Discrete functional analysis tools for some evolution equations

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Objective : To present discrete functional analysis tools for proving the convergence of numerical schemes, mainly for parabolic equations (Stefan problem, incompressible and compressible Navier-Stokes equations)

Works with many co-authors

First example, compressible Navier-Stokes Equations

Ω : bounded open connected set of $\mathbb{R}^3$
$T > 0$, $\gamma > 3/2$, $f \in L^2(]0, T[, L^2(\Omega))$

$$\partial_t \rho + \mathrm{div}(\rho u) = 0,$$
$$\partial_t(\rho u) + \mathrm{div}(\rho u \otimes u) - \Delta u + \nabla p = f,$$
$$p = \rho^\gamma.$$

Dirichlet boundary condition : $u = 0$
Initial condition on ρ and u (or on ρu).

First example, compressible Navier-Stokes Equations

Ω : bounded open connected set of $\mathbb{R}^3$
$T > 0$, $\gamma > 3/2$, $f \in L^2(]0, T[, L^2(\Omega))$

$$\begin{aligned} \partial_{n,t}\rho + \mathrm{div}_n(\rho_n u_n) = 0, \\ \partial_{n,t}(\rho_n u_n) + \mathrm{div}_n(\rho_n u_n \otimes u_n) - \Delta_n u_n + \nabla_n p_n = f_n, \\ p_n = \rho_n^\gamma. \end{aligned}$$

- Estimates on u_n, ρ_n, p_n
- Passing to the limit on $\rho_n u_n$ and $\rho_n u_n \otimes u_n$
- Passing to the limit on $p_n = \rho_n^\gamma$.

For nonlinear terms, weak convergences are not sufficient

Second example, Stefan problem

Ω : bounded open connected set of $\mathbb{R}^3$, $T > 0$

$\partial_t \rho - \Delta u = 0$, $u = \varphi(\rho)$

$\varphi \in C(\mathbb{R}, \mathbb{R})$ is nondecreasing $\varphi' = 0$ on $]a, b[$, $a < b$

Dirichlet boundary condition : $u = 0$
Initial condition on ρ

Second example, Stefan problem

Ω : bounded open connected set of $\mathbb{R}^3$, $T > 0$

$\partial_{n,t}\rho_n - \Delta_n u_n = 0$, $u_n = \varphi(\rho_n)$

$\varphi \in C(\mathbb{R},\mathbb{R})$ is nondecreasing $\varphi' = 0$ on $]a,b[$, $a < b$

- Estimates on u_n, ρ_n
- Passing to the limit on the equation, $\partial_t\rho - \Delta u = 0$
- Prove $u = \varphi(\rho)$

First step : Prove that $\displaystyle\int_{]0,T[\times\Omega} \rho_n u_n \to \int_{]0,T[\times\Omega} \rho u$

Second step : Minty trick, $u = \varphi(\rho)$

Common difficulty for this two examples

Ω : bounded open connected set of $\mathbb{R}^3$, $T > 0$,

$\rho_n \to \rho$ weakly in $L^2(]0, T[, L^q(\Omega))$
$u_n \to u$ weakly in $L^2(]0, T[, L^p(\Omega))$

$1 < p, q < +\infty$, $\frac{1}{p} + \frac{1}{q} = 1$

Question : $\displaystyle\int_0^T \int_\Omega \rho_n u_n \to \int_0^T \int_\Omega \rho u$?

In general, no. We need an additional hypothesis

Continuous setting, Stationary case

Discrete setting mimics continuous setting.

Ω bounded open set of $\mathbb{R}^3$. $1 < p, q < +\infty$, $\frac{1}{p} + \frac{1}{q} = 1$

$\rho_n \to \rho$ weakly in $L^q(\Omega)$
$u_n \to u$ weakly in $L^p(\Omega)$

Question : $\displaystyle\int_\Omega \rho_n u_n \to \int_\Omega \rho u$?

- in general, no.
- yes if $(u_n)_n$ is bounded in $H_0^1(\Omega)$ and $p < 6$

Two methods,

- Compactness on $(u_n)_n$ (M1)
- Compactness on $(\rho_n)_n$ (M2)

Continuous setting, Stationary case, M1

Ω bounded open set of $\mathbb{R}^3$, $1 < p < 6$, $q = p/(p-1)$

$\rho_n \to \rho$ weakly in $L^q(\Omega)$
$u_n \to u$ weakly in $L^p(\Omega)$
$(u_n)_n$ is bounded in $H_0^1(\Omega)$

Compact embedding of $H_0^1(\Omega)$ in $L^p(\Omega)$

Then
$u_n \to u$ in $L^p(\Omega)$
$\rho_n \to \rho$ weakly in $L^q(\Omega)$
and $\displaystyle\int_\Omega \rho_n u_n \to \int_\Omega \rho u$

Continuous setting, Stationary case, M2

Ω bounded open set of $\mathbb{R}^3$, $1 < p < 6$, $q = p/(p-1)$

$\rho_n \to \rho$ weakly in $L^q(\Omega)$
$u_n \to u$ weakly in $L^p(\Omega)$
$(u_n)_n$ is bounded in $H_0^1(\Omega)$

Identify $L^2(\Omega)'$ with $L^2(\Omega)$

Compact embedding of $L^q(\Omega)$ in $H^{-1}(\Omega)$

Then
$u_n \to u$ weakly in $H_0^1(\Omega)$
$\rho_n \to \rho$ in $H^{-1}(\Omega)$

and $\displaystyle\int_\Omega \rho_n u_n = \langle \rho_n, u_n \rangle_{H^{-1},H_0^1} \to \langle \rho, u \rangle_{H^{-1},H_0^1} = \int_\Omega \rho u$

Discrete setting, stationary case

It is possible to adapt the previous methods to a discrete setting where $H_0^1(\Omega)$ is replaced by a space H_n which depends on n (with a norm, depending on n, "close" to the H_0^1-norm).

Space discretization, Finite Volume scheme

Mesh $\mathcal{M}$.

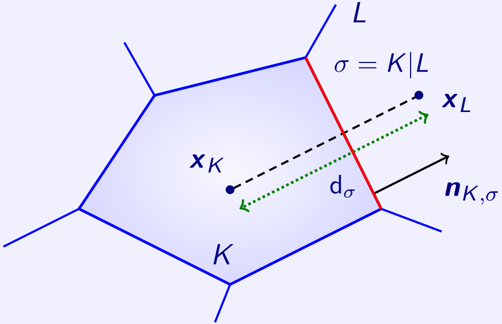

Figure: Here is an example of *admissible mesh*

$\mathrm{size}(\mathcal{M}) = \sup\{\mathrm{diam}(K), K \in \mathcal{M}\}$

$H_{\mathcal{M}}$: functions from Ω to $\mathbb{R}$, constant on each K, $K \in \mathcal{M}$

Discrete H_0^1-norm

Mesh: $\mathcal{M}$ (not necessarily admissible)

$u \in H_{\mathcal{M}}$ (that is u is a function constant on each K, $K \in \mathcal{M}$).

$$\|u\|_{1,2,n}^2 = \sum_{\sigma \in \mathcal{E}_{int}, \sigma = K|L} m_\sigma d_\sigma \left|\frac{u_K - u_L}{d_\sigma}\right|^2 + \sum_{\sigma \in \mathcal{E}_{ext}, \sigma \in \mathcal{E}_K} m_\sigma d_\sigma \left|\frac{u_K}{d_\sigma}\right|^2.$$

Discrete setting, Stationary case, M1

$\rho_n, u_n \in H_{\mathcal{M}_n}$, $\mathrm{size}(\mathcal{M}_n) \to 0$ as $n \to \infty$ (regularity of the meshes)

$p = q = 2$ (for simplicity)

$\rho_n \to \rho$ weakly in $L^2(\Omega)$
$u_n \to u$ weakly in $L^2(\Omega)$
$(u_n)_n$ is bounded in $H_{\mathcal{M}_n}, \|\cdot\|_{1,2,\mathcal{M}_n}$

"Compact embedding" of $(H_{\mathcal{M}_n}, \|\cdot\|_{1,2,\mathcal{M}_n})_n$ in $L^2(\Omega)$

Then
$u_n \to u$ in $L^2(\Omega)$
$\rho_n \to \rho$ weakly in $L^2(\Omega)$
and $\int_\Omega \rho_n u_n \to \int_\Omega \rho u$

Admissible meshes: Compactness follows from
$\|u(\cdot + \eta) - u\|_2 \leq C\sqrt{|\eta|}\|u\|_{1,2,\mathcal{M}_n}$ if $u \in H_{\mathcal{M}_n}$

Discrete setting, Stationary case, M1

$\rho_n \to \rho$ weakly in $L^2(\Omega)$
$u_n \to u$ weakly in $L^2(\Omega)$
$(u_n)_n$ is bounded in $H_{\mathcal{M}_n}, \|\cdot\|_{1,2,\mathcal{M}_n}$

"Compact embedding" of $(H_{\mathcal{M}_n}, \|\cdot\|_{1,2,\mathcal{M}_n})_n$ in $L^2(\Omega)$

Then $\int_\Omega \rho_n u_n \to \int_\Omega \rho u$

Non admissible meshes: Compactness follows from (d=3)
$\|u(\cdot+\eta)-u\|_2 \le C|\eta|^{\frac{2}{5}}\|u\|_{1,2,\mathcal{M}_n}$ if $u \in H_{\mathcal{M}_n}$

Proof using, for $u \in H_n$,
$\|u(\cdot+\eta)-u\|_{L^1(\mathbb{R}^3)} \le |\eta|\sqrt{d}\|u\|_{1,2,n}$ and
$\|u\|_{L^6(\mathbb{R}^3)} \le C\|u\|_{1,2,n}$ if $u \in H_n$ (Discrete Sobolev embedding)

Discrete setting, Stationary case, M2

$\rho_n, u_n \in H_{\mathcal{M}_n}$, $\mathrm{size}(\mathcal{M}_n) \to 0$ as $n \to \infty$ (regularity of the meshes)

$\rho_n \to \rho$ weakly in $L^2(\Omega)$
$u_n \to u$ weakly in $L^2(\Omega)$
$(u_n)_n$ is bounded in $H_{\mathcal{M}_n}, \|\cdot\|_{1,2,\mathcal{M}_n}$

This gives $(u_n)_n$ is bounded in $H^s(\Omega)$, $0 < s < 2/5$
Identify $L^2(\Omega)'$ with $L^2(\Omega)$, since $H^s(\Omega)$ is compact in $L^2(\Omega)$,
Compact embedding of $L^2(\Omega)$ in $H^{-s}(\Omega)$

Then
$u_n \to u$ weakly in $H^s(\Omega)$
$\rho_n \to \rho$ in $H^{-s}(\Omega)$
and $\displaystyle\int_\Omega \rho_n u_n = \langle \rho_n, u_n \rangle_{H^{-s},H^s} \to \langle \rho, u \rangle_{H^{-s},H^s} = \int_\Omega \rho u$

Continuous setting, evolution case

$\rho_n \to \rho$ weakly in $L^2(]0,T[, L^2(\Omega))$
$u_n \to u$ weakly in $L^2(]0,T[, L^2(\Omega))$

Question : $\displaystyle\int_{]0,T[\times\Omega[} \rho_n u_n \to \int_{]0,T[\times\Omega[} \rho u$?

- in general, no. Even if $(u_n)_n$ is bounded in $L^2(]0,T[, H_0^1(\Omega))$
  No compactness of $L^2(]0,T[, H_0^1(\Omega))$ in $L^2(]0,T[, L^2(\Omega))$
- yes if $(u_n)_n$ is bounded in $H^1(]0,T[, H_0^1(\Omega))$ since compactness of $H^1(]0,T[, H_0^1(\Omega))$ in $L^2(]0,T[, L^2(\Omega))$
- yes if $(\rho_n)_n$ is bounded in $H^1(]0,T[, L^2(\Omega))$ since compactness of $H^1(]0,T[, L^2(\Omega))$ in $L^2(]0,T[, H^{-1}(\Omega))$

Is it possible to use weaker hypotheses on $(\partial_t u_n)_n$ or $(\partial_t \rho_n)_n$?

Continuous setting, (Generalized) Aubin-Simon Compactness Lemma

X, B, Y are three Banach spaces, $X \subset B$, $X \subset Y$ such that

1. X compactly embedded in B
2. $\|w_n\|_X \leq C$, $\|w_n - w\|_B \to 0$, $\|w_n\|_Y \to 0$ implies $w = 0$

Let $T > 0$ $1 \leq p < +\infty$ and $(u_n)_{n\in\mathbb{N}}$ be a sequence such that

- $(u_n)_{n\in\mathbb{N}}$ is bounded in $L^p(]0, T[, X)$,
- $(\partial_t u_n)_{n\in\mathbb{N}}$ is bounded in $L^1(]0, T[, Y)$.

Then there exists $u \in L^p(]0, T[, B)$ such that, up to a subsequence, $u_n \to u$ in $L^p(]0, T[, B)$

Particular cases for hypothesis 2:

Easy case : $Y = X$ or B or, more generally, $\|\cdot\|_B \leq C\|\cdot\|_Y$
Aubin Simon : B continuously embedded in Y, $\|\cdot\|_Y \leq C\|\cdot\|_B$

Generalized Lions lemma (crucial if $\|\cdot\|_B \not\leq C\|\cdot\|_Y$)

X, B, Y are three Banach spaces, $X \subset B$, $X \subset Y$ such that

1. X compactly embedded in B
2. $\|w_n\|_X \leq C$, $\|w_n - w\|_B \to 0$, $\|w_n\|_Y \to 0$ implies $w = 0$

Then, for any $\varepsilon > 0$, there exists C_ε such that, for $w \in X$,

$$\|w\|_B \leq \varepsilon\|w\|_X + C_\varepsilon\|w\|_Y.$$

Proof: By contradiction

Classical Lions lemma, a particular case, simpler

B is a Hilbert space and X is a Banach space $X \subset B$. We define on X the dual norm of $\| \cdot \|_X$, with the scalar product of B, namely

$$\|u\|_Y = \sup\{(u|v)_B,\ v \in X, \|v\|_X \leq 1\}.$$

Then, for any $\varepsilon > 0$ and $w \in X$,

$$\|w\|_B \leq \varepsilon \|w\|_X + \frac{1}{\varepsilon}\|w\|_Y.$$

The proof is simple since

$$\|u\|_B = (u|u)_B^{\frac{1}{2}} \leq (\|u\|_Y \|u\|_X)^{\frac{1}{2}} \leq \varepsilon \|w\|_X + \frac{1}{\varepsilon}\|w\|_Y.$$

Compactness of X in B is not needed here (but this compactness is needed for Aubin-Simon Compactness Lemma).

Continuous setting, evolution case, compressible NS, M2

$\rho_n \to \rho$ weakly in $L^2(]0,T[,L^2(\Omega))$, if $\gamma \geq 2$
$u_n \to u$ weakly in $L^2(]0,T[,L^2(\Omega)^3)$
$(u_n)_n$ is bounded in $L^2(]0,T[,H_0^1(\Omega)^3)$
$\partial_t \rho_n + \mathrm{div}(\rho_n u_n) = 0$

Then $(\partial_t \rho_n)_n$ is bounded in $L^1(]0,T[,W^{-1,1}(\Omega))$

This gives compactness of $(\rho_n)_n$ in $L^2(]0,T[,H^{-1}(\Omega))$
(Aubin-Simon compactness Theorem with :
$X = L^2(\Omega)$, $B = H^{-1}(\Omega)$, $Y = W^{-1,1}(\Omega)$))

$u_n \to u$ weakly in $L^2(]0,T[,H_0^1(\Omega)^3)$
$\rho_n \to \rho$ in $L^2(]0,T[,H^{-1}(\Omega))$
and, for any regular φ,

$$\int_{]0,T[\times\Omega} \rho_n u_n \cdot \nabla\varphi = \langle \rho_n, u_n \cdot \nabla\varphi \rangle_{L^2(H^{-1}),L^2(H_0^1)} \to \int_{]0,T[\times\Omega} \rho u \cdot \nabla\varphi$$

which gives $\partial_t \rho + \mathrm{div}(\rho u) = 0$

Continuous setting, evolution case, compressible NS, M2

$u_n \to u$ weakly in $L^2(]0,T[,H_0^1(\Omega)^3)$
$\rho_n u_n \to \rho u$ weakly in $L^2(]0,T[,L^r(\Omega)^3)$
with $r = 6\gamma/(6+\gamma) \geq 2$ if $\gamma \geq 3$

$\partial_t(\rho_n u_n) + \mathrm{div}(\rho_n u_n \otimes u_n) - \Delta u_n + \nabla p_n = f$
Then $(\partial_t(\rho_n u_n))_n$ is bounded in $L^1(]0,T[,W^{-1,1}(\Omega)^3)$

This gives compactness of $(\rho_n u_n)_n$ in $L^2(]0,T[,H^{-1}(\Omega)^3)$
(Aubin-Simon compactness Theorem with :
$X = L^2(\Omega)$, $B = H^{-1}(\Omega)$, $Y = W^{-1,1}(\Omega)$))

$u_n \to u$ weakly in $L^2(]0,T[,H_0^1(\Omega)^3)$
$\rho_n u_n \to \rho u$ in $L^2(]0,T[,H^{-1}(\Omega)^3)$

Which gives the convergence (in the distributional sense) of $\rho_n u_n \otimes u_n$ to $\rho u \otimes u$ (and allows passing to the limit in the momentum equation)

Continuous setting, evolution case, Stefan, M1

$\rho_n \to \rho$ weakly in $L^2(]0,T[,L^2(\Omega))$
$u_n \to u$ weakly in $L^2(]0,T[,L^2(\Omega))$
$(u_n)_n$ is bounded in $L^2(]0,T[,H_0^1(\Omega))$
$\partial_t \rho_n - \Delta u_n = 0,\ u_n = \varphi(\rho_n)$

$\varphi \in C(\mathbb{R},\mathbb{R})$ is nondecreasing $\varphi' = 0$ on $]a,b[$, $a < b$

one has $\partial_t \rho - \Delta u = 0$, but $u = \varphi(\rho)$?
First step : pass to the limit on $\int \rho_n u_n$

no direct estimate on $\partial_t u_n$, but (Alt-Luckaus trick) estimate on the time-translates of u_n
Then compactness of $(u_n)_n$ in $L^2(]0,T[,L^2(\Omega))$

$u_n \to u$ in $L^2(]0,T[,L^2(\Omega))$
$\rho_n \to \rho$ weakly in $L^2(]0,T[,L^2(\Omega))$
and, $\int_{]0,T[\times\Omega} \rho_n u_n \to \int_{]0,T[\times\Omega} \rho u$
Second step :Minty trick, $u = \varphi(\rho)$

Minty trick

$\rho_n \to \rho$ weakly in L^2 ($L^2 = L^2(\Omega)$ or $L^2(]0,T[,L^2(\Omega))$)
$u_n \to u$ weakly in L^2
$\int \rho_n u_n \to \int \rho u$
$u_n = \varphi(\rho_n)$

$\varphi \in C(\mathbb{R},\mathbb{R})$ is nondecreasing, $|\varphi(s)| \le C|s|$

Question : $u = \varphi(\rho)$? for any $\bar\rho \in L^2$

$0 \le \int(\rho_n - \bar\rho)(\varphi(\rho_n) - \varphi(\bar\rho)) = \int(\rho_n - \bar\rho)(u_n - \varphi(\bar\rho))$
as $n \to \infty$, $0 \le \int(\rho - \bar\rho)(u - \varphi(\bar\rho))$
$\bar\rho = \rho - \varepsilon\psi$, $\varepsilon > 0$ and ψ regular function,

$$0 \le \int \psi(u - \varphi(\rho - \varepsilon\psi))$$

$\varepsilon \to 0$, ψ and $-\psi$ give $\int \psi(u - \varphi(\rho)) = 0$ and then $u = \varphi(\rho)$

Continuous setting, evolution case, Stefan, M2

$\rho_n \to \rho$ weakly in $L^2(]0,T[,L^2(\Omega))$
$u_n \to u$ weakly in $L^2(]0,T[,L^2(\Omega))$
$(u_n)_n$ is bounded in $L^2(]0,T[,H^1_0(\Omega))$
$\partial_t \rho_n - \Delta u_n = 0$, $u_n = \varphi(\rho_n)$

Then $(\partial_t \rho_n)_n$ bounded in $L^2(]0,T[,H^{-1}(\Omega))$
This gives compactness of $(\rho_n)_n$ in $L^2(]0,T[,H^{-1}(\Omega))$
(Aubin-Simon Theorem with :
$X = L^2(\Omega)$, $B = Y = H^{-1}(\Omega)$)

$u_n \to u$ weakly in $L^2(]0,T[,H^1_0(\Omega))$
$\rho_n \to \rho$ in $L^2(]0,T[,H^{-1}(\Omega))$
and, $\displaystyle\int_{]0,T[\times\Omega} \rho_n u_n \to \int_{]0,T[\times\Omega} \rho u$

which gives (Minty trick) $u = \varphi(\rho)$

Use of the compactness lemma in the previous examples

For compressible Navier Stokes eqs :
$B = H^{-1}(\Omega)$, $X = L^2(\Omega)$, $Y = W^{-1,1}(\Omega)$

For Stefan problem :
$X = L^2(\Omega)$, $B = Y = H^{-1}(\Omega)$

Is it possible to have discrete versions of these compactness results, for proving the convergence of numerical schemes ?

Space-Time discretization

$T > 0$, time step $k = \frac{T}{N}$

- $H_{\mathcal{M}}$ the space of functions from Ω to $\mathbb{R}$, constant on each K, $K \in \mathcal{M}$.
- The function u is constant on $K \times ((p-1)k, pk)$ with $K \in \mathcal{M}$ and $p \in \{1, \ldots, N\}$.
  $u(\cdot, t) = u^{(p)}$ for $t \in ((p-1)k, pk)$ and $u^{(p)} \in H_{\mathcal{M}}$.
- Discrete derivatives in time, $\partial_{t,k} u$, defined by:

$$\partial_{t,k} u(\cdot, t) = \partial_{t,k}^{(p)} u = \frac{1}{k}(u^{(p)} - u^{(p-1)}) \text{ for } t \in ((p-1)k, pk),$$

for $p \in \{2, \ldots, N\}$ (and $\partial_{t,k} u(\cdot, t) = 0$ for $t \in (0, k)$).

$\mathcal{M}$ can be different for ρ, p and each component of the velocity (MAC-scheme)

Discrete Lions lemma

B is a Banach space, $(B_n)_{n\in\mathbb{N}}$ is a sequence of finite dimensional subspaces of B. $\|\cdot\|_{X_n}$ and $\|\cdot\|_{Y_n}$ are two norms on B_n such that:
If $(\|w_n\|_{X_n})_{n\in\mathbb{N}}$ is bounded, then,

- up to a subsequence, there exists $w \in B$ s.t. $w_n \to w$ in B.
- If $\|w_n - w\|_B \to 0$ and $\|w_n\|_{Y_n} \to 0$, then $w = 0$.

Then, for any $\varepsilon > 0$, there exists C_ε such that, for $n \in \mathbb{N}$ and $w \in B_n$

$$\|w\|_B \leq \varepsilon\|w\|_{X_n} + C_\varepsilon\|w\|_{Y_n}.$$

Example: $B_n = H_{\mathcal{M}_n}$ (the finite dimensional space given by the mesh $\mathcal{M}_n$). We have to choose B, $\|\cdot\|_{X_n}$ and $\|\cdot\|_{Y_n}$.

Discrete Lions lemma, proof

Proof by contradiction. There exists $\varepsilon > 0$ and $(w_n)_{n\in\mathbb{N}}$ such that, for all n, $w_n \in B_n$ and

$$\|w_n\|_B > \varepsilon\|w_n\|_{X_n} + C_n\|w_n\|_{Y_n},$$

with $\lim_{n\to\infty} C_n = +\infty$.

It is possible to assume that $\|w_n\|_B = 1$. Then $(\|w_n\|_{X_n})_{n\in\mathbb{N}}$ is bounded and, up to a subsequence, $w_n \to w$ in B (so that $\|w\|_B = 1$). But $\|w_n\|_{Y_n} \to 0$, so that $w = 0$, in contradiction with $\|w\|_B = 1$.

Discrete Compactness Lemma

B a Banach, $1 \leq p < +\infty$, $(B_n)_{n\in\mathbb{N}}$ family of finite dimensional subspaces of B. $\|\cdot\|_{X_n}$ and $\|\cdot\|_{Y_n}$ two norms on B_n such that: If $(\|w_n\|_{X_n})_{n\in\mathbb{N}}$ is bounded, then,

- up to a subsequence, there exists $w \in B$ s.t. $w_n \to w$ in B.
- If $\|w_n - w\|_B \to 0$ and $\|w_n\|_{Y_n} \to 0$, then $w = 0$.

$X_n = B_n$ with norm $\|\cdot\|_{X_n}$, $Y_n = B_n$ with norm $\|\cdot\|_{Y_n}$. Let $T > 0$, $k_n > 0$ and $(u_n)_{n\in\mathbb{N}}$ be a sequence such that

- for all n, $u_n(\cdot, t) = u_n^{(p)} \in B_n$ for $t \in ((p-1)k_n, pk_n)$
- $(u_n)_{n\in\mathbb{N}}$ is bounded in $L^p((0,T), X_n)$,
- $(\partial_{t,k_n} u_n)_{n\in\mathbb{N}}$ is bounded in $L^1((0,T), Y_n)$.

Then there exists $u \in L^p((0,T), B)$ such that, up to a subsequence, $u_n \to u$ in $L^p((0,T), B)$.

Example: $B_n = H_{\mathcal{M}_n}$. We have to choose B, $\|\cdot\|_{X_n}$, $\|\cdot\|_{Y_n}$

Discrete setting, evolution case, compressible NS, M2

$\rho_n \to \rho$ weakly in $L^2(]0,T[,L^2(\Omega))$ $(\gamma \geq 2)$
$u_n \to u$ weakly in $L^2(]0,T[,L^2(\Omega)^3)$
$(u_n)_n$ is bounded in $L^2(]0,T[,H_n)$, with $\|\cdot\|_{1,2,\mathcal{M}_n^{(i)}}$
$\partial_{t,k_n}\rho_n + \mathrm{div}_n(\rho_n u_n) = 0$

Then $(\partial_{t,k_n}\rho_n)_n$ is bounded in $L^1(]0,T[,Y_n)$
where $Y_n = H_n^{(i)}$ with :

$$\|w\|_{Y_n} = \max\{\int_\Omega w\varphi;\ \varphi \in H_n^{(i)};\ \|\nabla_n\varphi\|_{L^\infty(\Omega)} + \|\varphi\|_{L^\infty(\Omega)} = 1\}.$$

Compactness Theorem with
$B = H^{-s}(\Omega)$ and $X_n = H_n^{(i)}$ with $L^2(\Omega)$-norm
gives compactness of $(\rho_n)_n$ in $L^2(]0,T[,H^{-s}(\Omega))$, $0 < s < 1/2$

$u_n \to u$ weakly in $L^2(]0,T[,H^s(\Omega)^3)$
$\rho_n \to \rho$ in $L^2(]0,T[,H^{-s}(\Omega))$
and, for any regular φ,

$$\int \rho_n u_n \cdot \nabla_{\mathcal{M}_n}\varphi = \langle \rho_n, u_n \cdot \nabla\varphi \rangle_{L^2(H^{-s}),L^2(H^s)} + R \to \int \rho u \cdot \nabla\varphi$$

Discrete setting, evolution case, compressible NS, M2

Similarly it is possible to prove the convergence of $\mathrm{div}_n \rho_n u_n \otimes u_n$ to $\mathrm{div} \rho u \otimes u$

$\rho_n u_n \to \rho u$ weakly in $L^2(]0,T[, L^2(\Omega)^3)$ (if $\gamma \geq 3$)
$u_n \to u$ weakly in $L^2(]0,T[, L^2(\Omega)^3)$
$(u_n)_n$ is bounded in $L^2(]0,T[, H_n)$, with $\|\cdot\|_{1,2,\mathcal{M}_n^{(i)}}$

Using the discrete momentum equation, one has essentially (for each component of u_n)
$(\partial_{t,k_n}(\rho_n u_n))_n$ is bounded in $L^1(]0,T[, Y_n)$
where $Y_n = H_n^{(i)}$ (mesh for a component of u_n) with :

$$\|w\|_{Y_n} = \max\{\int_\Omega w\varphi;\ \varphi \in H_n^{(i)};\ \|\nabla_n \varphi\|_{L^\infty(\Omega)} + \|\varphi\|_{L^\infty(\Omega)} = 1\}.$$

Compactness Theorem with
$B = H^{-s}(\Omega)$ and $X_n = H_n^{(i)}$ with $L^2(\Omega)$-norm
gives compactness of $(\rho_n u_n)_n$ in $L^2(]0,T[, H^{-s}(\Omega)^3)$, $0 < s < 1/2$
which allows to prove $\partial_t \rho u + \mathrm{div}(\rho u \otimes u) - \Delta u + \nabla p = f$

Discrete setting, evolution case, Stefan, M1

$\rho_n \to \rho$ weakly in $L^2(]0,T[, L^2(\Omega))$
$u_n \to u$ weakly in $L^2(]0,T[, L^2(\Omega))$
$(u_n)_n$ is bounded in $L^2(]0,T[, H_{\mathcal{M}_n}(\Omega))$ with $\|\cdot\|_{1,2,\mathcal{M}_n}$
$\partial_{,t,k_n}\rho_n - \Delta_n u_n = 0$, $u_n = \varphi(\rho_n)$

$\varphi \in C(\mathbb{R},\mathbb{R})$ is nondecreasing $\varphi' = 0$ on $]a,b[$, $a < b$

one has $\partial_t \rho - \Delta u = 0$, but $u = \varphi(\rho)$?
First step: pass to the limit on $\int \rho_n u_n$

no direct estimate on $\partial_{t,k_n} u_n$, but a discrete version of Alt-Luckaus trick gives an estimate on the time-translates of u_n
Then compactness of $(u_n)_n$ in $L^2(]0,T[, L^2(\Omega))$

$u_n \to u$ in $L^2(]0,T[, L^2(\Omega))$
$\rho_n \to \rho$ weakly in $L^2(]0,T[, L^2(\Omega))$
and, $\int_{]0,T[\times\Omega} \rho_n u_n \to \int_{]0,T[\times\Omega} \rho u$
Second step: Minty trick, $u = \varphi(\rho)$

Discrete setting, evolution case, Stefan, M2

$\rho_n \to \rho$ weakly in $L^2(]0,T[, L^2(\Omega))$
$u_n \to u$ weakly in $L^2(]0,T[, L^2(\Omega))$
$(u_n)_n$ is bounded in $L^2(]0,T[, H_{\mathcal{M}_n})$ with $\|\cdot\|_{1,2,\mathcal{M}_n}$
$\partial_{,t,k_n}\rho_n - \Delta_{\mathcal{M}_n} u_n = 0$, $u_n = \varphi(\rho_n)$
First step: pass to the limit on $\int \rho_n u_n$

$(\partial_{t,k_n}\rho_n)_n$ bounded in $L^2(]0,T[, H_{\mathcal{M}_n})$ with $\|\cdot\|_{-1,2,\mathcal{M}_n}$
This gives compactness of $(\rho_n)_n$ in $L^2(]0,T[, H^{-s}(\Omega))$

$B = H^{-s}(\Omega)$, $B_n = H_{\mathcal{M}_n}$, $\|\cdot\|_{X_n} = \|\cdot\|_{L^2(\Omega)}$,
$\|\cdot\|_{Y_n} = \|\cdot\|_{-1,2,\mathcal{M}_n}$ (the dual norm of the norm $\|\cdot\|_{1,2,\mathcal{M}_n}$)

$\rho_n \to \rho$ in $L^2(]0,T[, H^{-s}(\Omega))$ $(0 < s < 1/2)$
$u_n \to u$ weakly in $L^2(]0,T[, H^{s}(\Omega))$
and, $\int_{]0,T[\times\Omega} \rho_n u_n \to \int_{]0,T[\times\Omega} \rho u$

Second step: Minty trick, $u = \varphi(\rho)$

Spaces B, X_n, Y_n for compressible NS

$B = H^{-s}(\Omega)$, $0 < s < 1/2$
$Y_n = H_{\mathcal{M}_n}$ with $\| \cdot \|_{-1,1,\mathcal{M}_n}$
$X_n = H_{\mathcal{M}_n}$ with $L^2(\Omega)$-norm

- Compact embedding of $L^2(\Omega)$ in $H^{-s}(\Omega)$
- If $w_n \in H_{\mathcal{M}_n}$, $w_n \to w$ weakly in $L^2(\Omega)$ and $\|w_n\|_{-1,1,\mathcal{M}_n} \to 0$, then $w = 0$? Yes. . . Proof :
  Let $\varphi \in W_0^{1,\infty}(\Omega)$ and its "projection" $\pi_n\varphi \in H_{\mathcal{M}_n}$. One has $\|\pi_n\varphi\|_{1,\infty,\mathcal{M}_n} \le \|\varphi\|_{W^{1,\infty}(\Omega)}$ and then

$$|\int_\Omega w_n(\pi_n\varphi)dx| \le \|w_n\|_{-1,1,\mathcal{M}_n}\|\varphi\|_{W^{1,\infty}(\Omega)} \to 0,$$

  and, since $w_n \to w$ weakly in $L^1(\Omega)$ and $\pi_n\varphi \to \varphi$ uniformly,

$$\int_\Omega w_n(\pi_n\varphi)dx \to \int_\Omega w\varphi dx.$$

  This gives $\int_\Omega w\varphi dx = 0$ for all $\varphi \in W_0^{1,\infty}(\Omega)$ and then $w = 0$ a.e.