A parabolic equation with a flux limiter

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The complete problem

Model of erosion and sedimentation process

$$H_t(x,t) - \mathrm{div}[\overline{u}(x,t)\Lambda(x)\nabla H(x,t)] = 0,$$

$$H_t(x,t) \geq -F(x),$$

$$0 \leq \overline{u}(x,t) \leq 1,$$

$$(\overline{u}(x,t) - 1)\,(H_t(x,t) + F(x)) = 0.$$

$(x,t) \in \Omega \times (0,T)$, Ω : bounded open set of $\mathbb{R}^d$ ($d \geq 1$).
Initial and Boundary Conditions on H.
$F \geq 0$ a.e..

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The complete problem

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Time discretization of the complete problem

Time step : k, $t_n = nk$. $H_{n+1} = H(\cdot, t_{n+1})$, $u_{n+1} = \overline{u}(\cdot, t_{n+1})$.

$$\frac{H_{n+1} - H_n}{k} - \mathrm{div}[u_{n+1}\Lambda(x)\nabla H_{n+1}(x,t)] = 0,$$

$$\mathrm{div}[u_{n+1}\Lambda\nabla H_{n+1}] + F \geq 0,$$

$$0 \leq u_{n+1} \leq 1,$$

$$(u_{n+1} - 1)\,(\mathrm{div}[u_{n+1}\Lambda\nabla H_{n+1}] + F(x)) = 0.$$

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$$0 \leq u_{n+1} \leq 1,$$

$$(u_{n+1} - 1)\left(\mathrm{div}[u_{n+1}\Lambda\nabla H_n] + F(x)\right) = 0.$$

Intermediate problem

$g : \Omega \to \mathbb{R}^d$, Lipschitz continuous, $g \cdot n = 0$ on $\partial\Omega$.
$F \in L^\infty(\Omega)$, $F \geq 0$ a.e..

$$\mathrm{div}(ug) + F \geq 0, \ \text{in } \Omega,$$

$$0 \leq u \leq 1, \ \text{in } \Omega,$$

$$(u-1)\,(\mathrm{div}(ug) + F) = 0, \ \text{in } \Omega.$$

u is not unique (example : $g = 0$, $F = 0$ on ω).
Hyperbolic Inequality.

Associated evolution problem

$g : \Omega \to \mathbb{R}^d$, Lipschitz continuous, $g \cdot n = 0$ on $\partial\Omega$.
$F \in L^\infty(\Omega)$, $F \geq 0$ a.e..

$$u_t - \mathrm{div}(ug) - F \leq 0, \text{ in } \Omega \times (0, \infty),$$

$$0 \leq u \leq 1, \text{ in } \Omega \times (0, \infty),$$

$$(u-1)u_t - (u-1)\,(\mathrm{div}(ug) + F) = 0, \text{ in } \Omega \times (0, \infty),$$

with initial condition $u(x, 0) = 1$ for a.e. $x \in \Omega$.
Hyperbolic Inequality.

Associated evolution problem

$g : \Omega \to \mathbb{R}^d$, Lipschitz continuous, $g \cdot n = 0$ on $\partial\Omega$.
$F \in L^\infty(\Omega)$, $F \geq 0$ a.e..

$$u_t - \mathrm{div}(ug) - F = 0, \text{ in } \Omega \times (0, \infty),$$

$$0 \leq u \leq 1, \text{ in } \Omega \times (0, \infty),$$

$$(u - 1)u_t - (u - 1)\left(\mathrm{div}(ug) + F\right) = 0, \text{ in } \Omega \times (0, \infty),$$

with initial condition $u(x, 0) = 1 \text{ for a.e. } x \in \Omega$.
u may not exist (example : $\mathrm{div}(g) + F > 0 \text{ on } \omega$).

The intermediate problem

$g : \Omega \to \mathbb{R}^d$, Lipschitz continuous, $g \cdot n = 0$ on $\partial\Omega$.
$F \in L^\infty(\Omega)$, $F \geq 0$ a.e..

$$\mathrm{div}(ug) + F \geq 0, \text{ in } \Omega, \tag{1}$$

$$0 \leq u \leq 1, \text{ in } \Omega, \tag{2}$$

$$(u - 1)\,(\mathrm{div}(ug) + F) = 0, \text{ in } \Omega. \tag{3}$$

Existence of u, uniqueness of ug, computation of ug.

Weak solution of (1)-(3)

$u \in L^\infty(\Omega)$, $0 \le u \le 1$ a.e.,

$$\int_\Omega (\xi(u(x))(-g(x)\cdot\nabla\varphi(x)) +$$

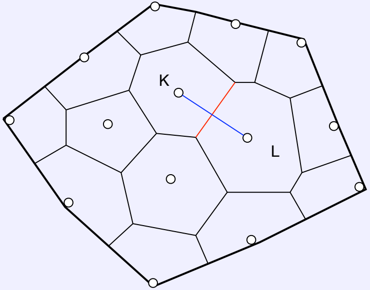
$$(\xi'(u(x))u(x) - \xi(u(x)))\varphi(x)\mathrm{div} g(x) + \tag{4}$$

$$\xi'(u(x))\varphi(x)F(x))dx \ge 0,$$

for all $\xi \in C^1(\mathbb{R})$, convex s.t. $\xi'(1) \ge 0$, and $\varphi \in C^1(\overline{\Omega}, \mathbb{R}_+)$.

$\xi(s) = s$ gives (1) and $\xi(s) = (s-1)^2$ gives (3).
(If gu is Lipschitz continuous (4) is equivalent to (1)-(3).)

Approximate solution of (1)-(3), mesh



$T_{K,L} = m_{K,L} / d_{K,L}$

$\mathrm{size}(\mathcal{T}) = \sup\{\mathrm{diam}(K), K \in \mathcal{T}\}$, m_K is the measure of K

Approximation of $\mathrm{div}(ug) + F$ on K

$\mathcal{N}_K$ is the subset of $\mathcal{T}$ of all the control volumes having a common interface with K.

$$g_{K,L} = \int_{K|L} g(x) \cdot n_{K,L} d\gamma(x), \ \forall K \in \mathcal{T}, \ \forall L \in \mathcal{N}_K.$$

or, if $g = \nabla h$,

$$g_{K,L} = \tau_{KL}(h_L - h_K), \ \forall K \in \mathcal{T}, \ \forall L \in \mathcal{N}_K.$$

$$F_K = \int_K F(x) dx.$$

Approximation of $\mathrm{div}(ug) + F$ on K with an upwind choice of u on $K|L$:

$$\sum_{L \in \mathcal{N}_K} (g^+_{K,L} u_L - g^-_{K,L} u_K) + F_K$$

Approximate solution of (1)-(3), scheme

For all K:

$$\sum_{L \in \mathcal{N}_K} (g^+_{K,L} u_L - g^-_{K,L} u_K) + F_K \geq 0,$$

$$0 \leq u_K \leq 1, \tag{5}$$

$$(\sum_{L \in \mathcal{N}_K} (g^+_{K,L} u_L - g^-_{K,L} u_K) + F_K)(u_K - 1) = 0.$$

Definition of the approximate solution, $u_{\mathcal{T}}$:

$$u_{\mathcal{T}}(x) = u_K, \ \forall x \in K, \ \forall K \in \mathcal{T}.$$

difficulties. . .

- Existence of $u_{\mathcal{T}}$, computation of $u_{\mathcal{T}}$,
- Estimates on $u_{\mathcal{T}}$,
- Convergence of $u_{\mathcal{T}}$ to a solution of (4) (weak formulation of (1)-(3)) as $\mathrm{size}(\mathcal{T}) \to 0$.

Existence of $u_{\mathcal{T}}$, computation of $u_{\mathcal{T}}$, 1

Initialization: $u_K^{(0)} = 1$ and $p_K^{(0)} = 1$, for all $K \in \mathcal{T}$.

Iterations: Let $n \in \mathbb{N}^\star$. Assume that $u_K^{(n-1)}$ and $p_K^{(n-1)}$ are known for all $K \in \mathcal{T}$.

1. Computation of $\{p_K^{(n)}, K \in \mathcal{T}\}$:

$$p_K^{(n)} = 0, \text{ if } \sum_{L \in \mathcal{N}_K} (g_{K,L}^+ u_L^{(n-1)} - g_{K,L}^- u_K^{(n-1)}) + F_K < 0,$$
$$p_K^{(n)} = p_K^{(n-1)}, \text{ if } \sum_{L \in \mathcal{N}_K} (g_{K,L}^+ u_L^{(n-1)} - g_{K,L}^- u_K^{(n-1)}) + F_K \geq 0.$$

2. Computation of $\{u_K^{(n)}, K \in \mathcal{T}\}$ (linear system):

$$\sum_{L \in \mathcal{N}_K} (g_{K,L}^+ u_L^{(n)} - g_{K,L}^- u_K^{(n)}) = -F_K, \text{ if } p_K^{(n)} = 0,$$
$$u_K^{(n)} = 1, \text{ if } p_K^{(n)} = 1.$$

Existence of $u_{\mathcal{T}}$, computation of $u_{\mathcal{T}}$, 2

1. There exists a unique family $\{(p_K^{(n)}, u_K^{(n)}), K \in \mathcal{T}, n \in \mathbb{N}\}$ solution of the preceding algorithm.
2. For all $K \in \mathcal{T}$ and all $n \in \mathbb{N}$, one has $u_K^{(n)} \geq 0$.
3. For all $K \in \mathcal{T}$, the sequence $(u_K^{(n)})_{n\in\mathbb{N}}$ is nonincreasing.
4. There exists $n \leq \text{card}(\mathcal{T})$ such that, setting $u_K = u_K^{(n)}$ for all $K \in \mathcal{T}$, the family $\{u_K, K \in \mathcal{T}\}$ is such that $u_K^{(p)} = u_K$ for all $K \in \mathcal{T}$ and $p \geq n$. This family is therefore a solution of (5)

Link with variational inequalities

Upwinding on u is related to add a diffusion term ($g = (g_1, \ldots, g_d)^t$):

$$\mathrm{size}(\mathcal{T}) \sum_{i=1}^{d} D_i(|g_i| D_i u) + \mathrm{div}(ug) + F \geq 0, \text{ in } \Omega,$$

$$0 \leq u \leq 1, \text{ in } \Omega,$$

$$(u - 1)\,(\mathrm{size}(\mathcal{T}) \sum_{i=1}^{d} D_i(|g_i| D_i u) + \mathrm{div}(ug) + F) = 0, \text{ in } \Omega.$$

which is a variational inequality (with a transport term).
The preceding algorithm works for many variational inequalities (see Herbin ,SINUM, 2003, for obstacle and Signorini problems).

Estimate on $u_{\mathcal{T}}$

1. L^∞-estimate: $\|u_{\mathcal{T}}\|_\infty \leq 1$,
2. Weak-BV inequality:

$$\sum_{(K,L)\in\mathcal{E}} |g_{K,L}|(u_K - u_L)^2 \leq C.$$

Only weak-$\star$ compactness in L^∞.
The weak-BV estimate looks like $\sum_{i=1}^{2} \|g_i D_i u\|_{L^2} \leq \frac{1}{\sqrt{\text{size}(\mathcal{T})}}$ (with $g = (g_1, \ldots, g_d)^t$).

Nonlinear weak convergence, young measures

$L^\infty(\Omega)$-estimate on $u_\mathcal{T}$ gives (up to subsequences of sequences of approximate solutions) that there exists $u \in L^\infty(\Omega \times (0,1))$ such that $u_\mathcal{T} \to u$, as $\mathrm{size}(\mathcal{T}) \to 0$ in the following sense:

$$\int_\Omega \xi(u_\mathcal{T}(x))\varphi(x)dx \to \int_0^1 \int_\Omega \xi(u(x,\alpha))\varphi(x)dx d\alpha,$$

for all $\varphi \in L^1(\Omega)$ and all $\xi \in C(\mathbb{R},\mathbb{R})$.

That is:

$$\xi(u_\mathcal{T}) \to \int_0^1 \xi(u(\cdot,\alpha))d\alpha, \ L^\infty(\Omega) \text{ weak-}\star.$$

Weak solution of (1)-(3)

$u \in L^\infty(\Omega)$, $0 \le u \le 1$ a.e.,

$$\int_\Omega (\xi(u(x))(-g(x)\cdot\nabla\varphi(x)) +$$

$$(\xi'(u(x))u(x) - \xi(u(x)))\varphi(x)\mathrm{div}g(x) +$$

$$\xi'(u(x))\varphi(x)F(x))dx \ge 0,$$

for all $\xi \in C^1(\mathbb{R})$, convex s.t. $\xi'(1) \ge 0$, and $\varphi \in C^1(\overline{\Omega}, \mathbb{R}_+)$.

Weak process solution of (1)-(3)

Assuming $u_{\mathcal{T}} \to u$, as $\mathrm{size}(\mathcal{T}) \to 0$, in the nonlinear weak sense, one proves (thanks to the weak BV estimate) that u is a weak process solution:

$u \in L^\infty(\Omega \times (0,1))$, $0 \leq u \leq 1$ a.e.,

$$\int_0^1 \int_\Omega (\xi(u(x,\alpha))(-g(x) \cdot \nabla\varphi(x))+$$

$$(\xi'(u(x,\alpha))u(x,\alpha) - \xi(u(x,\alpha)))\varphi(x)\mathrm{div} g(x)+$$

$$\xi'(u(x,\alpha))\varphi(x)F(x))dx d\alpha \geq 0,$$

for all $\xi \in C^1(\mathbb{R})$, convex s.t. $\xi'(1) \geq 0$, and $\varphi \in C^1(\overline{\Omega}, \mathbb{R}_+)$.

Uniqueness of the weak process solution

$g = \Lambda \nabla h$, g Lipschitz continuous and $g \cdot n = 0$ on $\partial\Omega$.
If $u \in L^\infty(\Omega \times (0,1))$, $0 \leq u \leq 1$ a.e., is a weak process solution of (1)-(3), then:

- $u(x,\alpha)$ does not depends on α for a.e. x in $\{g \neq 0\}$.
- $x \mapsto g(x)u(x)$ is the unique solution of (4) (weak formulation of (1)-(3)).
- $gu_{\mathcal{T}}$ converges to gu in $(L^p(\Omega))^d$ for all $p < \infty$.

The proof uses the doubling variables method of Krushkov.

Conclusion for the intermediate problem

$g = \Lambda \nabla h$, g Lipschitz continuous and $g \cdot n = 0$ on $\partial\Omega$.
$F \in L^\infty(\Omega)$, $F \geq 0$ a.e..

$$\mathrm{div}(ug) + F \geq 0, \text{ in } \Omega,$$

$$0 \leq u \leq 1, \text{ in } \Omega,$$

$$(u - 1)\,(\mathrm{div}(ug) + F) = 0, \text{ in } \Omega.$$

Existence of a weak solution u (in the sense of (4)) and uniqueness of ug. Computation of an approximate solution.

Another weak formulation of (1)-(3)

$g : \Omega \to \mathbb{R}^d$, g Lipschitz continuous and $g \cdot n = 0$ on $\partial\Omega$.
$F \in L^\infty(\Omega)$, $F \geq 0$ a.e..
$C(g, F)$ is the set of functions $v \in L^2(\Omega)$ such that:

$$0 \leq v \leq 1, \text{ a.e. in } \Omega$$

$$\int_\Omega (-vg \cdot \nabla\varphi + F\varphi) dx \geq 0, \ \forall \varphi \in C^1(\overline{\Omega}, \mathbb{R}_+).$$

$C(g, F)$ is a closed convex subset of $L^2(\Omega)$, and $0 \in C(g, F)$.
Then, $u_\mathcal{T} \to u$ in $L^p(\Omega)$, for all $p < \infty$, as $\text{size}(\mathcal{T}) \to 0$, and:

$$u = P_{C(g,F)} 1_\Omega.$$

u is the "mild" solution of (1)-(3).

Characterization of $u_{\mathcal{T}}$

$g : \Omega \to \mathbb{R}^d$, g Lipschitz continuous and $g \cdot n = 0$ on $\partial\Omega$.
$F \in L^\infty(\Omega)$, $F \geq 0$ a.e..
$C(g, F, \mathcal{T})$ is the set of functions $v \in L^2(\Omega)$ such that $v = v_K$ a.e. in K, with $v_K \in \mathbb{R}$, for all $K \in \mathcal{T}$ and:

$$0 \leq v_K \leq 1, \text{ for all } K \in \mathcal{T}$$

$$\sum_{L \in \mathcal{N}_K} (g^+_{K,L} v_L - g^-_{K,L} v_K) + F_K \geq 0.$$

$C(g, F, \mathcal{T})$ is a closed convex subset of $L^2(\Omega)$, and $0 \in C(g, F, \mathcal{T})$.

$$u_{\mathcal{T}} = P_{C(g,F,\mathcal{T})} 1_\Omega.$$

Open questions...

Existence, uniqueness, convergence of the numerical scheme for the complete problem

$$H_t(x,t) - \mathrm{div}[\overline{u}(x,t)\Lambda(x)\nabla H(x,t)] = 0,$$

$$H_t(x,t) \geq -F(x),$$

$$0 \leq \overline{u}(x,t) \leq 1,$$

$$(\overline{u}(x,t) - 1)\,(H_t(x,t) + F(x)) = 0.$$

$(x,t) \in \Omega \times (0,T)$, Ω : bounded open set of $\mathbb{R}^d$ $(d \geq 1)$.
Initial and Boundary Conditions on H.
$F \geq 0$ a.e..
Solve the intermediate problem with less regularity on $g = \Lambda\nabla h$.