Resonance and nonlinearities

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Notations

Notations:

$$(\cdot)_t = \frac{\partial(\cdot)}{\partial t},$$

$$(\cdot)_x = \frac{\partial(\cdot)}{\partial x},$$

$t \in \mathbb{R}_+$.

$x \in \mathbb{R}$, but extensions to $x \in \mathbb{R}^d$, $d = 2$ or 3 are possible.

Linear hyperbolic problems

data: $A \in M_p(\mathbb{R})$, $p \geq 1$, $W_0 \in L^q(\mathbb{R}, \mathbb{R}^p)$, $q \in [1, \infty]$.
Unknown: $W \; : \; \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}^p$,
Equation: $W_t + AW_x = 0$ in $\mathbb{R} \times \mathbb{R}_+$,
Initial condition : $W(x, 0) = W_0$.

"Genuine-hyperbolic" system : The eignevalues of A are real and A is diagonalizable. (simple case : strictly hyperbolic, the eigenvalues are real and simple.)

In this case, the problem has a unique (weak) solution.

"Resonant-hyperbolic" system : The eignevalues of A are real and A is not diagonalizable (then, $p > 1$).

Linear resonant problem

The Cauchy problem for a linear resonant problem is ill-posed in L^∞(or in L^1, $L^2 \dots$, but well posed in C^∞).

Riemann problem for a typical example:

$$\begin{bmatrix} u \\ v \end{bmatrix}_t + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}_x = 0,$$

$$\begin{bmatrix} u(x,0) \\ v(x,0) \end{bmatrix} = \begin{bmatrix} u_l \\ v_l \end{bmatrix}, \text{ if } x < 0, \text{ and } \begin{bmatrix} u_r \\ v_r \end{bmatrix}, \text{ if } x > 0,$$

The solution is , for all $t > 0$, $v(\cdot, t) = v(\cdot, 0)$ and

$$u(\cdot, t) = u_l 1_{\mathbb{R}_-} + u_r 1_{\mathbb{R}_+} + t(v_l - v_r)\delta_0.$$

Non linear hyperbolic system

Unknown: $W \ : \ \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}^p$, $p > 1$
Equation: $W_t + A(W)W_x = 0$

"Simple case": For all admissible $W \in \mathbb{R}^p$, $A(W)$ is genuine-hyperbolic (real eigenvalues and diagonalizable).

Questions :

1. Existence and uniqueness of a solution if $A(W)$ is only resonant-hyperbolic (real eigenvalues and not diagonalizable) for some admisible values of $W \in \mathbb{R}^p$.

2. Behaviour of numerical schemes using a linearization of the equation.

3. Other interesting question : systems where $A(W)$ is not hyperbolic for some admissible values of $W \in \mathbb{R}^p$.

Nonlinear resonant problem, academic example

Academic simple example:

$$\begin{aligned} u_t + (au)_x &= 0, \\ a_t &= 0, \end{aligned}$$

$$\begin{bmatrix} u(x,0) \\ a(x,0) \end{bmatrix} = \begin{bmatrix} u_l \\ a_l \end{bmatrix}, \text{ if } x < 0, \text{ and } \begin{bmatrix} u_r \\ a_r \end{bmatrix}, \text{ if } x > 0,$$

has no weak solution in L^∞ if $a_l > 0$, $a_r < 0$ and $a_l u_l \neq a_r u_r$ (and has infinetely many solution if $a_l < 0$ and $a_r > 0$).

Nonlinear resonant problem, academic example

$$u_t + (au)_x = 0,$$
$$a_t = 0,$$

is equivalent (for regular solution) to $W_t + A(W)W_x = 0$, with $W = \begin{bmatrix} u \\ a \end{bmatrix}$ and $A(W) = \begin{bmatrix} a & u \\ 0 & 0 \end{bmatrix}$. Resonance occurs when $a = 0$ and $u \neq 0$.

The Riemann problem for the nonlinear system is ill posed in L^∞ provided that 0 is between a_r and a_l (except if $u_r = u_l = 0$).

This nonlinear case is “worse” than the linear case.

Hyperbolic equation with a discontinuous coefficient

Two phase flow in an heterogeneous porous medium:
unknown: $u \ : \ \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$:

$$u_t(x,t) + (kg(u))_x = 0,$$

$k(x) = k_l$, for $x < 0$,
$k(x) = k_r$, for $x > 0$,
$k_l, k_r > 0$, $k_l \neq k_r$,
$g \ : \ [0,1] \to \mathbb{R}$, Lipschitz continuous and such that
$g(0) = g(1) = 0$. Example: $g(u) = u(1-u)$

This hyperbolic equation with a discontinuous coefficient can be viewed has a conservative 2×2 hyperbolic system, adding k has an unknown and the equation $k_t = 0$.

Hyperbolic equation with a discontinuous coefficient

$$\begin{aligned} u_t(x,t) + (kg(u))_x &= 0, \\ k_t &= 0. \end{aligned}$$

$$W = \begin{bmatrix} u \\ k \end{bmatrix} \text{ and } F(W) = \begin{bmatrix} kg(u) \\ 0 \end{bmatrix},$$

$$W_t + (F(W))_x = 0,$$

or equivalently (for regular solutions), with $A(W) = DF(W)$:

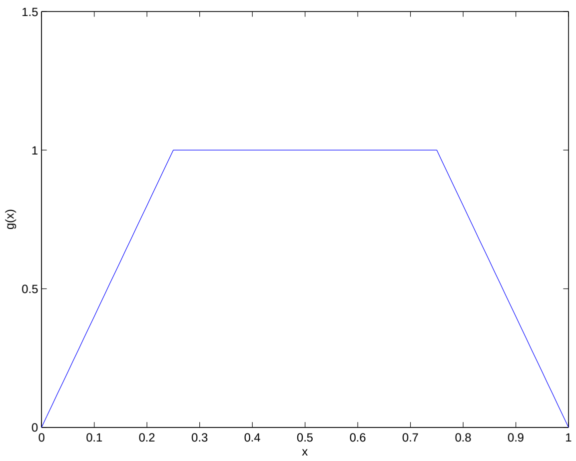
$$W_t + A(W)W_x = 0.$$

Hyperbolic equation with a discontinuous coefficient

Unknown: $W \ : \ \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}^p$, $p = 2$
Equation: $W_t + A(W)W_x = 0$

$W = \begin{bmatrix} u \\ k \end{bmatrix}$, $A(W) = \begin{bmatrix} kg'(u) & g(u) \\ 0 & 0 \end{bmatrix}$ which is not diagonalizable if $g'(u) = 0$ and $g(u) \neq 0$ (if $g(u) = u(1-u)$, this is the case for $u = \frac{1}{2}$).

Hyperbolic equation with a discontinuous coefficient



Resonance occurs for all (k, u) with $u \in (\frac{1}{4}, \frac{3}{4})$.

Hyperbolic system with a source term

Saint Venant Equations with topography (nonflat bottom)
unknowns: $h, u \;:\; \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$ (with $h > 0$):

$$\begin{aligned} & h_t + (hu)_x = 0, \\ & (hu)_t + (hu^2 + \tfrac{1}{2} g h^2)_x = -g h z_x, \end{aligned}$$

g is a given constant and z is a given function of x.
This 2×2 conservative hyperbolic system with a source term can be viewed has a nonconservative 3×3 hyperbolic system, adding z has an unknown and the equation $z_t = 0$.

Hyperbolic system with a source term

$$\begin{aligned} &h_t + (hu)_x = 0, \\ &(hu)_t + (hu^2 + \tfrac{1}{2}gh^2)_x + ghz_x = 0, \\ &z_t = 0. \end{aligned}$$

$$W = \begin{bmatrix} u \\ hu \\ z \end{bmatrix}, F(W) = \begin{bmatrix} hu \\ \frac{(hu)^2}{h} + \frac{1}{2}gh^2 \\ 0 \end{bmatrix} \text{ and}$$

$$B(W) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & gh \\ 0 & 0 & 0 \end{bmatrix},$$

$$W_t + (F(W))_x + B(W)W_x = 0,$$

or equivalently (for regular solutions):

$W_t + A(W)W_x = 0,$ (with $A(W) = DF(W) + B(W)$).

Hyperbolic system with a source term

Unknown: $W \ : \ \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}^p$, $p = 3$
Equation: $W_t + A(W)W_x = 0$

$$W = \begin{bmatrix} u \\ hu \\ z \end{bmatrix}, A(W) = \begin{bmatrix} 0 & 1 & 0 \\ -u^2 + gh & 2u & gh \\ 0 & 0 & 0 \end{bmatrix}.$$

Eigenvalues of $A(W)$ are $u \pm c$ and 0, with $c = \sqrt{gh}$.
$A(W)$ is not diagonalizable if $u - c = 0$ or $u + c = 0$ (and $h > 0$).

Nonlinear resonant problem, porous media

First example

$$\begin{aligned} &u_t + (kg(u))_x = 0, \\ &k_t = 0, \end{aligned}$$

$g : [0,1] \to \mathbb{R}_+$, Lipschitz continuous and such that $g(0) = g(1) = 0$. The Cauchy problem is well posed in the following sense:

$k(x) = k_l > 0$ if $x < 0$, $k(x) = k_r > 0$ if $x > 0$, $u(\cdot, 0) = u_0 \in L^\infty$, $0 \le u_0 \le 1$. Then, the Cauchy problem has a unique entropy weak solution.

Karlsen-Risebro-Towers

Seguin-Vovelle, Bachmann-Vovelle

Bachmann: $k(x)g(u) \rightsquigarrow g(x,u)$, $g(x,0) = g(x,1) = 0$.

Entropy weak solution

$u_0 \in L^\infty(\mathbb{R})$, $0 \le u_0 \le 1$ a.e.. $u \in L^\infty(\mathbb{R}_+ \times \mathbb{R})$ is an entropy weak solution if:
$\forall \kappa \in [0,1]$, $\forall \varphi \in C_c^\infty(\mathbb{R}_+ \times \mathbb{R}, \mathbb{R}_+)$,

$$
\begin{aligned}
&\int_0^\infty \int_{\mathbb{R}} |u(t,x) - \kappa|\, \varphi_t(t,x)\, dt\, dx \\
+ &\int_0^\infty \int_{\mathbb{R}} \big(k(x)\Phi(x, u(t,x), \kappa)\big)\, \varphi_x(t,x)\, dx\, dt \\
+ &\int_{\mathbb{R}} |u_0(x) - \kappa|\, \varphi(0,x)\, dx + |k_L - k_R| \int_0^\infty g(\kappa)\, \varphi(t,0)\, dt \ge 0,
\end{aligned}
$$

$$\Phi(u,\kappa) = \mathrm{sgn}(u-\kappa)(g(u) - g(\kappa))$$

sketch of proof (1)

Similar methods for proving existence:

1. Replace k par k_ϵ (regular) and pass to limit as $\epsilon \to 0$.
2. Add $-\epsilon u_{xx}$ and pass to limit as $\epsilon \to 0$.
3. Pass to the limit on (monotone) numerical schemes.

An L^∞ estimate on u_ϵ is easy, $0 \leq u_\epsilon \leq 1$ a.e..

Main difficulty for existence (even if u_0 regular): Prove that $g(u_\epsilon)$ converge to $g(u)$ (a.e. convergence of u_ϵ to u)

Main difficulty for uniqueness : existence of trace for u on $\{x = 0\}$.

sketch of proof (2)

For this 2 difficulties, a technical (but not necessary) assumption is used by some authors : $g \in C^2$ and $\text{mes}(\{s \in [0,1] \, : \, g''(s) = 0\}) = 0$.
Existence by Temple function, compensated compactness,
Uniqueness thanks to trace for u on $\{x = 0\}$.

without this assumption of nonlinearity on g. Existence and uniqueness result are obtain using entropy process solution (or Young measures) and a kinetic formulation.

Let Ω be an open set of $\mathbb{R}^N$ and $(u_n)_{n\in\mathbb{N}}$ bounded in $L^\infty(\Omega)$.
Then, there exists a subsequence, still denoted $(u_n)_{n\in\mathbb{N}}$ and $u \in L^\infty(\Omega \times (0,1))$ such that:

$$\int_\Omega \theta(u_n(x))\psi(x)\,dx \to \int_0^1 \int_\Omega \theta(u(x,\alpha))\psi(x)\,dx\,d\alpha, \text{ quand } n \to +\infty$$
$$\forall \psi \in L^1(\Omega),\ \forall \theta \in \mathcal{C}(\mathbb{R},\mathbb{R}).$$

DiPerna, Tartar

Entropy process solution

$u_0 \in L^\infty(\mathbb{R})$, $0 \le u_0 \le 1$ a.e..
$u \in L^\infty(\mathbb{R}_+ \times \mathbb{R} \times [0,1])$ is an entropy process solution if:
$\forall \kappa \in [0,1]$, $\forall \varphi \in \mathcal{C}_c^\infty(\mathbb{R}_+ \times \mathbb{R}, \mathbb{R}_+)$,

$$\begin{aligned}
&\int_0^1 \int_0^\infty \int_{\mathbb{R}} |u(t,x,\alpha)-\kappa)|\,\varphi_t(t,x)\,dt\,dx\,d\alpha \\
+ &\int_0^1 \int_0^\infty \int_{\mathbb{R}} \big(k(x)\Phi(x,u(t,x,\alpha),\kappa)\big)\varphi_x(t,x)\,dx\,dt\,d\alpha \\
+ &\int_{\mathbb{R}} |u_0(x)-\kappa)|\,\varphi(0,x)\,dx \; + |k_L - k_R| \int_0^\infty g(\kappa)\,\varphi(t,0)\,dt \ge 0,
\end{aligned}$$

$\Phi(u,\kappa) = \mathrm{sgn}(u-\kappa)(g(u)-g(\kappa))$.
If k is regular, uniqueness is obtained by doubling variable technique (Krushkov)

Kinetic process solution

$u_0 \in L^\infty(\mathbb{R})$, $0 \le u_0 \le 1$ a.e.. $u \in L^\infty(\mathbb{R}_+ \times \mathbb{R} \times (0,1))$ is a kinetic process solution if it exists $m_\pm \in \mathcal{C}_{w-\star}(\mathbb{R}, \mathcal{M}_+(\mathbb{R}_+ \times \mathbb{R}))$ such that :
$\forall \varphi \in C_c^\infty(\mathbb{R}^3)$,

$$
\begin{aligned}
&\int_0^1 \int_{\mathbb{R}_+\times\mathbb{R}\times\mathbb{R}} h_\pm(t,x,\alpha,\xi)(\varphi_t + k(x)a(\xi)\varphi_x)\, dt\, dx\, d\xi\, d\alpha \\
&+ \int_{\mathbb{R}\times\mathbb{R}} h_\pm^0(x,\xi)\varphi_{|_{t=0}}\, d\xi\, dx - \int_{\mathbb{R}_+\times\mathbb{R}} (k_L - k_R)^\pm a(\xi)\varphi_{|_{x=0}}\, d\xi\, dt \\
&= \int_{\mathbb{R}_+\times\mathbb{R}\times\mathbb{R}} \partial_\xi \varphi\, dm_\pm .
\end{aligned}
$$

with $a(\xi) = g'(\xi)$.
and
$h_\pm(t,x,\alpha,\xi) = \mathrm{sgn}_\pm(u(t,x,\alpha) - \xi)$
$h_\pm^0(x,\xi) = \mathrm{sgn}_\pm(u_0(x) - \xi)$.

Kinetic process solution

Entropy process solution is equivalent to kinetic process solution

Uniqueness following the case "k constant" and without α

Brenier; Lions, Perthame et Tadmor; Perthame

Nonlinear resonant problem, Saint Venant

Second example

Saint Venant Equations with topography. The Riemann problem:

$$
\begin{aligned}
&h_t + (hu)_x = 0,\\
&(hu)_t + (hu^2 + \tfrac{1}{2}gh^2)_x + ghz_x = 0,\\
&z_t = 0
\end{aligned}
$$

$$
\begin{bmatrix} h \\ hu \\ z \end{bmatrix}(x,0) = \begin{bmatrix} h_l \\ (hu)_l \\ z_l \end{bmatrix}, \text{ if } x < 0, \text{ and } \begin{bmatrix} h_r \\ (hu)_r \\ z_r \end{bmatrix}, \text{ if } x > 0,
$$

has one (sometimes three...) solution (composed of constant states and waves), satisfying a classical entropy condition, assuming continuity of (the Riemann invariants) hu and ψ at the contact discontinuity (at $x = 0$) with $\psi = \frac{1}{2}u^2 + g(h + z)$.
Chinnayya-LeRoux-Seguin, Goatin-LeFloch

Nonlinear resonant problem, phase transition

Third example

Isentropic Euler Equations with an EOS taking into account a simple model of “phase transition”, that is:

$$
\begin{aligned}
p &= a_1\rho, \text{ if } 0 < \rho < \rho_1,\\
p &= a_1\rho_1, \text{ if } \rho_1 \le \rho \le \rho_2,\\
p &= a_2\rho, \text{ if } \rho_2 < \rho,
\end{aligned}
$$

with ρ_1, ρ_2, a_1, a_2 given constants, $0 < \rho_1 < \rho_2$, $0 < a_1 < a_2$.

$$
\begin{aligned}
&\rho_t + (\rho u)_x = 0,\\
&(\rho u)_t + (\rho u^2 + p)_x = 0.
\end{aligned}
$$

For $\rho_1 \le \rho \le \rho_2$ and any u, the system is resonant (with u as eigenvalue, and the 2 genuinely nonlinear fields lead to a linear degenerate field). But the Riemann problem is well posed. Recent result of Godlewski-Seguin.

$$W_t + (F(W))_x + B(W)W_x = 0,$$

$$W(\cdot, 0) = W_0.$$

Time step: k, $t_n = nk$

Space step: h, $x_{i+\frac{1}{2}} = (i + \frac{1}{2})h$

Approximate solution for $x \in (x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}})$ and $t = t_n$: W_i^n

$$W_i^0 = \frac{1}{h} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} W_0(x)dx, \; i \in \mathbb{Z}.$$

Riemann problem

$$W_t + (F(W))_x + B(W)W_x = 0,$$

$$W(x,0) = \begin{cases} W_l \text{ if } x < 0, \\ W_r \text{ if } x > 0. \end{cases}$$

Let W be the (or a) self similar solution, that is:

$$W(x,t) = R(\frac{x}{t}, W_l, W_r)$$

and set

$$W^{\star,\pm}(W_l, W_r) = R(0^{\pm}, W_l, W_r).$$

For $i \in \mathbb{Z}$, $n \geq 0$:

$$\frac{W_i^{n+1} - W_i^n}{k} + F_{i+\frac{1}{2}}^{n,-} - F_{i-\frac{1}{2}}^{n,+} + B(W_i^n)(W_{i+\frac{1}{2}}^{n,-} - W_{i-\frac{1}{2}}^{n,+}) = 0,$$

with $F_{i+\frac{1}{2}}^{n,\pm} = F(W_{i+\frac{1}{2}}^{n,\pm})$,
$W_{i+\frac{1}{2}}^{n,\pm} = W^{\star,\pm}(W_i^n, W_{i+1}^n)$.

CFL condition: $k \leq Ch$.

Godunov scheme, particular cases

$$\frac{W_i^{n+1}-W_i^n}{k}+F^{n,-}_{i+\frac12}-F^{n,+}_{i-\frac12}+B(W_i^n)(W^{n,-}_{i+\frac12}-W^{n,+}_{i-\frac12})=0,$$

1. conservative case: $F^{n,+}_{i+\frac12}=F^{n,-}_{i+\frac12}$, even if $W^{n,+}_{i+\frac12}\neq W^{n,-}_{i+\frac12}$ (thanks to Rankine-Hugoniot condition on the Riemann problem).
2. In the case of Saint Venant equations with topography, one has $z^{n,-}_{i+\frac12}=z^{n,+}_{i-\frac12}=z_i^n$ and then $B(W_i^n)(W^{n,-}_{i+\frac12}-W^{n,+}_{i-\frac12})=0$. The non conservativity of the equation (that is the source term) appears only in the fact that, generally, $F^{n,+}_{i+\frac12}\neq F^{n,-}_{i+\frac12}$.

Linearized system

- Initial system: $W_t + A(W)W_x = 0$, $A(W) = DF(W) + B(W)$.
- Linearized system: $W_t + A(\overline{W})W_x = 0$, with some fixed $\overline{W} \in \mathbb{R}^p$.
- Initial system with a change of unknown: $Y = \phi(W)$. ϕ is not necessarily invertible, but one assumes that there exists C, G, D, such that $A(W) = C(Y)$, $F(W) = G(Y)$, $B(W)W_x$=$D(W)Y_x$. Then $W_t + A(W)W_x$ leads to

$$Y_t + C(W)Y_x = 0.$$

- Linearized system with a change of unknown: $Y = \phi(W)$,

$$Y_t + C(\overline{W})Y_x = 0,$$

with some fixed $\overline{W} \in \mathbb{R}^p$.

linearized Riemann problem

$$Y_t + C(\overline{W})Y_x = 0,$$

$$Y(x,0) = \begin{cases} Y_l = \phi(W_l) \text{ if } x < 0, \\ Y_r = \phi(W_r) \text{ if } x > 0. \end{cases}$$

For instance, $\overline{W} = \frac{W_r + W_l}{2}$ or another mean value between W_l and W_r.
Let Y be the self similar solution of this problem (when it exists...), that is:

$$Y(x,t) = LR(\frac{x}{t}, Y_l, Y_r)$$

and set $Y^{\star,\pm}(W_l, W_r) = LR(0^{\pm}, Y_l, Y_r)$.

VFRoe-ncv scheme

For $i \in \mathbb{Z}$, $n \geq 0$:

$$\frac{W_i^{n+1} - W_i^n}{k} + F_{i+\frac{1}{2}}^{n,-} - F_{i-\frac{1}{2}}^{n,+} + D(W_i^n)(Y_{i+\frac{1}{2}}^{n,-} - Y_{i-\frac{1}{2}}^{n,+}) = 0,$$

with:

$F_{i+\frac{1}{2}}^{n,\pm} = G(Y_{i+\frac{1}{2}}^{n,\pm})$ ($F(W) = G(\phi(W))$, $B(W)W_x = D(W)Y_x$),

$Y_{i+\frac{1}{2}}^{n,\pm} = Y^{\star,\pm}(W_i^n, W_{i+1}^n)$

CFL condition: $k \leq Ch$.

VFroe-ncv scheme

1. Modification for conservativity (null eigenvalue)
2. Discontinuity of the numerical flux when there is 0 as eigenvalue. $F(W^\star(W_l, W_r))$ is a discontinuous fonction of W_l and W_r).
3. Choice of Y.
   $Y = (kg(u), k)^t$ for porous media,
   $Y = (2c, u, z)^t$ or $Y = (q, \psi, z)^t$ for Saint Venant with topography.

Remark on resonance, porous media

$$u_t(x,t) + (kg(u))_x = 0,$$
$$k_t = 0.$$

With the choice $Y = (kg(u), k)^t$ the linearized system is
$Y_t + C(\overline{W})Y_x = 0$, $\overline{W} = (\overline{u}, \overline{k})^t$,

$$C(\overline{W}) = \begin{bmatrix} \overline{k}g'(\overline{u}) & 0 \\ 0 & 0 \end{bmatrix},$$

which is never resonant…
Eigenvalues: $\lambda_1 = \overline{k}g'(\overline{u})$, $\lambda_2 = 0$
Eigenvectors: $e_1 = (1,0)^t$, $e_2 = (0,1)^t$

$$\begin{aligned} &h_t + (hu)_x = 0, \\ &(hu)_t + (hu^2 + \tfrac{1}{2}gh^2)_x + ghz_x = 0, \\ &z_t = 0 \end{aligned}$$

With the choice $Y = (q, \psi, z)^t$, the linearized system is

$Y_t + C(\overline{W})Y_x = 0$, $\overline{W} = (\overline{h}, \overline{h}\overline{u}, \overline{z})^t$, $\overline{c} = \sqrt{g\overline{h}}$,

$$C(\overline{W}) = \begin{bmatrix} \overline{u} & \overline{h} & 0 \\ g & \overline{u} & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

which is never resonant (for $\overline{h} > 0$)...
Eigenvalues: $\lambda_1 = 0$, $\lambda_2 = \overline{u} + \overline{c}$, $\lambda_2 = \overline{u} - \overline{c}$
Eigenvectors: $e_1 = (0, 0, 1)^t$, $e_2 = (\overline{h}, \overline{c}, 0)^t$, $e_3 = (-\overline{h}, \overline{c}, 0)^t$

With the choice $Y = (2c, u, z)^t$, the linearized system is

$Y_t + C(\overline{W})Y_x = 0$, $\overline{W} = (\overline{h}, \overline{h}\overline{u}, \overline{z})^t$, $\overline{c} = \sqrt{g\overline{h}}$,

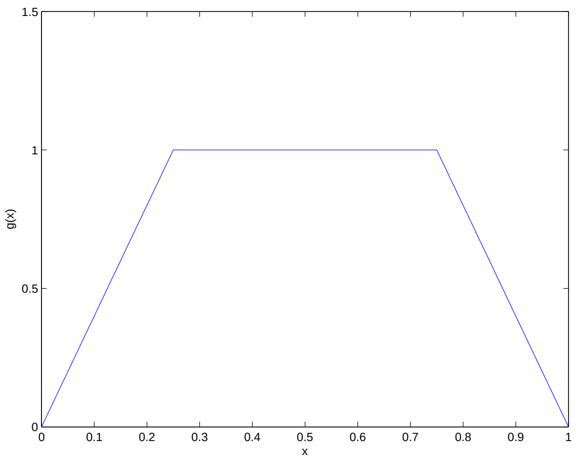
$$C(\overline{W}) = \begin{bmatrix} \overline{u} & \overline{c} & 0 \\ \overline{c} & \overline{u} & g \\ 0 & 0 & 0 \end{bmatrix},$$

which is resonant if $\overline{u} + \overline{c} = 0$ or $\overline{u} - \overline{c} = 0$.
Eigenvalues: $\lambda_1 = 0$, $\lambda_2 = \overline{u} + \overline{c}$, $\lambda_2 = \overline{u} - \overline{c}$
Eigenvectors:

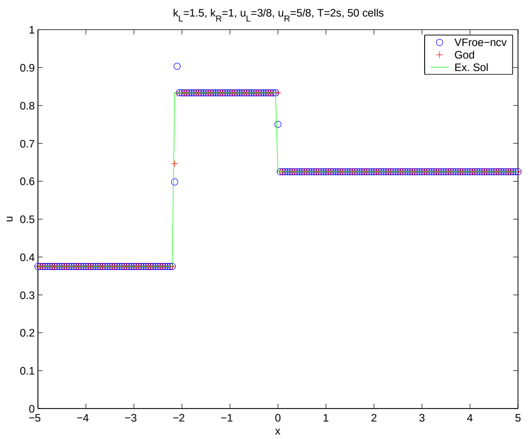
- If $\overline{u} \pm \overline{c} \neq 0$,
  $e_1 = (\overline{c}g, -\overline{u}g, \overline{u}^2 - \overline{c}^2)^t$, $e_2 = (1, 1, 0)^t$, $e_3 = (1, -1, 0)^t$
- If $\overline{u} = \overline{c}$, $e_1 = (1, -1, 0)^t$, $e_2 = (1, 1, 0)^t$
- If $\overline{u} = -\overline{c}$, $e_1 = (1, 1, 0)^t$, $e_3 = (1, -1, 0)^t$

Numerical results

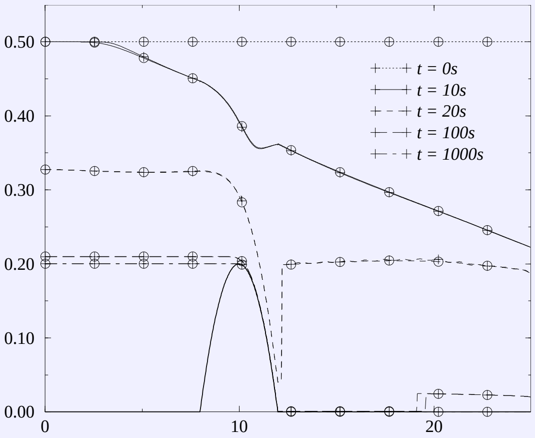
1. Perfect with Godunov. In particular, for Saint Venant with topography, one has preservation of all steady state solutions : $q = hu$ and $\psi = \frac{u^2}{2} + g(h + z)$ constant (and not only those with $u = 0$ which are called ”lake at rest”)
2. Perfect also with VFRoe-ncv, sometimes with one “incorrect point”. For Saint Venant with topography: Preservation of steady state solution with $u = 0$ (for any choice of the variable of linearization$Y$) and preservation of all steady state solutions for $Y = (q, \psi, z)^t$



Resonance occurs for all (k, u) with $u \in (\frac{1}{4}, \frac{3}{4})$.



Resonance occurs for all (k, u) with $u \in (\frac{1}{4}, \frac{3}{4})$.



A case with $h = 0$, steady state, sonic point. . . VFRoe-ncv with $Y = (2c, u, z)^t$ (best choice for the problem of vanishing h).

References

1. Porous medium: Towers, Seguin-Vovelle, Bachmann-Vovelle, Bachmann
2. Saint Venant: Kurganov-Levy, Simeoni-Perthame, Chinnayya-LeRoux-Seguin, Goutal, Karni Hérard-Gallouët-Seguin. . .