)\, dx = \int_\Omega f_n \cdot u_n \, dx.$$

But $p_n = \rho_n^\gamma$ a.e. and $\mathrm{div}(\rho_n u_n) = 0$, then $\int_\Omega p_n \mathrm{div}(u_n)\, dx = 0$.
This gives an estimate on u_n:

$$\|u_n\|_{(H_0^1(\Omega))^d} \leq C_1.$$

Estimate on p_n, Nečas Lemma

Let $q \in L^2(\Omega)$ s.t. $\int_\Omega q dx = 0$.
Then, there exists $v \in (H_0^1(\Omega))^d$ s.t.

$$\mathrm{div}(v) = q \text{ a.e. in } \Omega,$$

$$\|v\|_{(H_0^1(\Omega))^d} \leq C_2 \|q\|_{L^2(\Omega)},$$

where C_2 only depends on Ω.

Estimate on p_n

$m_n = \frac{1}{|\Omega|}\int_\Omega p_n dx$, $v_n \in H_0^1(\Omega)^d$, $\mathrm{div}(v_n) = p_n - m_n$.
Taking v_n as test function in $-\Delta u_n + \nabla p_n = f_n$:

$$\int_\Omega \nabla u_n : \nabla v_n \, dx - \int_\Omega p_n \mathrm{div}(v_n) \, dx = \int_\Omega f_n \cdot v_n \, dx.$$

Using $\int_\Omega \mathrm{div}(v_n) dx = 0$:

$$\int_\Omega (p_n - m_n)^2 dx = \int_\Omega (f_n \cdot v_n - \nabla u_n : \nabla v_n) dx.$$

Since $\|v_n\|_{(H_0^1(\Omega))^d} \leq C_2 \|p_n - m_n\|_{L^2(\Omega)}$ and $\|u_n\|_{(H_0^1(\Omega))^d} \leq C_1$, the preceding inequality leads to:

$$\|p_n - m_n\|_{L^2(\Omega)} \leq C_3.$$

where C_3 only depends on the L^2-bound of $(f_n)_{n\in\mathbb{N}}$ and on Ω.

Estimates on p_n and ρ_n

$$\|p_n - m_n\|_{L^2(\Omega)} \leq C_3.$$

$$\int_\Omega p_n^{\frac{1}{\gamma}} dx = \int_\Omega \rho_n dx \leq \sup\{M_p, p \in \mathbb{N}\}.$$

Then:

$$\|p_n\|_{L^2(\Omega)} \leq C_4;$$

where C_4 only depends on the L^2-bound of $(f_n)_{n\in\mathbb{N}}$, the bound of $(M_n)_{n\in\mathbb{N}}$, γ and Ω.

$p_n = \rho_n^\gamma$ a.e. in Ω, then:

$$\|\rho_n\|_{L^{2\gamma}(\Omega)} \leq C_5 = C_4^{\frac{1}{\gamma}}.$$

Weak-convergence on u_n, p_n, ρ_n

Thanks to the estimates on u_n, p_n, ρ_n, it is possible to assume (up to a subsequence) that, as $n \to \infty$:

$$u_n \to u \text{ in } L^2(\Omega)^d \text{ and weakly in } H_0^1(\Omega)^d,$$

$$p_n \to p \text{ weakly in } L^2(\Omega),$$

$$\rho_n \to \rho \text{ weakly in } L^{2\gamma}(\Omega).$$

Passing to the limit on the equations, except EOS

Linear equation :

$$-\Delta u + \nabla p = f \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega,$$

Strong times weak convergence

$$\mathrm{div}(\rho u) = 0 \text{ in } \Omega,$$

L^1-weak convergence of ρ_n gives positivity of ρ and convergence of mass:

$$\rho \geq 0 \text{ in } \Omega, \int_\Omega \rho(x)dx = M.$$

Question (if $\gamma > 1$):

$$p = \rho^\gamma \text{ in } \Omega \text{ ?}$$

Idea : prove $\int_\Omega p_n \rho_n dx \to \int_\Omega p\rho dx$ and deduce a.e. convergence (of p_n and ρ_n) and $p = \rho^\gamma$.

$\nabla : \nabla = \mathrm{div}\,\mathrm{div} + \mathrm{curl} \cdot \mathrm{curl}$

For all $\bar{u}, \bar{v}$ in $H_0^1(\Omega)^d$,

$$\int_\Omega \nabla \bar{u} : \nabla \bar{v} dx = \int_\Omega \mathrm{div}(\bar{u})\mathrm{div}(\bar{v}) dx + \int_\Omega \mathrm{curl}(\bar{u}) \cdot \mathrm{curl}(\bar{v}) dx.$$

Then, the weak form of $-\Delta u_n + \nabla p_n = f_n$ gives for all $\bar{v}$ in $H_0^1(\Omega)^d$

$$\int_\Omega \mathrm{div}(u_n)\mathrm{div}(\bar{v}) dx + \int_\Omega \mathrm{curl}(u_n) \cdot \mathrm{curl}(\bar{v}) dx - \int_\Omega p_n \mathrm{div}(\bar{v})\, dx = \int_\Omega f_n \cdot \bar{v}\, dx$$

Choice of $\bar{v}$? $\mathrm{curl}(\bar{v}) = 0$, $\mathrm{div}(\bar{v}) = \rho_n \ldots$

Curl-free test function

Let B be a ball containing Ω and $w_n \in H_0^1(B)$, $-\Delta w_n = \rho_n$,

$$v_n = \nabla w_n$$

- $v_n \in (H^1(\Omega))^d$,
- $\mathrm{div}(v_n) = \rho_n$ a.e. in Ω,
- $\mathrm{curl}(v_n) = 0$ a.e. in Ω,
- $\|v_n\|_{(H^1(\Omega))^d} \leq C_6 \|\rho_n\|_{L^2(\Omega)}$, where C_6 only depends on Ω.

Then, up to a subsequence,

$v_n \to v$ in $L^2(\Omega)$ and weakly in $H^1(\Omega)$,

$\mathrm{curl}(v) = 0$, $\mathrm{div}(v) = \rho$.

(Remark : $\|v_n\|_{(H^2(\Omega))^d} \leq C_6 \|\rho_n\|_{H^1(\Omega)}$)

Proving $\int_\Omega (p_n - \mathrm{div}(u_n))\rho_n \varphi dx \to \int_\Omega (p - \mathrm{div}(u))\rho\varphi dx$

Let $\varphi \in C_c^\infty(\Omega)$ (so that $v_n\varphi \in H_0^1(\Omega)^d$)). Taking $\bar{v} = v_n\varphi$:

$$\int_\Omega \mathrm{div}(u_n)\mathrm{div}(v_n\varphi)dx + \int_\Omega \mathrm{curl}(u_n)\cdot \mathrm{curl}(v_n\varphi)dx - \int_\Omega p_n \mathrm{div}(v_n\varphi)\, dx$$
$$= \int_\Omega f_n \cdot (v_n\varphi)\, dx.$$

But, $\mathrm{div}(v_n\varphi) = \rho_n\varphi + v_n\cdot\nabla\varphi$ and $\mathrm{curl}(v_n\varphi) = L(\varphi)v_n$, where $L(\varphi)$ is a matrix involving the first order derivatives of φ. Then:

$$\int_\Omega (\mathrm{div}(u_n) - p_n)\rho_n\varphi dx = \int_\Omega f_n \cdot (v_n\varphi)dx$$
$$- \int_\Omega \mathrm{div}(u_n)v_n\cdot\nabla\varphi dx - \int \mathrm{curl}(u_n)\cdot L(\varphi)v_n + \int_\Omega p_n v_n\cdot\nabla\varphi dx.$$

Weak convergence of u_n in $H_0^1(\Omega)^d$, weak convergence of p_n in $L^2(\Omega)$ and convergence of v_n and f_n in $L^2(\Omega)^d$:

$$\lim_{n\to\infty} \int_\Omega (\mathrm{div}(u_n) - p_n)\rho_n\varphi dx = \int_\Omega f\cdot(v\varphi)dx$$
$$- \int_\Omega \mathrm{div}(u)v\cdot\nabla\varphi dx - \int \mathrm{curl}(u)\cdot L(\varphi)v + \int_\Omega pv\cdot\nabla\varphi dx.$$

Proving $\int_\Omega (p_n - \mathrm{div}(u_n))\rho_n\varphi dx \to \int_\Omega (p - \mathrm{div}(u))\rho\varphi dx$

But, since $-\Delta u + \nabla p = f$:

$$\int_\Omega \mathrm{div}(u)\mathrm{div}(v\varphi)dx + \int_\Omega \mathrm{curl}(u)\cdot\mathrm{curl}(v\varphi)dx - \int_\Omega p\mathrm{div}(v\varphi)\,dx \\ = \int_\Omega f\cdot(v\varphi)\,dx.$$

which gives (using $\mathrm{div}(v) = \rho$ and $\mathrm{curl}(v) = 0$):

$$\int_\Omega (\mathrm{div}(u) - p)\rho\varphi dx = \int_\Omega f\cdot(v\varphi)dx \\ - \int_\Omega \mathrm{div}(u)v\cdot\nabla\varphi dx - \int \mathrm{curl}(u)\cdot L(\varphi)v + \int_\Omega pv\cdot\nabla\varphi dx.$$

Then:

$$\lim_{n\to\infty}\int_\Omega (p_n - \mathrm{div}(u_n))\rho_n\varphi dx = \int_\Omega (p - \mathrm{div}(u))\rho\varphi dx.$$

Proving $\int_\Omega (p_n - \mathrm{div}(u_n))\rho_n dx \to \int_\Omega (p - \mathrm{div}(u))\rho dx$

Lemma : $F_n \to F$ in $D'(\Omega)$, $(F_n)_{n\in\mathbb{N}}$ bounded in L^q for some $q > 1$. Then $F_n \to F$ weakly in L^1.

With $F_n = (p_n - \mathrm{div}(u_n))\rho_n$, $F = (p - \mathrm{div}(u))\rho$ and since $\gamma > 1$, the lemma gives

$$\int_\Omega (p_n - \mathrm{div}(u_n))\rho_n dx \to \int_\Omega (p - \mathrm{div}(u))\rho dx.$$

Proving $\int_\Omega p_n \rho_n dx \to \int_\Omega p \rho dx$

$$\int_\Omega (p_n - \mathrm{div}(u_n))\rho_n dx \to \int_\Omega (p - \mathrm{div}(u))\rho dx.$$

But since $\mathrm{div}(\rho_n u_n) = 0$, $\mathrm{div}(\rho u) = 0$, the preliminary lemma gives:

$$\int_\Omega \mathrm{div}(u_n)\rho_n dx = 0, \ \int_\Omega \mathrm{div}(u)\rho dx = 0;$$

Then:

$$\int_\Omega p_n \rho_n dx \to \int_\Omega p \rho dx.$$

a.e. convergence of ρ_n and p_n

Let $G_n = (\rho_n^\gamma - \rho^\gamma)(\rho_n - \rho) \in L^1(\Omega)$ and $G_n \geq 0$ a.e. in Ω.
Futhermore $G_n = (p_n - \rho^\gamma)(\rho_n - \rho) = p_n\rho_n - p_n\rho - \rho^\gamma\rho_n + \rho^\gamma\rho$ and:

$$\int_\Omega G_n dx = \int_\Omega p_n\rho_n dx - \int_\Omega p_n\rho dx - \int_\Omega \rho^\gamma\rho_n dx + \int_\Omega \rho^\gamma\rho dx.$$

Using the weak convergence in $L^2(\Omega)$ of p_n and ρ_n and $\int_\Omega p_n\rho_n dx \to \int_\Omega p\rho dx$:

$$\lim_{n\to\infty} \int_\Omega G_n dx = 0,$$

Then (up to a subsequence), $G_n \to 0$ a.e. and then $\rho_n \to \rho$ a.e. (since $y \mapsto y^\gamma$ is an increasing function on $\mathbb{R}_+$). Finally:

$\rho_n \to \rho \text{ in } L^q(\Omega) \text{ for all } 1 \leq q < 2\gamma$,

$p_n = \rho_n^\gamma \to \rho^\gamma \text{ in } L^q(\Omega) \text{ for all } 1 \leq q < 2$,

and $p = \rho^\gamma$.

Additional difficulty for stat. comp. NS equations

Ω is a bounded open set of $\mathbb{R}^d$, $d = 2$ or 3, with a Lipschitz continuous boundary, $\gamma > 1$, $f \in L^2(\Omega)^d$ and $M > 0$

$$-\Delta u + \mathrm{div}(\rho u \otimes u) + \nabla p = f \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega,$$

$$\mathrm{div}(\rho u) = 0 \text{ in } \Omega, \ \rho \geq 0 \text{ in } \Omega, \ \int_\Omega \rho(x) dx = M,$$

$$p = \rho^\gamma \text{ in } \Omega$$

$d = 2$: no aditional difficulty

$d = 3$: no additional difficulty if $\gamma \geq 3$. But for $\gamma < 3$, no estimate on p in $L^2(\Omega)$.

Estimates in the case of NS equations, $\frac{3}{2} < \gamma < 3$

Estimate on u : Taking u as test function in the momentum leads to an estimate on u in $(H_0^1(\Omega)^d$ since

$$\int_\Omega \rho u \otimes u : \nabla u dx = 0.$$

Then, we have also an estimate on u in $L^6(\Omega)^d$ (using Sobolev embedding).

Estimate on p in $L^q(\Omega)$, with some $1 < q < 2$ and $q = 1$ when $\gamma = \frac{3}{2}$ (using Nečas Lemma in some L^r instead of L^2).

Estimate on ρ in $L^q(\Omega)$, with some $\frac{3}{2} < q < 6$ and $q = \frac{3}{2}$ when $\gamma = \frac{3}{2}$ (since $p = \rho^\gamma$).

Remark : $\rho u \otimes u \in L^1(\Omega)$, since $u \in L^6(\Omega)^d$ and $\rho \in L^{\frac{3}{2}}(\Omega)$ (and $\frac{1}{6} + \frac{1}{6} + \frac{2}{3} = 1$).

NS equations, $\gamma < 3$, how to pass to the limit in the EOS

We prove

$$\lim_{n\to\infty} \int_\Omega p_n \rho_n^\theta dx = \int_\Omega p \rho^\theta dx,$$

with some convenient choice of $\theta > 0$ instead of $\theta = 1$.

This gives, as for $\theta = 1$, the a.e. convergence (up to a subsequence) of p_n and ρ_n.